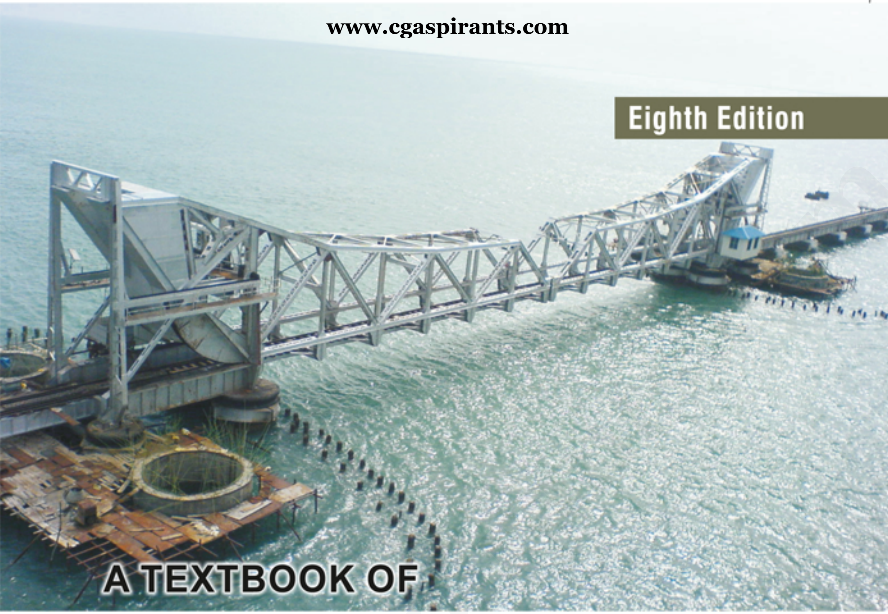


Eighth Edition

A TEXTBOOK OF ENGINEERING MATHEMATICS

(For U.P. Technical University, Lucknow)

SEMESTER-III/IV

N.P. Bali

Dr. Manish Goyal

A TEXTBOOK OF

ENGINEERING MATHEMATICS

A TEXTBOOK OF

ENGINEERING MATHEMATICS

For

B.TECH. 2nd YEAR, *SEMESTER III/IV*

Strictly according to the latest revised syllabi (NAS-301/NAS-401)

FOR UTTAR PRADESH TECHNICAL UNIVERSITY (U.P.T.U.), LUCKNOW

By

N.P. BALI
Former Principal
S.B. College, Gurgaon
Haryana

Dr. MANISH GOYAL
M.Sc. (Mathematics), Ph.D., CSIR-NET
Associate Professor
Department of Mathematics
Institute of Applied Sciences & Humanities
G.L.A. University, Mathura
U.P.

UNIVERSITY SCIENCE PRESS
(An Imprint of Laxmi Publications Pvt. Ltd.)
An ISO 9001:2008 Company
BENGALURU • CHENNAI • COCHIN • GUWAHATI • HYDERABAD
JALANDHAR • KOLKATA • LUCKNOW • MUMBAI • RANCHI • NEW DELHI
BOSTON (USA) • ACCRA (GHANA) • NAIROBI (KENYA)

A TEXTBOOK OF ENGINEERING MATHEMATICS

Printed and bound in India
Typeset at Excellent Graphics, Delhi
First Edition: 2003; Second Edition : 2005; Third Edition: 2006; Fourth Edition: 2010; Revised Fourth Edition : 2010
Fifth Edition : 2011; Sixth Edition: 2012; Reprint : 2013, 2014; Seventh Edition: 2015, Eighth Edition : 2016
ISBN 978-93-80386-53-9

Published in India by

UNIVERSITY SCIENCE PRESS
(An Imprint of Laxmi Publications Pvt.Ltd.)

An ISO 9001:2008 Company
113, GOLDEN HOUSE, DARYAGANJ,
NEW DELHI - 110002, INDIA
Telephone : 91-11-4353 2500, 4353 2501
Fax : 91-11-2325 2572, 4353 2528
www.laxmipublications.com info@laxmipublications.com

Branches

Bengaluru	080-26 75 69 30	
Chennai	044-24 34 47 26,	24 35 95 07
Cochin	0484-237 70 04,	405 13 03
Guwahati	0361-254 36 69,	251 38 81
Hyderabad	040-27 55 53 83,	27 55 53 93
Jalandhar	0181-222 12 72	
Kolkata	033-22 27 43 84	
Lucknow	0522-220 99 16	
Mumbai	022-24 91 54 15,	24 92 78 69
Ranchi	0651-220 44 64	

C—
Printed at:

Dedicated

To

Lord Krishna

CONTENTS

Preface		...	*(ix)*
Syllabus		...	*(x)*
Standard Results		...	*(xi)–(xv)*
Units			*Pages*
1.	Function of Complex Variable	...	1
2.	Integral Transforms	...	131
3.	Statistical Techniques	...	214
4.	Numerical Techniques–I	...	379
5.	Numerical Techniques–II	...	470
	Examination Papers	...	**555–566**
	Appendix	...	**(*i*)–(*iv*)**

PREFACE TO THE EIGHTH EDITION

मय्यावेश्य मनो ये मां नित्ययुक्ता उपासते।

श्रद्धया परयोपेतास्ते मे युक्ततमा मता:।।

The Eighth edition of this textbook is an outcome of new syllabus of **Engineering Mathematics III (NAS-301/NAS-401)** for the students of B. Tech. II year (Both III and IV semester in all branches) proposed and implemented by the U.P. Technical University (U.P.T.U.), Lucknow recently. The book has been renovated in the light of the latest syllabus. It will work as the latest ready reckoner for the readers.

The subject matter has been made more lucid and easier to understand. A large number of new solved examples and questions have been added. All the answers have been checked and verified. All the questions of latest university papers have been added in the body of the text. The suggestions from our colleagues and readers have been incorporated at the proper places. An appreciably heavy demand of the book ensures its utility to the users.

Separate exercise (TEST YOUR KNOWLEDGE) have been given at the end of each unit.

This book is written with a unique style only meant for the welfare of dear students. We hope that this book will be a strength for them and it will serve their very purpose of attaining excellent results.

We are highly obliged to **Dr. Hari Kishan,** Ex-Head, Department of Mathematics, K.R. (P.G.) College, Mathura, also an eminent author, for his valuable help round the clock in making this book **PERFECT** in all senses.

We are indebted to GOD for shower of blessing. The suggestions with a view to enhance the utility of the book are always welcome.

—AUTHORS

SYLLABUS

U.P. TECHNICAL UNIVERSITY, LUCKNOW

MATHEMATICS-III

NAS-301/NAS-401 L T P

3 1 0

Unit I: Function of Complex variable **8**

Analytic functions, C-R equations. Harmonic Functions, Cauchy's integral theorem, Cauchy's integral formula. Derivatives of analytic functions, Taylor's and Laurent's series, Singularities, Zeroes and Poles, Residue theorem, Evaluation of real integrals of the type

$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ and $\int_{-\infty}^{+\infty} f(x)\, dx$.

Unit II: Integral Transforms **8**

Fourier integral, Complex Fourier transform, Inverse Transforms, Convolution Theorems, Fourier sine and cosine transform, Applications of Fourier transform to simple one dimensional heat transfer equations, wave equations and Laplace equations, Z-transform and its application to solve difference equations.

Unit III: Statistical Techniques **8**

Moments, Moment generating functions, Skewness, Kurtosis, Curve fitting, Method of least squares, Fitting of straight lines, Polynomials, Exponential curves, Correlation, Linear, Non-linear and multiple regression analysis, Binomial, Poisson and Normal distributions, Tests of significations: Chi-square test, t-test.

Unit IV: Numerical Techniques–I **8**

Zeroes of transcendental and polynomial equations using Bisection method, Regula-falsi method and Newton-Raphson method, Rate of convergence of above methods.

Interpolation: Finite differences, Newton's forward and backward interpolation, Lagrange's and Newton's divided difference formula for unequal intervals.

Unit V: Numerical Techniques–II **8**

Solution of system of linear equations, Matrix Decomposition methods, Jacobi method, Gauss Seidal method.

Numerical differentiation, Numerical integration, Trapezoidal rule, Simpson's one third and three-eight rules.

Solution of ordinary differential equations (first order, second order and simultaneous) by Euler's, Picard's and fourth-order Runge-Kutta methods.

STANDARD RESULTS

1. $\dfrac{d}{dx}(x^n) = nx^{n-1}$

2. $\dfrac{d}{dx}(a^x) = a^x \log_e a$

3. $\dfrac{d}{dx}(e^x) = e^x$

4. $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

5. $\dfrac{d}{dx}(\log_{10} x) = \dfrac{1}{x}\log_{10} e$

6. $\dfrac{d}{dx}(\sin x) = \cos x$

7. $\dfrac{d}{dx}(\cos x) = -\sin x$

8. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

9. $\dfrac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

10. $\dfrac{d}{dx}(\cot x) = -\text{cosec}^2 x$

11. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

12. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

13. $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$

14. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

15. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$

16. $\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1+x^2}$

17. $\dfrac{d}{dx}(\text{cosec}^{-1} x) = -\dfrac{1}{x\sqrt{x^2-1}}$

18. $\sinh x = \dfrac{e^x - e^{-x}}{2}$

19. $\cosh x = \dfrac{e^x + e^{-x}}{2}$

20. $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

21. $\cosh^2 x - \sinh^2 x = 1$, $\text{sech}^2 x + \tanh^2 x = 1$, $\coth^2 x = 1 + \text{cosech}^2 x$

22. $\cosh^2 x + \sinh^2 x = \cosh 2x$

23. $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$, $\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$

24. $\dfrac{d}{dx}(\sinh x) = \cosh x$

25. $\dfrac{d}{dx}(\cosh x) = \sinh x$

26. $\dfrac{d}{dx}(\tanh x) = \text{sech}^2 x$

27. $\dfrac{d}{dx}(\coth x) = -\text{cosech}^2 x$

28. $\dfrac{d}{dx}(\text{sech } x) = -\text{sech } x \tanh x$

29. $\dfrac{d}{dx}(\text{cosech } x) = -\text{cosech } x \coth x$

30. Product rule: $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

31. Quotient rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

32. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx}$ if $y = f_1(t)$ and $x = f_2(t)$

33. $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$, $\sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}$

34. $\tan^{-1}\left(\dfrac{a-b}{1+ab}\right) = \tan^{-1} a - \tan^{-1} b$, $\tan^{-1}\left(\dfrac{a+b}{1-ab}\right) = \tan^{-1} a + \tan^{-1} b$

35. $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = 2\tan^{-1} x$

36. $\sin 3x = 3\sin x - 4\sin^3 x$, $\cos 3x = 4\cos^3 x - 3\cos x$, $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

$\sin 2x = 2\sin x \cos x$, $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$,

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

37. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$, $\qquad \cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots ; \; |x| < 1 \qquad (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \qquad (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$

38. $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$, $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

$\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$, $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$

39. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$, $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$, $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

40. $\sin(A+B) = \sin A\cos B + \cos A\sin B$, $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$, $\cos(A-B) = \cos A\cos B + \sin A\sin B$

41. $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$, $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$

$\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}$, where $|x| < 1$, $\qquad \dfrac{d}{dx}(\coth^{-1} x) = \dfrac{1}{x^2-1}$, where $|x| > 1$

$\dfrac{d}{dx}(\operatorname{sech}^{-1} x) = -\dfrac{1}{x\sqrt{1-x^2}}$, $\dfrac{d}{dx}(\operatorname{cosech}^{-1} x) = -\dfrac{1}{x\sqrt{x^2+1}}$

42. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

43. $\sin^2\theta + \cos^2\theta = 1$, $\sec^2\theta - \tan^2\theta = 1$, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

44.

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	-1	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	-1	0	1
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	∞	0

45.

θ	$90° - \theta$	$90° + \theta$	$\pi - \theta$	$\pi + \theta$
$\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$
$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$
$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$

46. sine formula: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$; cosine formula: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

47. Area of triangle $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$

48. ${}^{n}C_{r} = \dfrac{n!}{r!\,n-r!}$

49. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c;\ n \neq -1$

$\int \dfrac{1}{x}\,dx = \log_e x + c;\ \int e^x\,dx = e^x + c;\ \int a^x\,dx = \dfrac{a^x}{\log_e a} + c$

$\int \sin x\,dx = -\cos x + c;\ \int \cos x\,dx = \sin x + c$

$\int \tan x\,dx = \log \sec x + c;\ \int \cot x\,dx = \log \sin x + c$

$\int \sec x\,dx = \log(\sec x + \tan x) + c = \log\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right) + c$

$\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x) + c = \log\tan\dfrac{x}{2} + c$

$\int \sec x\tan x\,dx = \sec x + c;\ \int \operatorname{cosec} x\cot x\,dx = -\operatorname{cosec} x + c$

50. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + c;\ \int \dfrac{-dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\dfrac{x}{a}\right) + c$

$\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + c;\ \int \dfrac{-dx}{a^2 + x^2} = \dfrac{1}{a}\cot^{-1}\left(\dfrac{x}{a}\right) + c$

$\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a}\log\left(\dfrac{a+x}{a-x}\right) + c;\ \int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\log\left(\dfrac{x-a}{x+a}\right) + c$

$\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left(\dfrac{x}{a}\right) + c;\ \int \dfrac{-dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\operatorname{cosec}^{-1}\left(\dfrac{x}{a}\right) + c$

51. $\int \operatorname{sech}^2 x\, dx = \tanh x + c$, $\int \operatorname{cosech}^2 x\, dx = -\coth x + c$

$\int \sinh x\, dx = \cosh x + c$, $\int \cosh x\, dx = \sinh x + c$

$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + c$, $\int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x + c$

52. $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} + c$

$\int \sqrt{a^2 + x^2}\, dx = \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{1}{2} a^2 \log (x + \sqrt{a^2 + x^2}) + c$

$\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \log (x + \sqrt{x^2 - a^2}) + c$

$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left(\frac{x}{a}\right) + c$; $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \left(\frac{x}{a}\right) + c$

53. $\int_a^b f(x)\, dx = \int_a^b f(y)\, dy$; $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$; $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(x) \text{ is even function} \\ 0, & \text{if } f(x) \text{ is odd function} \end{cases}$$

$$\int_0^{2a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$$

54. Leibnitz rule for differentiation under the integral sign

$$\frac{d}{dx} \int_{\phi(\alpha)}^{\psi(\alpha)} f(x, \alpha)\, dx = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial}{\partial \alpha}\{f(x, \alpha)\}\, dx + f\{\psi(\alpha), \alpha\} \frac{d\psi(\alpha)}{d\alpha} - f\{\phi(\alpha), \alpha\} \frac{d\phi(\alpha)}{d\alpha}$$

55. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ and $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

56. $\overrightarrow{AB}$ = position vector of B-position vector of A = $\overrightarrow{OB} - \overrightarrow{OA}$

57. $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos \theta$; work done = $\int_c \vec{F} \cdot d\vec{r}$

58. $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin \theta\, \hat{n}$

59. Area of parallelogram = $\vec{a} \times \vec{b}$, Moment of force = $\vec{r} \times \vec{F}$

60. $\vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\vec{a}\times\vec{b})\cdot\vec{c}$

where $\vec{a} = \Sigma\, a_1\hat{i}$, $\vec{b} = \Sigma\, b_1\hat{i}$ and $\vec{c} = \Sigma c_1\hat{i}$

If $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$, then $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

61. $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

62. $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$

63. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\ \vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{d}$

64. A (Adj. A) = | A | I

65. $AA^{-1} = I = A^{-1}A$

66. AI = A = IA

67. $(ABC)' = C'B'A'$

68. (AB)C = A(BC); A(B + C) = AB + AC

69. A + B = B + A; A + (B + C) = (A + B) + C

70. $(AB)^{-1} = B^{-1}A^{-1}$

71. **Walli's formula**

$$\int_0^{\pi/2} \sin^n\theta\, d\theta = \int_0^{\pi/2} \cos^n\theta\, d\theta = \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2} & \text{if } n \text{ is even} \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\cdots\dfrac{4}{5}\cdot\dfrac{2}{3} & \text{if } n \text{ is odd} \end{cases}$$

72. $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

$\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

73. $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(-1/2) = -2\sqrt{\pi}$

74. $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dfrac{x^6}{6} + \ldots\ldots$

$\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \dfrac{x^5}{5} - \dfrac{x^6}{6} - \ldots\ldots$

75. $\sin n\pi = 0$; $\cos n\pi = (-1)^n$, $\sin\left(n+\dfrac{1}{2}\right)\pi = (-1)^n$; $\cos\left(n+\dfrac{1}{2}\right)\pi = 0$, where $n \in I$

76. $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$.

UNIT 1

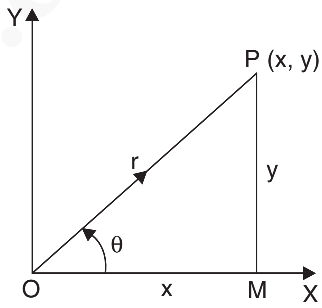
Function of Complex Variable

1.1 INTRODUCTION

A complex number z is an ordered pair (x, y) of real numbers and is written as

$$z = x + iy, \quad \text{where } i = \sqrt{-1}.$$

The real numbers x and y are called the real and imaginary parts of z. In the Argand's diagram, the complex number z is represented by the point P(x, y). If (r, θ) are the polar coordinates of P, then $r = \sqrt{x^2 + y^2}$ is called the modulus of z and is denoted by $|z|$. Also $\theta = \tan^{-1} \dfrac{y}{x}$ is called the argument of z and is denoted by arg. z. Every non-zero complex number z can be expressed as



$$z = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

If $z = x + iy$, then the complex number $x - iy$ is called the conjugate of the complex number z and is denoted by $\bar{z}$.

Clearly, $$|\bar{z}| = |z|, |z|^2 = z\bar{z},$$

$$\text{Re}(z) = \frac{z + \bar{z}}{2}, \quad \text{Im}(z) = \frac{z - \bar{z}}{2i}.$$

1.2 DEFINITIONS

Let S be a non-empty set of complex numbers and δ be a positive real number.

1. Circle. $|z - a| = r$ represents a circle C with centre at the point a and radius r.

2. Open disk. The set of points which satisfies the equation $|z - z_0| < \delta$ defines an open disk of radius δ with centre at $z_0 = (x_0, y_0)$. This set consists of all points which lie inside circle C.

3. Closed disk. The set of points which satisfies the equation $|z - z_0| \le \delta$ defines a closed disk of radius δ with centre at $z_0 = (x_0, y_0)$. This set consists of all points which lie inside and on the boundary of circle C.

4. Annulus. The set of points which lie between two concentric circles $C_1 : |z - a| = r_1$ and $C_2 : |z - a| = r_2$ defines an open annulus *i.e.,* the set of points which satisfies the inequality $r_1 < |z - a| < r_2$.

The set of points which satisfies the inequality $r_1 \le |z - a| \le r_2$ defines a closed annulus.

It is to be noted that $r_1 \le |z - a| < r_2$ is neither open nor closed.

5. Neighbourhood. δ-Neighbourhood of a point z_0 is the set of all points z for which $|z - z_0| < \delta$ where δ is a positive constant. If we exclude the point z_0 from the open disk $|z - z_0| < \delta$ then it is called the deleted neighbourhood of the point z_0 and is written as $0 < |z - z_0| < \delta$.

6. Interior and exterior points. A point z is an interior point of S if all the points in some δ-neighbourhood of z are in S and an exterior point of S if they are outside S.

7. Boundary point. A point z is a boundary point of S if every δ-neighbourhood of z contains at least one point of S and at least one point not in S. For example, the points on the circle $|z - z_0| = r$ are the boundary points for the disk $|z - z_0| \le r$.

8. Open and closed sets. A set S is open if every point of S is an interior point while a set S is closed if every boundary point of S belongs to S. *e.g.* $S = \{z : |z - z_0| < r\}$ is open set while $S = \{z : |z - z_0| \le r\}$ is closed set.

9. Bounded set. An open set S is bounded if ∃ a positive real number M such that $|z| \le M$ for all $z \in S$ otherwise unbounded.

For example: the set $S = \{z : |z - z_0| < r\}$ is a bounded set while the set $S = \{z : |z - z_0| > r\}$ is an unbounded set.

10. Connected set. An open set S is connected if any two points z_1 and z_2 belonging to S can be joined by a polygonal line which is totally contained in S.

11. Domain. An open connected set is called a domain denoted by D.

12. Region. A region is a domain together with all, some or none of its boundary points. Thus a domain is always a region but a region may or may not be a domain.

13. Finite complex plane. The complex plane without the point $z = \infty$ is called the finite complex plane.

14. Extended complex plane. The complex plane to which the point $z = \infty$ has been added is called the extended complex plane.

1.3 FUNCTION OF A COMPLEX VARIABLE

If x and y are real variables, then $z = x + iy$ is called a **complex variable.** If corresponding to each value of a complex variable $z(= x + iy)$ in a given region R, there correspond one or more values of another complex variable w $(= u + iv)$, then w is called a **function of the complex variable** z and is denoted by

$$w = f(z) = u + iv$$

For example, if $w = z^2$ where $z = x + iy$ and $w = f(z) = u + iv$

then $$u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)$$

$$\Rightarrow \quad u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

Thus u and v, the real and imaginary parts of w, are functions of the real variables x and y.

$$\therefore \quad w = f(z) = u(x, y) + iv(x, y)$$

If to each value of z, there corresponds one and only one value of w, then w is called a *single-valued function* of z. If to each value of z, there correspond more than one values of w, then w is called a *multi-valued function of* z. For example, $w = \sqrt{z}$ is a multi-valued function.

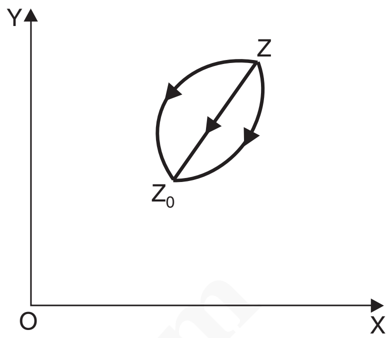
To represent $w = f(z)$ graphically, we take two Argand diagrams: one to represent the point z and the other to represent w. The former diagram is called the XOY-plane or the z-plane and the latter UOV-plane or the w-plane.

1.4 LIMIT OF f(z)

A function $f(z)$ tends to the limit l as z tends to z_0 *along any path,* if to each positive arbitrary number ε, however small, there corresponds a positive number δ, such that

$$| f(z) - l | < \varepsilon \quad \text{whenever} \quad 0 < | z - z_0 | < \delta$$

and we write $\underset{z \to z_0}{\text{Lt}} f(z) = l$, where l is finite



Note. In real variables, $x \to x_0$ implies that x approaches x_0 along the number line, either from left or from right. In complex variables, $z \to z_0$ implies that z approaches z_0 along any path, straight or curved, since the two points representing z and z_0 in a complex plane can be joined by an infinite number of curves.

1.5 CONTINUITY OF f(z)

A single-valued function $f(z)$ is said to be continuous at a point $z = z_0$ if $f(z_0)$ exists, $\lim\limits_{z \to z_0} f(z)$ exists and $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$.

A function $f(z)$ is said to be continuous in a region R of the z-plane if it is continuous at every point of the region. A function $f(z)$ which is not continuous at z_0 is said to be discontinuous at z_0.

If the function $f(z) = u + iv$ is continuous at $z_0 = x_0 + iy_0$ then the real functions u and v are also continuous at the point (x_0, y_0). Therefore, we can discuss the continuity of a complex valued function by studying the continuity of its real and imaginary parts. If $f(z)$ and $g(z)$ are continuous at a point z_0 then the functions $f(z) \pm g(z), f(z)\, g(z)$ and $\dfrac{f(z)}{g(z)}$, where $g(z_0) \neq 0$ are also continuous at z_0.

If $f(z)$ is continuous in a closed region S then it is bounded in S *i.e.,* $|f(z)| \leq \text{M} \quad \forall\, z \in \text{S}$.

Also, the function $f(z)$ is continuous at $z = \infty$ if the function $f\left(\dfrac{1}{\xi}\right)$ is continuous at $\xi = 0$

1.6 DERIVATIVE OF f(z)

Let $w = f(z)$ be a single-valued function of the variable $z(= x + iy)$, then the derivative or differential co-efficient of $w = f(z)$ is defined as

$$\boxed{\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z}}$$

provided the limit exists, independent of the manner in which $\delta z \to 0$.

1.7 ANALYTIC FUNCTION [*G.B.T.U. 2012, M.T.U. 2012, U.P.T.U. 2014*]

A function $f(z)$ is said to be **analytic** at a point z_0 if it is one-valued and differentiable not only at z_0 but at every point of some neighbourhood of z_0. For example: $e^x (\cos y + i \sin y)$. A function $f(z)$ is said to be analytic in a certain domain D if it is analytic at every point of D.

The terms 'regular', 'holomorphic' and 'monogenic' are also sometimes used as synonymous with the term analytic.

A function $f(z)$ is said to be analytic at $z = \infty$ if the function $f\left(\frac{1}{z}\right)$ is analytic at $z = 0$.

Here it should be noted that analyticity implies differentiability but not vice versa. For example, the function $f(z) = |z|^2$ is differentiable only at $z = 0$ and nowhere else therefore $f(z)$ is differentiable at $z = 0$ but not analytic anywhere.

A function $f(z)$ may be differentiable in a domain except for a finite number of points. These points are called **singular points** or **singularities** of $f(z)$ in that domain.

1.8 ENTIRE FUNCTION

A function $f(z)$ which is analytic at every point of the finite complex plane is called an entire function. Since the derivative of a polynomial exists at every point, a polynomial of any degree is an entire function. Rational functions with non-zero denominators are also entire functions.

1.9 NECESSARY AND SUFFICIENT CONDITIONS FOR f(z) TO BE ANALYTIC

[*M.T.U. 2012, U.P.T.U. (C.O.) 2008*]

The necessary and sufficient conditions for the function

$$w = f(z) = u(x, y) + iv(x, y)$$

to be analytic in a region R, are

(*i*) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *are continuous functions of x and y in the region R.*

(*ii*) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The conditions in (*ii*) are known as **Cauchy-Riemann equations** or briefly **C-R equations.**

Proof. (*a*) **Necessary Condition.** Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region R, then $\frac{dw}{dz} = f'(z)$ exists uniquely at every point of that region.

Let δx and δy be the increments in x and y respectively. Let δu, δv and δz be the corresponding increments in u, v and z respectively. Then,

$$f'(z) = \operatorname*{Lt}_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z} = \operatorname*{Lt}_{\delta z \to 0} \frac{(u + \delta u) + i(v + \delta v) - (u + iv)}{\delta z}$$

$$= \operatorname*{Lt}_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \qquad \text{...(1)}$$

Since the function $w = f(z)$ is analytic in the region R, the limit (1) must exist independent of the manner in which $\delta z \to 0$, *i.e.*, along whichever path δx and $\delta y \to 0$.

First, let $\delta z \to 0$ along a line parallel to x-axis so that $\delta y = 0$ and $\delta z = \delta x$.

[since $z = x + iy$, $z + \delta z = (x + \delta x) + i(y + \delta y)$ and $\delta z = \delta x + i\delta y$]

$$\therefore \quad \text{From (1)}, \quad f'(z) = \operatorname*{Lt}_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{...(2)}$$

Now, let $\delta z \to 0$ along a line parallel to y-axis so that $\delta x = 0$ and $\delta z = i\delta y$.

$\therefore$ From (1), $$f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\frac{\delta u}{i\,\delta y} + i\,\frac{\delta v}{i\,\delta y} \right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y} \qquad \text{...(3)} \qquad \Big|\ \because\ \frac{1}{i} = -i$$

From (2) and (3), we have $\dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - i\,\dfrac{\partial u}{\partial y}$

Equating the real and imaginary parts, $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}}$ and $\boxed{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$ (*U.P.T.U. 2015*)

Hence the necessary condition for f(z) to be analytic is that the C-R equations must be satisfied.

(*b*) **Sufficient Condition.** Let $f(z) = u + iv$ be a single-valued function possessing partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ at each point of a region R and satisfying C-R equations.

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We shall show that $f(z)$ is analytic, *i.e.,* $f'(z)$ exists at every point of the region R.

By Taylor's theorem for functions of two variables, we have, on omitting second and higher degree terms of δx and δy.

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= \left[u(x, y) + \left(\frac{\partial u}{\partial x}\,\delta x + \frac{\partial u}{\partial y}\,\delta y \right) \right] + i \left[v(x, y) + \left(\frac{\partial v}{\partial x}\,\delta x + \frac{\partial v}{\partial y}\,\delta y \right) \right]$$

$$= [u(x, y) + iv(x, y)] + \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} \right) \delta y$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} \right) \delta y$$

or $$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} \right) \delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta x + \left(-\frac{\partial v}{\partial x} + i\,\frac{\partial u}{\partial x} \right) \delta y \qquad |\ \text{Using C-R equations}$$

$$= \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) i\delta y \qquad |\ \because\ -1 = i^2$$

$$= \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) (\delta x + i\delta y) = \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta z \qquad |\ \because\ \delta x + i\delta y = \delta z$$

$\Rightarrow$ $$\frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$\therefore$ $$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

Thus $f'(z)$ exists, because $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$ exist.

Hence $f(z)$ is analytic.

Note 1. The real and imaginary parts of an analytic function are called **conjugate functions.** Thus, if $f(z) = u(x, y) + iv(x, y)$ is an analytic function, then $u(x, y)$ and $v(x, y)$ are conjugate functions. The relation between two conjugate functions is given by C-R equations.

Note 2. When a function $f(z)$ is known to be analytic, it can be differentiated in the ordinary way as if z is a real variable.

Thus, $f(z) = z^2 \Rightarrow f'(z) = 2z$

$f(z) = \sin z \Rightarrow f'(z) = \cos z$ etc.

1.10 CAUCHY-RIEMANN EQUATIONS IN POLAR COORDINATES (*U.P.T.U. 2008*)

Let (r, θ) be the polar coordinates of the point whose cartesian coordinates are (x, y), then

$$x = r\cos\theta,\ y = r\sin\theta,$$

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\therefore \quad u + iv = f(z) = f(re^{i\theta}) \quad \text{...(1)}$$

Differentiating (1) partially w.r.t. r, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \,.\, e^{i\theta} \quad \text{...(2)}$$

Differentiating (1) partially w.r.t. θ, we have

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \,.\, ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \quad | \text{ Using (2)}$$

$$= -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

or $\boxed{\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}}$ and $\boxed{\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}}$, which is the polar form of C-R equations.

1.11 DERIVATIVE OF w, i.e., f′(z) IN POLAR COORDINATES

$$w = f(z)$$

$$\therefore \quad \frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u + iv) = \frac{\partial w}{\partial x}$$

$$= \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \cos\theta\,\frac{\partial w}{\partial r} - \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{\sin\theta}{r}$$

$$= \cos\theta\,\frac{\partial w}{\partial r} - \left(-r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}\right)\frac{\sin\theta}{r}$$

$\because\ r^2 = x^2 + y^2$
$\therefore\ \partial r/\partial x = \cos\theta$ as $x = r\cos\theta$
and $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$
$\therefore\ \dfrac{\partial \theta}{\partial x} = \dfrac{-\sin\theta}{r}$ as $y = r\sin\theta$

$$= \cos\theta \frac{\partial w}{\partial r} - i\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\sin\theta = \cos\theta \frac{\partial w}{\partial r} - i\sin\theta \frac{\partial w}{\partial r}$$

$$\Rightarrow \quad \boxed{\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}} \qquad \text{...(1)}$$

which is the result in terms of $\frac{\partial w}{\partial r}$.

Again, $$\frac{dw}{dz} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta - \frac{\partial w}{\partial \theta}\cdot\frac{\sin\theta}{r}$$

$$= \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta} = -\frac{i}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$= -\frac{i}{r}\frac{\partial w}{\partial \theta}\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$\Rightarrow \quad \boxed{\frac{dw}{dz} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}}$$

which is the result in terms of $\frac{\partial w}{\partial \theta}$.

1.12 HARMONIC FUNCTION [*M.T.U. 2014, G.B.T.U. 2012, U.P.T.U. 2007, 2009*]

A function of x, y which possesses continuous partial derivatives of the first and second orders and satisfies Laplace's equation is called a Harmonic function.

1.13 THEOREM

If $f(z) = u + iv$ is an analytic function then u and v are both harmonic functions.

Proof. Let $f(z) = u + iv$ be analytic in some region of the z-plane, then u and v satisfy C-R equations.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{...(1)}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{...(2)}$$

Differentiating eqn. (1) partially w.r.t. x and eqn. (2) w.r.t. y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{...(3)}$$

and $$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad \text{...(4)}$$

Assuming $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ and adding equations (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(5)$$

Now, differentiating eqn. (1) partially w.r.t. y and eqn. (2) w.r.t. x, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \qquad ...(6)$$

and
$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \qquad ...(7)$$

Assuming $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ and subtracting eqn. (7) from eqn. (6), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad ...(8)$$

Equations (5) and (8) show that the real and imaginary parts u and v of an analytic function satisfy the Laplace's equation. Hence u and v are harmonic functions.

Note. Here u and v are called conjugate harmonic functions.

1.14 ORTHOGONAL SYSTEM [*M.T.U. 2012, U.P.T.U. 2009*]

Every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.

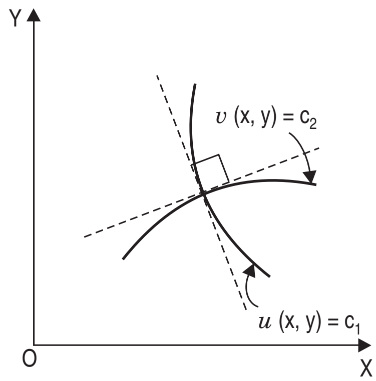
Consider the two families of curves

$$u(x, y) = c_1 \qquad ...(1)$$

and
$$v(x, y) = c_2 \qquad ...(2)$$

Differentiating eqn. (1) w.r.t. x, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}} = m_1 \text{ (say)}$$



Similarly, from eqn. (2), we get $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial y}} = m_2$ (say)

$$\therefore \qquad m_1 m_2 = \frac{\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial v}{\partial x}}{\dfrac{\partial u}{\partial y} \cdot \dfrac{\partial v}{\partial y}} \qquad ...(3)$$

Since $f(z)$ is analytic, u and v satisfy C-R equations

i.e.,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \quad \text{From (3),} \quad m_1 m_2 = \frac{\dfrac{\partial v}{\partial y} \cdot \dfrac{\partial v}{\partial x}}{-\dfrac{\partial v}{\partial x} \cdot \dfrac{\partial v}{\partial y}} = -1$$

Thus, the product of the slopes of the curves (1) and (2) is –1. Hence the curves intersect at right angles, *i.e.,* they form an orthogonal system.

1.15 THEOREM (*U.P.T.U. 2008*)

An analytic function with constant modulus is constant.

Proof. Let $f(z) = u + iv$ be an analytic function with constant modulus. Then,

$$|f(z)| = |u + iv| = \text{constant}$$

$$\Rightarrow \quad \sqrt{u^2 + v^2} = \text{constant} = c \text{ (say)}$$

Squaring both sides, we get

$$u^2 + v^2 = c^2 \qquad \ldots(1)$$

Differentiating eqn. (1) partially w.r.t. x, we get

$$2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \quad u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} = 0 \qquad \ldots(2)$$

Again, differentiating eqn. (1) partially w.r.t. y, we get

$$2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad u\left(-\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial u}{\partial x}\right) = 0 \qquad \ldots(3) \qquad \left|\ \because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\right.$$

Squaring and adding eqns. (2) and (3), we get

$$(u^2 + v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} = 0$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \qquad |\ \because \ u^2 + v^2 = c^2 \neq 0$$

$$\Rightarrow \quad |f'(z)|^2 = 0 \qquad \left|\ \because \ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right.$$

$$\Rightarrow \quad |f'(z)| = 0$$

$\Rightarrow$ $f(z)$ is constant.

1.16 APPLICATION OF ANALYTIC FUNCTIONS TO FLOW PROBLEMS

Since the real and imaginary parts of an analytic function satisfy the Laplace's equation in two variables, these conjugate functions provide solutions to a number of field and flow problems.

For example, consider the two dimensional irrotational motion of an incompressible fluid, in planes parallel to xy-plane.

Let $\vec{V}$ be the velocity of a fluid particle, then it can be expressed as

$$\vec{V} = v_x \hat{i} + v_y \hat{j} \qquad \ldots(1)$$

Since the motion is irrotational, there exists a scalar function $\phi(x, y)$, such that

$$\vec{V} = \nabla\phi(x, y) = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} \qquad ...(2)$$

From (1) and (2), we have $\quad v_x = \dfrac{\partial\phi}{\partial x}$ and $v_y = \dfrac{\partial\phi}{\partial y}$ ...(3)

The scalar function $\phi(x, y)$, which gives the velocity components, is called the **velocity potential function** or simply the **velocity potential.**

Also the fluid being incompressible, div $\vec{V} = 0$

$$\Rightarrow \qquad \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)(v_x\hat{i} + v_y\hat{j}) = 0$$

$$\Rightarrow \qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad ...(4)$$

Substituting the values of v_x and v_y from (3) in (4), we get

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = 0 \quad \text{or} \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Thus, the function ϕ is harmonic and can be treated as real part of an analytic function

$$w = f(z) = \phi(x, y) + i\,\psi(x, y)$$

For interpretation of conjugate function $\psi(x, y)$, the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\frac{dy}{dx} = -\frac{\dfrac{\partial\psi}{\partial x}}{\dfrac{\partial\psi}{\partial y}} = \frac{\dfrac{\partial\phi}{\partial y}}{\dfrac{\partial\phi}{\partial x}} \qquad \text{| By C-R equations}$$

$$= \frac{v_y}{v_x} \qquad \text{| By (3)}$$

This shows that the resultant velocity $\sqrt{{v_x}^2 + {v_y}^2}$ of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$ *i.e.,* the fluid particles move along this curve. Such curves are known as **stream lines** and $\psi(x, y)$ is called the **stream function**. The curves represented by $\phi(x, y) = c$ are called **equipotential lines**.

Since $\phi(x, y)$ and $\psi(x, y)$ are conjugate functions of analytic function $w = f(z)$, the equipotential lines $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$, intersect each other orthogonally.

Now,
$$\frac{dw}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y} \qquad \text{| By C-R equations}$$

$$= v_x - iv_y \qquad \text{| By (3)}$$

$\therefore$ The magnitude of resultant velocity $= \left|\dfrac{dw}{dz}\right| = \sqrt{{v_x}^2 + {v_y}^2}$

The function $w = f(z)$ which fully represents the flow pattern is called the **complex potential**.

In the study of electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are called **equipotential lines** and **lines of force** respectively. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as **isothermals** and **heat flow lines** respectively.

1.17 DETERMINATION OF THE CONJUGATE FUNCTION

If $f(z) = u + iv$ is an analytic function where both $u(x, y)$ and $v(x, y)$ are conjugate functions, then we determine the other function v when one of these say u is given as follows:

$$\because \quad v = v(x, y)$$

$$\therefore \quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\Rightarrow \quad dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \qquad \text{...(1)} \quad | \text{ By C-R eqns.}$$

$$M = -\frac{\partial u}{\partial y}, \quad N = \frac{\partial u}{\partial x}$$

$$\therefore \quad \frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

Now, $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ gives}$$

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$$

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which is true as u being a harmonic function satisfies Laplace's equation.

$\therefore$ dv is exact.

$\therefore$ dv can be integrated to get v.

However, if we are to construct $f(z) = u + iv$ when only u is given, we first of all find v by above procedure and then write $f(z) = u + iv$.

Similarly, if we are to determine u and only v is given then we use $du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$ and integrate it to find u. Consequently $f(z) = u + iv$ can also be determined.

1.18 MILNE'S THOMSON METHOD

With the help of this method, we can directly construct $f(z)$ in terms of z without first finding out v when u is given or u when v is given.

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$\Rightarrow \quad x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z})$$

$$\therefore \quad f(z) = u(x, y) + iv(x, y)$$

$$= u\left\{\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right\} + iv\left\{\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right\} \qquad \text{...(1)}$$

Relation (1) is an identity in z and $\bar{z}$. Putting $\bar{z} = z$, we get

$$f(z) = u(z, 0) + iv(z, 0) \qquad \text{...(2)}$$

Now, $f(z) = u + iv$

$$\Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{| By C-R eqns.}$$

$$= \phi_1(x, y) - i\,\phi_2(x, y)$$

| where $\phi_1(x, y) = \dfrac{\partial u}{\partial x}$ and $\phi_2(x, y) = \dfrac{\partial u}{\partial y}$

Now, $f'(z) = \phi_1(z, 0) - i\,\phi_2(z, 0)$ | Replacing x by z and y by 0

Integrating, we get

$$\boxed{f(z) = \int \{\phi_1(z, 0) - i\,\phi_2(z, 0)\}\, dz + c} \qquad \text{| } c \text{ is an arbitrary constant.}$$

Hence the function is constructed directly in terms of z.

Similarly if $v(x, y)$ is given, then

$$\boxed{f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]\, dz + c} \qquad \Bigg|\ \psi_1(x, y) = \frac{\partial v}{\partial y} \text{ and } \psi_2(x, y) = \frac{\partial v}{\partial x}$$

Milne's Thomson method can easily be grasped by going through the steps involved in following various cases.

Case I. *When only real part* **u(x, y)** *is given.*

To construct analytic function $f(z)$ directly in terms of z when only real part u is given, we use the following steps:

1. Find $\dfrac{\partial u}{\partial x}$

2. Write it as equal to $\phi_1(x, y)$

3. Find $\dfrac{\partial u}{\partial y}$

4. Write it as equal to $\phi_2(x, y)$

5. Find $\phi_1(z, 0)$ by replacing x by z and y by 0 in $\phi_1(x, y)$.

6. Find $\phi_2(z, 0)$ by replacing x by z and y by 0 in $\phi_2(x, y)$.

7. $f(z)$ is obtained by the formula

$$f(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\}\, dz + c \text{ directly in terms of } z.$$

Case II. *When only imaginary part* **v(x, y)** *is given.*

To construct analytic function $f(z)$ directly in terms of z when only imaginary part v is given, we use the following steps :

1. Find $\dfrac{\partial v}{\partial y}$

2. Write it as equal to $\psi_1(x, y)$

3. Find $\dfrac{\partial v}{\partial x}$

4. Write it as equal to $\psi_2(x, y)$

5. Find $\psi_1(z, 0)$ by replacing x by z and y by 0 in $\psi_1(x, y)$

6. Find $\psi_2(z, 0)$ by replacing x by z and y by 0 in $\psi_2(x, y)$

7. $f(z)$ is obtained by the formula

$$f(z) = \int \{\psi_1(z, 0) + i\psi_2(z, 0)\}\, dz + c \text{ directly in terms of } z.$$

Case III. *When* **u – v** *is given.*

To construct analytic function $f(z)$ directly in terms of z when $u - v$ is given, we follow the following steps:

1. $f(z) = u + iv$ ...(1)

2. $i\,f(z) = iu - v$ ...(2)

3. Add (1) and (2) to get

$$(1 + i)\,f(z) = (u - v) + i(u + v)$$

or, $$F(z) = U + iV$$

where $$F(z) = (1 + i)\,f(z),\ U = u - v \text{ and } V = u + v$$

4. Since $u - v$ is given hence $U(x, y)$ is given

5. Find $\dfrac{\partial U}{\partial x}$

6. Write it as equal to $\phi_1(x, y)$

7. Find $\dfrac{\partial U}{\partial y}$

8. Write it as equal to $\phi_2(x, y)$

9. Find $\phi_1(z, 0)$

10. Find $\phi_2(z, 0)$

11. $F(z)$ is obtained by the formula

$$F(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\}\, dz + c$$

12. $f(z)$ is determined by $f(z) = \dfrac{F(z)}{1+i}$ directly in terms of z.

Case IV. *When* **u + v** *is given.*

To construct analytic function $f(z)$ directly in terms of z when $u + v$ is given, we follow the following steps:

1. $f(z) = u + iv$ ...(1)

2. $i\,f(z) = iu - v$ ...(2)

3. Add (1) and (2) to get

$$(1 + i)\,f(z) = (u - v) + i(u + v)$$

$\Rightarrow$ $$F(z) = U + iV$$

where, $$F(z) = (1 + i)\,f(z),\ U = u - v \text{ and } V = u + v$$

4. Since $u + v$ is given hence $V(x, y)$ is given

5. Find $\dfrac{\partial V}{\partial y}$

6. Write it as equal to $\psi_1(x, y)$

7. Find $\dfrac{\partial V}{\partial x}$

8. Write it as equal to $\psi_2(x, y)$

9. Find $\psi_1(z, 0)$

10. Find $\psi_2(z, 0)$

11. F(z) is obtained by the formula

$$F(z) = \int \{\psi_1(z, 0) + i\psi_2(z, 0)\}\, dz + c$$

12. $f(z)$ is determined by $f(z) = \dfrac{F(z)}{1+i}$ directly in terms of z.

EXAMPLES

Example 1. *Find the values of c_1 and c_2 such that the function*

$$f(z) = x^2 + c_1 y^2 - 2xy + i\,(c_2 x^2 - y^2 + 2xy)$$

is analytic. Also find $f'(z)$. (U.K.T.U. 2011)

Sol. Here $\quad f(z) = (x^2 + c_1 y^2 - 2xy) + i\,(c_2 x^2 - y^2 + 2xy)$ ...(1)

Comparing (1) with $f(z) = u(x, y) + iv(x, y)$, we get

$$u(x, y) = x^2 + c_1 y^2 - 2xy \qquad ...(2)$$

and $$v(x, y) = c_2 x^2 - y^2 + 2xy \qquad ...(3)$$

For the function $f(z)$ to be analytic, it should satisfy Cauchy-Riemann equations.

Now from (2), $\quad \dfrac{\partial u}{\partial x} = 2x - 2y \quad$ and $\quad \dfrac{\partial u}{\partial y} = 2c_1 y - 2x$

Also, from (3), $\quad \dfrac{\partial v}{\partial x} = 2c_2 x + 2y \quad$ and $\quad \dfrac{\partial v}{\partial y} = -2y + 2x$

Cauchy-Riemann eqns. are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$\Rightarrow \quad 2x - 2y = -2y + 2x \quad$ which is true.

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow \quad 2c_1 y - 2x = -2c_2 x - 2y$ ...(4)

Comparing the coefficients of x and y in eqn. (4), we get

$$2c_1 = -2 \quad \Rightarrow \quad c_1 = -1$$

and $$-2 = -2c_2 \quad \Rightarrow \quad c_2 = 1$$

Hence $\quad c_1 = -1 \quad$ and $\quad c_2 = 1$

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x - 2y + i(2c_2 x + 2y)$$

$$= 2x - 2y + i(2x + 2y) \qquad | \because \; c_2 = 1$$

$$= 2(x + iy) + 2i(x + iy)$$

$$= 2z + 2iz = 2(1 + i)z.$$

Example 2. *Find p such that the function $f(z)$ expressed in polar coordinates as $f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$ is analytic.*

Sol. Let $f(z) = u + iv$, then $u = r^2 \cos 2\theta$, $v = r^2 \sin p\theta$

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta, \ \frac{\partial v}{\partial r} = 2r \sin p\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta, \ \frac{\partial v}{\partial \theta} = pr^2 \cos p\theta$$

For $f(z)$ to be analytic, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$\therefore$ $2r \cos 2\theta = pr \cos p\theta$ and $2r \sin p\theta = 2r \sin 2\theta$

Both these equations are satisfied if $p = 2$.

Example 3. (*i*) *Prove that the function sinh z is analytic and find its derivative.*

(U.K.T.U. 2010)

(*ii*) *Show that f(z) = log z is analytic everywhere in the complex plane except at the origin and that its derivative is* $\left(\frac{1}{z}\right)$.

Sol. (*i*) Here $f(z) = u + iv = \sinh z = \sinh (x + iy) = \sinh x \cos y + i \cosh x \sin y$

$\therefore$ $u = \sinh x \cos y$ and $v = \cosh x \sin y$

$$\frac{\partial u}{\partial x} = \cosh x \cos y, \ \frac{\partial u}{\partial y} = -\sinh x \sin y$$

$$\frac{\partial v}{\partial x} = \sinh x \sin y, \ \frac{\partial v}{\partial y} = \cosh x \cos y$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus C-R equations are satisfied.

Since $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous functions, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R equations.

Hence $f(z)$ is analytic everywhere.

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cosh x \cos y + i \sinh x \sin y = \cosh (x + iy) = \cosh z$.

(*ii*) Here $f(z) = u + iv = \log z = \log (x + iy)$

Let $x = r \cos \theta$ and $y = r \sin \theta$ so that

$$x + iy = r (\cos \theta + i \sin \theta) = re^{i\theta}$$

$$\log (x + iy) = \log (r\, e^{i\theta}) = \log r + i\theta = \frac{1}{2} \log (x^2 + y^2) + i \tan^{-1}\left(\frac{y}{x}\right)$$

Separating real and imaginary parts, we get

$$u = \frac{1}{2} \log (x^2 + y^2) \quad \text{and} \quad v = \tan^{-1}\left(\frac{y}{x}\right)$$

Now, $$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

and $$\frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2}, \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

We observe that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied except when $x^2 + y^2 = 0$ *i.e.*, when $x = 0, y = 0$

Also derivatives are continuous except at origin.

Hence the function $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin.

Also, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x - iy}{x^2 + y^2} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{x + iy} = \frac{1}{z}$$

Example 4. *Show that the function e^x (cos y + i sin y) is holomorphic and find its derivative.*

Sol. $$f(z) = e^x \cos y + i\, e^x \sin y = u + iv$$

Here, $$u = e^x \cos y, \quad v = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

Since, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hence, C-R equations are satisfied. Also first order partial derivatives of u and v are continuous everywhere. Therefore $f(z)$ is analytic.

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i\, e^x \sin y$$
$$= e^x (\cos y + i \sin y) = e^x \, . \, e^{iy} = e^{x+iy} = e^z$$

Example 5. *If n is real, show that r^n (cos $n\theta$ + i sin $n\theta$) is analytic except possibly when r = 0 and that its derivative is*

$$nr^{n-1} [\cos (n-1)\,\theta + i \sin (n-1)\,\theta].$$

Sol. Let $$w = f(z) = u + iv = r^n (\cos n\theta + i \sin n\theta)$$

Here, $$u = r^n \cos n\theta, \qquad v = r^n \sin n\theta$$

then, $$\frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta \qquad \frac{\partial v}{\partial r} = nr^{n-1} \sin n\theta$$

$$\frac{\partial u}{\partial \theta} = -nr^n \sin n\theta \qquad \frac{\partial v}{\partial \theta} = nr^n \cos n\theta$$

Thus, we see that, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$\therefore$ Cauchy-Riemann equations are satisfied. Also first order partial derivatives of u and v are continuous everywhere.

Hence $f(z)$ is analytic if $f'(z)$ or $\frac{dw}{dz}$ exists for all finite values of z.

We have,
$$\begin{aligned}\frac{dw}{dz} &= (\cos\theta - i\sin\theta)\,\frac{\partial w}{\partial r}\\ &= (\cos\theta - i\sin\theta)\,.\,nr^{n-1}(\cos n\theta + i\sin n\theta)\\ &= nr^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta]\end{aligned}$$

This exists for all finite values of r including zero, except when $r = 0$ and $n \le 1$.

Example 6. *Show that if f(z) is analytic and Re f(z) = constant then f(z) is a constant.*

(U.P.T.U. 2006)

Sol. Since the function $f(z) = u\ (x, y) + iv\ (x, y)$ is analytic, it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also, Re $f(z)$ = constant, therefore $u(x, y) = c_1$

$$\therefore \quad \frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}.$$

Using C-R equations, $\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$

Hence $v(x, y) = c_2$ = a real constant

Therefore $f(z) = u(x, y) + iv(x, y) = c_1 + ic_2$ = a complex constant.

Example 7. *Given that* $u(x, y) = x^2 - y^2$ *and* $v(x, y) = -\left(\frac{y}{x^2 + y^2}\right)$.

Prove that both u and v are harmonic functions but u + iv is not an analytic function of z.

Sol.
$$u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -2$$

Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Hence $u(x, y)$ is harmonic.

Also,
$$v = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} = \frac{2y^3 - 6x^2 y}{(x^2 + y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial y^2} = \frac{6x^2 y - 2y^3}{(x^2 + y^2)^3}$$

Since $\quad \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$. Hence $v(x, y)$ is also harmonic.

But, $\quad \dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y} \quad$ and $\quad \dfrac{\partial v}{\partial x} \neq -\dfrac{\partial u}{\partial y}$

Therefore $u + iv$ is not an analytic function of z.

Example 8. *If ϕ and ψ are functions of x and y satisfying Laplace's equation, show that $s + it$ is analytic, where*

$$s = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \ \textit{ and } \ t = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}.$$

[U.K.T.U. 2010, G.B.T.U. (C.O.) 2011]

Sol. Since ϕ and ψ are functions of x and y satisfying Laplace's equations,

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{...(1)}$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \quad \text{...(2)}$$

For the function $s + it$ to be analytic,

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \quad \text{...(3)}$$

and

$$\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x} \quad \text{...(4)}$$

must satisfy.

Now,

$$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \quad \text{...(5)}$$

$$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \quad \text{...(6)}$$

$$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} \quad \text{...(7)}$$

and

$$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}. \quad \text{...(8)}$$

From (3), (5) and (6), we have

$$\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \quad \Rightarrow \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

which is true by (2).

Again from (4), (7) and (8), we have

$$\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} = -\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial y} \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

which is also true by (1).

Hence the function $s + it$ is analytic.

Example 9. *Verify if* $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$, *$z \neq 0$; $f(0) = 0$ is analytic or not?*

[U.P.T.U. (C.O.) 2008]

Sol. $$u + iv = \frac{xy^2(x+iy)}{x^2+y^4} \; ; z \neq 0$$

$\therefore$ $$u = \frac{x^2y^2}{x^2+y^4}, \; v = \frac{xy^3}{x^2+y^4}$$

At the origin, $$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{0-0}{y} = 0$$

Since $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x}$

Hence Cauchy-Riemann equations are satisfied at the origin.

But $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{xy^2\,(x+iy)}{x^2+y^4} - 0 \right] \cdot \frac{1}{x+iy} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy^2}{x^2+y^4}$$

Let $z \to 0$ along the real axis $y = 0$, then

$$f'(0) = 0$$

Again let $z \to 0$ along the curve $x = y^2$ then

$$f'(0) = \lim_{x \to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

which shows that $f'(0)$ does not exist since the limit is not unique along two different paths. Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

Example 10. *Show that the function defined by $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied there.* [G.B.T.U. (C.O.) 2011]

Sol. Let $f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$ then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

Clearly, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Hence C-R equations are satisfied at the origin.

Now $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z} = \underset{z \to 0}{\text{Lt}} \frac{\sqrt{|xy|} - 0}{x + iy}$$

If $z \to 0$ along the line $y = mx$, we get

$$f'(0) = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|m|}}{1 + im}$$

Now this limit is not unique since it depends on m. Therefore, $f'(0)$ does not exist.

Hence the function $f(z)$ is not regular at the origin.

Example 11. *Prove that the function f(z) defined by*

$$f(z) = \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2}, z \neq 0 \text{ and } f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. (U.P.T.U. 2015)

Sol. Here, $$f(z) = \frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2}, z \neq 0$$

Let $$f(z) = u + iv = \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2},$$

then $$u = \frac{x^3 - y^3}{x^2 + y^2}, v = \frac{x^3 + y^3}{x^2 + y^2}$$

Since $z \neq 0 \Rightarrow x \neq 0, y \neq 0$

$\therefore$ u and v are rational functions of x and y with non-zero denominators. Thus, *u, v* and hence $f(z)$ *are continuous functions when $z \neq 0$*. To test them for continuity at $z = 0$, on changing u, v to polar co-ordinates by putting $x = r \cos \theta$, $y = r \sin \theta$, we get

$$u = r(\cos^3 \theta - \sin^3 \theta) \text{ and } v = r(\cos^3 \theta + \sin^3 \theta)$$

When $z \to 0$, $r \to 0$

$\therefore$ $$\underset{z \to 0}{\text{Lt}}\, u = \underset{r \to 0}{\text{Lt}}\, r(\cos^3 \theta - \sin^3 \theta) = 0$$

Similarly, $$\underset{z \to 0}{\text{Lt}}\, v = 0$$

$\therefore$ $$\underset{z \to 0}{\text{Lt}}\, f(z) = 0 = f(0)$$

$\Rightarrow$ **f(z) is continuous at z = 0**.

Hence $f(z)$ is continuous for all values of z.

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x - 0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{-y - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x - 0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{y - 0}{y} = 1$$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence C-R equations are satisfied at the origin.

Now $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z} = \underset{z \to 0}{\text{Lt}} \frac{(x^3 - y^3) + i(x^3 + y^3) - 0}{(x^2 + y^2)(x + iy)}$$

Let $z \to 0$ along the line $y = x$, then

$$f'(0) = \underset{x \to 0}{\text{Lt}} \frac{0 + 2ix^3}{2x^3(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{2} = \frac{1+i}{2} \qquad \text{...(1)}$$

Also, let $z \to 0$ along the x-axis (*i.e.* $y = 0$), then

$$f'(0) = \underset{x \to 0}{\text{Lt}} \frac{x^3 + ix^3}{x^3} = 1 + i \qquad \text{...(2)}$$

Since the limits (1) and (2) are different, $f'(0)$ does not exist.

Example 12. *Show that the function $f(z) = e^{-z^{-4}}$, $z \neq 0$ and $f(0) = 0$ is not analytic at $z = 0$, although Cauchy-Riemann equations are satisfied at this point.* [U.P.T.U. (C.O.) 2008]

Sol. Here, $$f(z) = e^{-z^{-4}} = e^{-(x+iy)^{-4}}$$

$$= e^{-\frac{1}{(x+iy)^4} \cdot \frac{(x-iy)^4}{(x-iy)^4}} = e^{-\left\{\frac{(x-iy)^4}{(x^2+y^2)^4}\right\}}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}[(x^4+y^4-6x^2y^2) - 4ixy(x^2-y^2)]}$$

$$\Rightarrow \qquad u + iv = e^{-\left[\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\right]}\left[\cos\frac{4xy\,(x^2-y^2)}{(x^2+y^2)^4} + i\sin\frac{4xy\,(x^2-y^2)}{(x^2+y^2)^4}\right]$$

$$\therefore \qquad u = e^{-\left[\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\right]}\cos\frac{4xy\,(x^2-y^2)}{(x^2+y^2)^4}$$

and
$$v = e^{-\left[\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\right]}\sin\frac{4xy\,(x^2-y^2)}{(x^2+y^2)^4}$$

At $z = 0$,
$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u\,(x, 0) - u\,(0, 0)}{x}$$

$$= \lim_{x\to 0}\frac{e^{-x^{-4}} - 0}{x} = \lim_{x\to 0}\frac{1}{xe^{x^{-4}}}$$

$$= \lim_{x\to 0}\frac{1}{x\left[1 + \frac{1}{x^4} + \frac{1}{2x^8} + \cdots\cdots\right]} = \lim_{x\to 0}\frac{1}{x + \frac{1}{x^3} + \frac{1}{2x^7} + \cdots\cdots} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0, y) - u(0, 0)}{y} = \lim_{y\to 0}\frac{-e^{-y^{-4}}}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x, 0) - v(0, 0)}{x} = \lim_{x\to 0}\frac{0}{x} = 0$$

and
$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0, y) - v(0, 0)}{y} = \lim_{y\to 0}\frac{0}{y} = 0.$$

Hence Cauchy-Riemann Conditions are satisfied at $z = 0$.

But
$$f'(0) = \lim_{z\to 0}\frac{f(z) - f(0)}{z} = \lim_{z\to 0}\frac{e^{-z^{-4}}}{z}$$

$$= \lim_{r\to 0}\frac{e^{-(re^{i\pi/4})^{-4}}}{re^{i\pi/4}} \text{ ; if } z \to 0 \text{ along } z = re^{i\pi/4}$$

$$= \lim_{r\to 0}\frac{e^{r^{-4}}}{re^{i\pi/4}} = \infty$$

which shows that $f'(z)$ does not exist at $z = 0$. Hence $f(z)$ is not analytic at $z = 0$.

Example 13. *(i) Examine the nature of the function*

$$f(z) = \begin{cases} \dfrac{x^2y^5\,(x+iy)}{x^4 + y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases} \text{ in the region including the origin.}$$

(ii) If $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$, *prove that* $\dfrac{f(z) - f(0)}{z} \to 0$ *as* $z \to 0$ *along any radius vector but not as* $z \to 0$ *in any manner and also that* $f(z)$ *is not analytic at* $z = 0$.

[G.B.T.U. 2013, U.K.T.U. 2010]

Sol. (*i*) Here, $$u + iv = \frac{x^2y^5(x+iy)}{x^4+y^{10}} \; ; z \neq 0$$

$$\therefore \quad u = \frac{x^3y^5}{x^4+y^{10}}, \; v = \frac{x^2y^6}{x^4+y^{10}}$$

At the origin, $$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x \to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y \to 0} \frac{0-0}{y} = 0$$

Similarly, $$\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$$

Since $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at the origin

But $$f'(0) = \lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{x^2y^5(x+iy)}{x^4+y^{10}} - 0\right] \cdot \frac{1}{x+iy}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2y^5}{x^4+y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{m^5x^7}{x^4+m^{10}x^{10}} = \lim_{x \to 0} \frac{m^5x^3}{1+m^{10}x^6} = 0$$

Again let $z \to 0$ along the curve $y^5 = x^2$

$$f'(0) = \lim_{x \to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$$

which shows that $f'(0)$ does not exist. Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

(*ii*) $$\frac{f(z)-f(0)}{z} = \left[\frac{x^3y(y-ix)}{x^6+y^2} - 0\right] \cdot \frac{1}{x+iy} = \frac{-ix^3y(x+iy)}{(x^6+y^2)} \cdot \frac{1}{x+iy} = -i \frac{x^3y}{x^6+y^2}$$

Let $z \to 0$ along radius vector $y = mx$ then,

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{x \to 0} \frac{-ix^3(mx)}{x^6+m^2x^2} = \lim_{x \to 0} \frac{-imx^2}{x^4+m^2} = 0$$

Hence $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector.

Now let $z \to 0$ along a curve $y = x^3$ then,

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{x \to 0} \frac{-ix^3 . x^3}{x^6+x^6} = \frac{-i}{2}$$

Hence $\dfrac{f(z)-f(0)}{z}$ does not tend to zero as $z \to 0$ along the curve $y = x^3$.

We observe that $f'(0)$ does not exist hence $f(z)$ is not analytic at $z = 0$.

Example 14. *Show that the following functions are harmonic and find their harmonic conjugate functions.*

(*i*) $u = \frac{1}{2} \log(x^2 + y^2)$ (U.P.T.U. 2015) (*ii*) $v = \sinh x \cos y$.

Sol. (*i*)
$$u = \frac{1}{2} \log (x^2 + y^2) \quad ...(1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad ...(2)$$

Also,
$$\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2y = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad ...(3)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \text{[From (2) and (3)]}$$

Since u satisfies Laplace's equation hence u is a harmonic function.

Let
$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy \quad \text{[Using C-R equations]}$$

$$= \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy$$

$$= \frac{x\,dy - y\,dx}{(x^2 + y^2)} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Integration yields,
$$v = \tan^{-1}\left(\frac{y}{x}\right) + c \quad | \ c \text{ is a constant}$$

which is the required harmonic conjugate function of u.

(*ii*)
$$v = \sinh x \cos y \quad ...(1)$$

$$\frac{\partial v}{\partial x} = \cosh x \cos y \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} = \sinh x \cos y \quad ...(2)$$

$$\frac{\partial v}{\partial y} = -\sinh x \sin y \quad \Rightarrow \quad \frac{\partial^2 v}{\partial y^2} = -\sinh x \cos y \quad ...(3)$$

Since,
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Hence v is harmonic.

Now, $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$
$$= -\sinh x \sin y\, dx - \cosh x \cos y\, dy$$
$$= -[\sinh x \sin y\, dx + \cosh x \cos y\, dy]$$
$$= -d(\cosh x \sin y).$$

Integration yields, $u = -\cosh x \sin y + c$ | c is a constant

Example 15. (*i*) *Show that the function $u(x, y) = x^4 - 6x^2y^2 + y^4$ is harmonic. Also find the analytic function $f(z) = u(x, y) + iv(x, y)$.* (U.P.T.U. 2007)

(*ii*) *Show that the function $u = x^3 - 3xy^2$ is harmonic and find the corresponding analytic function.* [U.P.T.U. (C.O.) 2008]

(*iii*) *Show that $e^x \cos y$ is a harmonic function, find the analytic function of which it is real part.* [U.P.T.U. (C.O.) 2008]

Sol. (*i*) $$u = x^4 - 6x^2y^2 + y^4$$

$\therefore$ $$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$$
$$\frac{\partial u}{\partial y} = -12x^2y + 4y^3 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$$

Since, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \therefore \quad u(x, y)$ is a harmonic function.

Now, let $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \left(-\frac{\partial u}{\partial y}\right)dx + \frac{\partial u}{\partial x}dy$$ | By C-R eqns.
$$= (12x^2y - 4y^3)\,dx + (4x^3 - 12xy^2)\,dy$$
$$= (12x^2y\,dx + 4x^3dy) - (4y^3\,dx + 12xy^2\,dy)$$
$$= d(4x^3y) - d(4xy^3)$$

Integration yields, $v = 4x^3y - 4xy^3 + c$

Hence $$f(z) = u + iv = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3 + c)$$
$$= (x + iy)^4 + c_1 = z^4 + c_1$$ | where $c_1 = ic$

(*ii*) $$u = x^3 - 3xy^2$$

$\therefore$ $$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x$$
$$\frac{\partial u}{\partial y} = -6xy \Rightarrow \frac{\partial^2 u}{\partial y^2} = -6x$$

Since, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \therefore \quad u$ is a harmonic function.

Now, $$dv = \frac{dv}{dx}dx + \frac{\partial v}{\partial y}dy = \left(-\frac{\partial u}{\partial y}\right)dx + \frac{\partial u}{\partial x}dy$$ | By C-R eqns.

$$= 6xy\,dx + (3x^2 - 3y^2)\,dy = (6xy\,dx + 3x^2\,dy) - 3y^2\,dy$$
$$= d(3x^2y) - d(y^3)$$

Integration yields,

$$v = 3x^2y - y^3 + c$$

$\therefore$ $$f(z) = u + iv = x^3 - 3xy^2 + i(3x^2y - y^3 + c)$$
$$= (x + iy)^3 + ic = z^3 + c_1 \qquad \text{(where } c_1 = ic\text{)}$$

(*iii*) Let $$u = e^x \cos y$$

$\therefore$ $$\frac{\partial u}{\partial x} = e^x \cos y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -\,e^x \sin y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -\,e^x \cos y$$

Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \therefore \quad u$ is a harmonic function.

Let $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy \qquad |\text{ By C-R eqns.}$$
$$= e^x \sin y\,dx + e^x \cos y\,dy = d\,(e^x \sin y)$$

Integration yields,

$$v = e^x \sin y + c$$

Hence $$f(z) = u + iv = e^x \cos y + i\,(e^x \sin y + c)$$
$$= e^x(\cos y + i \sin y) + c_1 \qquad |\text{ where } c_1 = ic$$
$$= e^{x+iy} + c_1 = e^z + c_1.$$

Example 16. *(i) In a two-dimensional fluid flow, the stream function is* $\psi = -\dfrac{y}{x^2 + y^2}$, *find the velocity potential* ϕ. [M.T.U. 2014]

(*ii*) *An electrostatic field in the xy-plane is given by the potential function* $\phi = 3x^2y - y^3$, *find the stream function and hence find complex potential.* (G.B.T.U. 2011, 2013)

Sol. (*i*) $$\psi = -\frac{y}{x^2 + y^2} \qquad \text{...(1)}$$

$$\frac{\partial \psi}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}, \qquad \frac{\partial \psi}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

We know that, $$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = \frac{\partial \psi}{\partial y}dx - \frac{\partial \psi}{\partial x}dy$$

$$= \frac{(y^2 - x^2)}{(x^2 + y^2)^2}dx - \frac{2xy}{(x^2 + y^2)^2}dy$$

$$= \frac{(x^2 + y^2)dx - 2x^2\,dx - 2xy\,dy}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)\, d(x) - x(2x\, dx + 2y\, dy)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)\, d(x) - xd\,(x^2 + y^2)}{(x^2 + y^2)^2} = d\left(\frac{x}{x^2 + y^2}\right).$$

Integration yields, $\phi = \dfrac{x}{x^2 + y^2} + c$ where c is a constant.

(*ii*) Let $\psi\,(x, y)$ be the stream function.

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy = \left(-\frac{\partial \phi}{\partial y}\right) dx + \left(\frac{\partial \phi}{\partial x}\right) dy$$

$$= \{-\,(3x^2 - 3y^2)\}\, dx + 6xy\, dy$$

$$= -\,3x^2\, dx + (3y^2\, dx + 6xy\, dy)$$

$$= -\,d\,(x^3) + 3d\,(xy^2)$$

Integrating, we get $\psi = -\,x^3 + 3xy^2 + c$ | c is a constant

Complex potential is given by

$$w = \phi + i\psi = 3x^2y - y^3 + i(-x^3 + 3xy^2 + c)$$

or, $w = -i[x^3 - iy^3 + 3ix^2y - 3xy^2 - c]$

or, $w = -i\,[(x + iy)^3 - c]$

$\Rightarrow$ $w = -iz^3 + c_1$ | where $c_1 = ic$

Example 17. (*i*) *If $u = e^x(x \cos y - y \sin y)$ is a harmonic function, find an analytic function $f(z) = u + iv$ such that $f(1) = e$.*

(*ii*) *Determine an analytic function $f(z)$ in terms of z whose real part is $e^{-x}(x \sin y - y \cos y)$.*

[M.T.U. 2012, G.B.T.U. 2011, U.P.T.U. 2006, 2008, 2014]

Sol. (*i*) We have, $u = e^x(x \cos y - y \sin y)$

$$\frac{\partial u}{\partial x} = e^x(x \cos y - y \sin y) + e^x \cos y = \phi_1\,(x, y) \quad \text{| say}$$

$$\frac{\partial u}{\partial y} = e^x\,[-\,x \sin y - y \cos y - \sin y] = \phi_2\,(x, y) \quad \text{| say}$$

$\therefore$ $\phi_1(z, 0) = e^z\, z + e^z = (z + 1)\, e^z$

$\phi_2\,(z, 0) = 0$

By Milne's Thomson method,

$$f(z) = \int \{\phi_1(z, 0) - i\, \phi_2(z, 0)\}\, dz + c \quad \text{| } c \text{ is a constant}$$

$$= \int (z + 1)\, e^z\, dz + c = (z - 1)\, e^z + e^z + c = ze^z + c \quad ...(1)$$

$f(1) = e + c$ | From (1)

$e = e + c$ | $f(1) = e$ (given)

$\Rightarrow$ $c = 0$

$\therefore$ From (1), $f(z) = ze^z$.

(*ii*) $u = e^{-x}(x \sin y - y \cos y)$

$$\frac{\partial u}{\partial x} = e^{-x} \sin y - e^{-x}\,(x \sin y - y \cos y) = \phi_1\,(x, y) \quad \text{| say}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y - \cos y + y \sin y) = \phi_2(x, y) \quad | \text{ say}$$

$\therefore \qquad \phi_1 (z, 0) = 0 \quad$ and $\quad \phi_2(z, 0) = e^{-z}(z - 1)$

By Milne's Thomson method,

$$f(z) = \int \{\phi_1 (z, 0) - i\phi_2 (z, 0)\} \, dz + c$$

$$= - i \int e^{-z} (z - 1) \, dz + c$$

$$= - i \left[(z - 1) (- e^{-z}) - \int (- e^{-z}) \, dz\right] + c$$

$$= - i \, [(1 - z) \, e^{-z} - e^{-z}] + c$$

$\Rightarrow \qquad f(z) = ize^{-z} + c \qquad$ | where c is a constant

Example 18. (*i*) *Determine the analytic function whose real part is e^{2x} (x cos 2y – y sin 2y).* (*ii*) *Find an analytic function whose imaginary part is e^{-x}(x cos y + y sin y).*

(U.P.T.U. 2009)

Sol. (*i*) Let $f(z) = u + iv$ be the required analytic function.

Here, $\qquad u = e^{2x} (x \cos 2y - y \sin 2y)$

$$\therefore \qquad \frac{\partial u}{\partial x} = e^{2x} (2x \cos 2y - 2y \sin 2y + \cos 2y) = \phi_1 (x, y) \quad | \text{ say}$$

and
$$\frac{\partial u}{\partial y} = - e^{2x} (2x \sin 2y + \sin 2y + 2y \cos 2y) = \phi_2 (x, y) \quad | \text{ say}$$

Now, $\qquad \phi_1 (z, 0) = e^{2z} (2z + 1)$

$\qquad \phi_2 (z, 0) = - e^{2z} (0) = 0$

By Milne's Thomson method,

$$f(z) = \int \{\phi_1(z, 0) - i \, \phi_2(z, 0)\} \, dz + c = \int e^{2z} (2z + 1) \, dz + c$$

$$= (2z + 1) \frac{e^{2z}}{2} - \int 2 \, . \frac{e^{2z}}{2} \, dz + c \qquad | \text{Integrating by parts}$$

$$= (2z + 1) \frac{e^{2z}}{2} - \frac{1}{2} e^{2z} + c$$

$f(z) = ze^{2z} + c$ where c is an arbitrary constant.

(*ii*) Let $f(z) = u + iv$ be the required analytic function.

Here $\qquad v = e^{-x}(x \cos y + y \sin y)$

$$\frac{\partial v}{\partial y} = e^{-x} (- x \sin y + y \cos y + \sin y) = \psi_1(x, y) \quad | \text{ say}$$

$$\frac{\partial v}{\partial x} = e^{-x} \cos y - e^{-x} (x \cos y + y \sin y) = \psi_2 (x, y) \quad | \text{ say}$$

$\therefore \qquad \psi_1 (z, 0) = 0$

$\qquad \psi_2(z, 0) = e^{-z} - e^{-z} z = (1 - z) e^{-z}$

By Milne's Thomson method,

$$f(z) = \int [\psi_1(z, 0) + i \, \psi_2(z, 0)] \, dz + c = i \int (1 - z) \, e^{-z} \, dz + c$$

$$= i\left[(1-z)(-e^{-z}) - \int (-1)(-e^{-z})\,dz\right] + c$$

$$= i\,[(z-1)\,e^{-z} + e^{-z}] + c$$

$$\Rightarrow \qquad f(z) = ize^{-z} + c$$

Example 19. (*i*) *Let* $f(z) = u(r, \theta) + iv(r, \theta)$ *be an analytic function. If* $u = -r^3 \sin 3\theta$, *then construct the corresponding analytic function* $f(z)$ *interms of* z.

(*ii*) *Find the analytic function* $f(z) = u + iv$, *given that* $v = \left(r - \frac{1}{r}\right) \sin \theta \,;\, r \neq 0$

Sol. (*i*) $u = -r^3 \sin 3\theta$

$$\frac{\partial u}{\partial r} = -3r^2 \sin 3\theta, \quad \frac{\partial u}{\partial \theta} = -3r^3 \cos 3\theta$$

we know that
$$dv = \frac{\partial v}{\partial r}\,dr + \frac{\partial v}{\partial \theta}\,d\theta = \left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right) dr + \left(r\,\frac{\partial u}{\partial r}\right) d\theta$$

$$= (3r^2 \cos 3\theta)\,dr - (3r^3 \sin 3\theta)\,d\theta$$

$$\Rightarrow \qquad dv = d\,(r^3 \cos 3\theta)$$

Integration yields,

$$v = r^3 \cos 3\theta + c$$

$$\therefore \qquad f(z) = u + iv = -r^3 \sin 3\theta + ir^3 \cos 3\theta + ic$$

$$= ir^3 (\cos 3\theta + i \sin 3\theta) + c_1 \qquad | \; c_1 = ic$$

$$= i(re^{i\theta})^3 + c_1$$

$$\Rightarrow \qquad f(z) = iz^3 + c_1 \qquad | \because \; z = re^{i\theta}$$

(*ii*)
$$v = \left(r - \frac{1}{r}\right) \sin \theta$$

$$\frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right) \sin \theta, \quad \frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right) \cos \theta$$

we know that,

$$du = \frac{\partial u}{\partial r}\,dr + \frac{\partial u}{\partial \theta}\,d\theta = \left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) dr + \left(-r\,\frac{\partial v}{\partial r}\right) d\theta$$

$$= \left(1 - \frac{1}{r^2}\right) \cos \theta\,dr - \left(r + \frac{1}{r}\right) \sin \theta\,d\theta$$

$$\Rightarrow \qquad du = d\,(r \cos \theta) + d\left(\frac{1}{r} \cos \theta\right)$$

Integration yields, $u = \left(r + \frac{1}{r}\right) \cos \theta + c$

$$\therefore \qquad f(z) = u + iv = \left(r + \frac{1}{r}\right) \cos \theta + c + i\left(r - \frac{1}{r}\right) \sin \theta$$

$$= re^{i\theta} + \frac{1}{r} e^{-i\theta} + c$$

$$\Rightarrow \qquad f(z) = z + \frac{1}{z} + c.$$

Example 20. *If $u - v = (x - y)\ (x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.*

Sol. Here, $f(z) = u + iv$

$\therefore \qquad if(z) = iu - v$

Adding $\qquad (1 + i)\, f(z) = (u - v) + i(u + v)$

Let $\qquad (1 + i)\, f(z) = \mathrm{F}(z)$, $u - v = \mathrm{U}$, $u + v = \mathrm{V}$, then

$$\mathrm{F}(z) = \mathrm{U} + i\mathrm{V}$$

Now, $\qquad \mathrm{U} = u - v = (x - y)\,(x^2 + 4xy + y^2)$

$$\Rightarrow \qquad \frac{\partial \mathrm{U}}{\partial x} = x^2 + 4xy + y^2 + (x - y)(2x + 4y) = 3x^2 + 6xy - 3y^2 = \phi_1(x, y)$$

| say

and
$$\frac{\partial \mathrm{U}}{\partial y} = -\,(x^2 + 4xy + y^2) + (x - y)(4x + 2y) = 3x^2 - 6xy - 3y^2 = \phi_2(x, y)$$

| say

Now, $\qquad \phi_1(z, 0) = 3z^2, \quad \phi_2(z, 0) = 3z^2$

By Milne's Thomson method,

$$\mathrm{F}(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz + c = \int [3z^2 - i(3z^2)]\, dz + c$$

$$\mathrm{F}(z) = (1 - i)\, z^3 + c$$

$$\Rightarrow \qquad (1 + i)\, f(z) = (1 - i)\, z^3 + c$$

or,
$$f(z) = \left(\frac{1-i}{1+i}\right) z^3 + \frac{c}{1+i} = \left(\frac{-2i}{2}\right) z^3 + c_1 \qquad \left(\text{where } c_1 = \frac{c}{1+i}\right)$$

or,
$$f(z) = -\,iz^3 + c_1.$$

Example 21. *If $u + v = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.*

Sol. Let $\qquad f(z) = u + iv \qquad$ ...(1)

Multiplying both sides by i

$$i\, f(z) = iu - v \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$(1 + i)\, f(z) = (u - v) + i(u + v) \qquad \text{...(3)}$$

$$\Rightarrow \qquad \mathrm{F}(z) = \mathrm{U} + i\mathrm{V} \qquad \text{...(4)}$$

where
$$\mathrm{F}(z) = (1 + i)\, f(z) \qquad \text{...(5)}$$

$$\mathrm{U} = u - v \quad \text{and} \quad \mathrm{V} = u + v \qquad \text{...(6)}$$

It means that we have been given

$$V = \frac{\sin 2x}{\cosh 2y - \cos 2x} \quad ...(7) \quad \Big| \because \quad e^{2y} + e^{-2y} = 2 \cosh 2y$$

Now $$\frac{\partial V}{\partial y} = \frac{-2 \sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = \psi_1(x, y) \quad | \text{ say}$$

and $$\frac{\partial V}{\partial x} = \frac{2 \cos 2x (\cosh 2y - \cos 2x) - 2 \sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2 \cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} = \psi_2(x, y) \quad | \text{ say}$$

$\therefore$ $$\psi_1(z, 0) = 0$$

$$\psi_2(z, 0) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z} = \frac{-2}{1 - 1 + 2 \sin^2 z} = -\operatorname{cosec}^2 z$$

By Milne's Thomson method, we have

$$F(z) = \int \{\psi_1(z, 0) + i\, \psi_2(z, 0)\}\, dz + c$$

$$= \int -i \operatorname{cosec}^2 z\, dz + c = i \cot z + c$$

Replacing F(z) by $(1 + i) f(z)$, from eqn. (5), we get

$$(1 + i) f(z) = i \cot z + c$$

$\Rightarrow$ $$f(z) = \frac{i}{1+i} \cot z + \frac{c}{1+i}$$

$\therefore$ $$f(z) = \tfrac{1}{2} (1 + i) \cot z + c_1, \qquad \text{where } c_1 = \frac{c}{1+i}.$$

Example 22. *If $f(z) = u + iv$ is an analytic function of z and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2 \cos x - 2 \cosh y}$, prove that $f(z) = \dfrac{1}{2}\left[1 - \cot \dfrac{z}{2}\right]$ when $f\left(\dfrac{\pi}{2}\right) = 0$.*

Sol. Let $f(z) = u + iv$ ...(1)

$\therefore$ $i\, f(z) = iu - v$

Add, $(1 + i) f(z) = (u - v) + i(u + v)$ ...(2)

$\Rightarrow$ $F(z) = U + iV$ ...(3)

where $u - v = U$, $u + v = V$ and $(1 + i) f(z) = F(z)$.

We have, $$u - v = \frac{\cos x + \sin x - e^{-y}}{2 \cos x - 2 \cosh y}$$

or, $$U = \frac{\cos x + \sin x - \cosh y + \sinh y}{2 \cos x - 2 \cosh y} \quad [\because \quad e^{-y} = \cosh y - \sinh y]$$

$$= \frac{1}{2} + \frac{\sin x + \sinh y}{2(\cos x - \cosh y)} \quad ...(4)$$

Diff. (4) w.r.t. x partially, we get

$$\frac{\partial U}{\partial x} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)\cos x - (\sin x + \sinh y)(-\sin x)}{(\cos x - \cosh y)^2}\right]$$

$$\phi_1(x, y) = \frac{1}{2}\left[\frac{1 - \cosh y \cos x + \sinh y \sin x}{(\cos x - \cosh y)^2}\right]$$

$$\phi_1(z, 0) = \frac{1}{2}\left[\frac{1 - \cos z}{(\cos z - 1)^2}\right] = \frac{1}{2(1 - \cos z)}. \qquad \text{...(5)}$$

Diff. (4) partially w.r.t. y, we get

$$\frac{\partial U}{\partial y} = \frac{1}{2}\left[\frac{(\cos x - \cosh y) . \cosh y - (\sin x + \sinh y)(-\sinh y)}{(\cos x - \cosh y)^2}\right]$$

$$\phi_2(x, y) = \frac{1}{2}\left[\frac{\cos x \cosh y + \sin x \sinh y - 1}{(\cos x - \cosh y)^2}\right]$$

$$\therefore \quad \phi_2(z, 0) = \frac{1}{2}\left[\frac{\cos z - 1}{(\cos z - 1)^2}\right] = \frac{1}{2} \cdot \left(\frac{-1}{1 - \cos z}\right). \qquad \text{...(6)}$$

By Milne's Thomson Method,

$$F(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz + c$$

$$= \int \left[\frac{1}{2} \cdot \frac{1}{(1 - \cos z)} + \frac{i}{2} \cdot \frac{1}{1 - \cos z}\right] dz + c$$

$$= \frac{1+i}{2} \int \frac{1}{2\sin^2 z/2}\, dz + c = \frac{1+i}{4} \int \operatorname{cosec}^2 (z/2)\, dz + c$$

$$= \left(\frac{1+i}{4}\right) \cdot \frac{(-\cot z/2)}{\left(\frac{1}{2}\right)} + c = -\left(\frac{1+i}{2}\right) \cot \frac{z}{2} + c$$

or, $$(1 + i) f(z) = -\left(\frac{1+i}{2}\right) \cot \frac{z}{2} + c$$

$$\Rightarrow \quad f(z) = -\frac{1}{2} \cot \frac{z}{2} + \frac{c}{1+i} \qquad \text{...(7)}$$

$$f\left(\frac{\pi}{2}\right) = -\frac{1}{2} \cot \frac{\pi}{4} + \frac{c}{1+i} \qquad \text{[From (7)]}$$

$$0 = -\frac{1}{2} + \frac{c}{1+i} \quad \Rightarrow \quad \frac{c}{1+i} = \frac{1}{2} \qquad \text{...(8)}$$

$$\therefore \quad \text{From (7)}, \; f(z) = -\frac{1}{2} \cot \frac{z}{2} + \frac{1}{2} = \frac{1}{2}\left(1 - \cot \frac{z}{2}\right). \qquad \text{[Using (8)]}$$

Example 23. (*i*) *If f(z) is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\ |f'(z)|^2. \qquad \text{(U.P.T.U. 2007, 2015)}$$

(*ii*) *If f(z) is a harmonic function of z, show that*

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2 \qquad \text{(U.P.T.U. 2009)}$$

Sol. (*i*) Let $f(z) = u + iv$ so that $|f(z)| = \sqrt{u^2+v^2}$

or $$|f(z)|^2 = u^2+v^2 = \phi(x, y) \text{ (say)}$$

$\therefore$ $$\frac{\partial\phi}{\partial x} = 2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2\phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2+v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly, $$\frac{\partial^2\phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2+v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

Adding, we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right] \quad ...(1)$$

Since $f(z) = u + iv$ is a regular function of z, u and v satisfy C-R equations and Laplace's equation.

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$$

$\therefore$ From (1), we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 2\left[0+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+0+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right] \quad ...(2)$$

Now $$f(z) = u + iv$$

$\therefore$ $$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \quad \text{and} \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

From (2), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = 4\ |f'(z)|^2 \quad \text{or} \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\ |f'(z)|^2.$$

(*ii*) We have, $f(z) = u + iv$ ...(1)

$\therefore$ $|f(z)| = \sqrt{u^2 + v^2}$ ...(2)

Partially differentiating eqn. (2) w.r.t. x and y, we get

$$\frac{\partial}{\partial x}|f(z)| = \frac{1}{2}(u^2+v^2)^{-1/2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right) = \frac{u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x}}{|f(z)|} \quad ...(3)$$

Similarly, $$\frac{\partial}{\partial y}|f(z)| = \frac{u\dfrac{\partial u}{\partial y} + v\dfrac{\partial v}{\partial y}}{|f(z)|} \quad ...(4)$$

Squaring and adding (3) and (4), we get

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x}\right)^2 + \left(u\dfrac{\partial u}{\partial y} + v\dfrac{\partial v}{\partial y}\right)^2}{|f(z)|^2}$$

$$= \frac{\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x}\right)^2 + \left(-u\dfrac{\partial v}{\partial x} + v\dfrac{\partial u}{\partial x}\right)^2}{|f(z)|^2} \qquad | \text{ Using C-R eqns.}$$

$$= \frac{(u^2+v^2)\left[\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right]}{|f(z)|^2}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \qquad \left| \because \ |f(z)|^2 = u^2 + v^2 \right.$$

$$= |f'(z)|^2 \qquad \left| \because \ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right.$$

ASSIGNMENT

1. (*i*) Determine a, b, c, d so that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic.

(*ii*) Find the constants a, b, c such that the function $f(z)$ where

$f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$ is analytic. Express $f(z)$ in terms of z.

(*M.T.U. 2013*)

(*iii*) Find the value of the constants a and b such that the following function $f(z)$ is analytic.

$f(z) = \cos x(\cosh y + a \sinh y) + i \sin x(\cosh y + b \sinh y)$

(*iv*) Determine p such that the function $f(z) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{px}{y}$ is an analytic function.

Also find $f'(z)$. (*M.T.U. 2012*)

2. Show that

(*a*) $f(z) = xy + iy$ is everywhere continuous but is not analytic.

(*b*) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

(*c*) $f(z) = z\,|\,z\,|$ is not analytic anywhere. (*U.K.T.U. 2010*)

3. Discuss the analyticity of the following functions:

(i) $\sin z$ (ii) $\cosh z$ (iii) $\frac{1}{z}$ (iv) z^3.

4. (i) Define analytic function. Discuss the analyticity and differentiability of $f(z) = |z|^4$ at $z = 0$. (*G.B.T.U. 2012*)

(ii) Define anlaytic function. Discuss the analyticity of $f(z) = \text{Re}\,(z^3)$ in the complex plane. (*U.P.T.U. 2014*)

5. Show that the polar form of Cauchy-Riemann equations are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$. Deduce that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

6. (i) Show that an analytic function $f(z)$, whose derivative is identically zero, is constant.

(ii) It is given that a function $f(z)$ and its conjugate $\overline{f(z)}$ are both analytic. Determine the function $f(z)$.

(iii) Show that if $f(z)$ is analytic and Im $f(z)$ = constant then $f(z)$ is a constant.

(iv) Show that if $f(z)$ is differentiable at a point z, then $|f'(z)|^2 = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

7. (i) Show that the function $f(z)$ defined by $f(z) = \frac{x^3 y^5 (x + iy)}{x^6 + y^{10}}$, $z \neq 0$, $f(0) = 0$, is not analytic at the origin even though it satisfies Cauchy-Riemann equations at the origin. (*G.B.T.U. 2011*)

(ii) Show that for the function

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

the Cauchy-Riemann equations are satisfied at the origin. Does $f'(0)$ exist?

(iii) Show that for the function

$$f(z) = \begin{cases} \dfrac{2xy\,(x + iy)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

the C-R equations are satisfied at origin but derivative of $f(z)$ does not exist at origin.

8. (i) If u is a harmonic function then show that $w = u^2$ is not a harmonic function unless u is a constant.

(ii) If $f(z)$ is an analytic function, show that $|f(z)|$ is not a harmonic function.

9. (i) Show that the function $u(x, y) = 2x + y^3 - 3x^2y$ is harmonic. Find its conjugate harmonic function $v(x, y)$ and the corresponding analytic function $f(z)$.

(ii) Show that the function $v(x, y) = e^x \sin y$ is harmonic. Find its conjugate harmonic function $u(x, y)$ and the corresponding analytic function $f(z)$.

(iii) Define a harmonic function and conjugate harmonic function. Find the harmonic conjugate of the function $u(x, y) = 2x(1 - y)$. (*U.P.T.U. 2009*)

(iv) Show that the function $u = e^{-2xy} \sin (x^2 - y^2)$ is harmonic. (*U.K.T.U. 2011*)

(v) Show that $u(x, y) = x^3 - 4xy - 3xy^2$ is harmonic. Find its harmonic conjugate $v(x, y)$ and the corresponding analytic function $f(z) = u + iv$. (*G.B.T.U. 2013*)

10. (i) Show that the function $u(r, \theta) = r^2 \cos 2\theta$ is harmonic. Find its conjugate harmonic function and the corresponding analytic function $f(z)$.

(ii) Determine constant 'b' such that $u = e^{bx} \cos 5y$ is harmonic.

(*iii*) Define Harmonic function. Show that the function $v = \log(x^2 + y^2) + x - 2y$ is harmonic. Also find the analytic function $f(z) = u + iv$. (*G.B.T.U. 2012*)

(*iv*) Show that $v(x, y) = e^{-x}(x \cos y + y \sin y)$ is harmonic. Find its harmonic conjugate. (*U.P.T.U. 2014*)

11. Determine the analytic function $f(z)$ in terms of z whose real part is

(*i*) $\frac{1}{2}\log(x^2 + y^2)$ (*U.K.T.U. 2011*) (*ii*) $\cos x \cosh y$

(*iii*) $e^{-x}(x \cos y + y \sin y)$; $f(0) = 1$ (*iv*) $(x - y)(x^2 + 4xy + y^2)$ (*G.B.T.U. 2012*)

(*v*) $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ (*vi*) $\dfrac{\sin 2x}{\cosh 2y + \cos 2x}$.

12. Find the regular function $f(z)$ in terms of z whose imaginary part is

(*i*) $\dfrac{x - y}{x^2 + y^2}$ (*ii*) $\cos x \cosh y$ (*iii*) $\sinh x \cos y$

(*iv*) $6xy - 5x + 3$ (*v*) $\dfrac{x}{x^2 + y^2} + \cosh x \cos y$. (*vi*) $e^x(x \sin y + y \cos y)$ (*U.P.T.U. 2015*)

13. Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of z.

14. (*i*) An electrostatic field in the xy-plane is given by the potential function $\phi = x^2 - y^2$, find the stream function.

(*ii*) If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function.

15. (*i*) In a two dimensional fluid flow, the stream function is $\psi = \tan^{-1}\left(\dfrac{y}{x}\right)$, find the velocity potential ϕ.

(*ii*) If $w = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, determine the function ϕ.

(*iii*) If $u = (x - 1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x + iy$. [*U.K.T.U. 2010*]

16. If $f(z)$ is an analytic function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |\text{Re } f(z)|^2 = 2\, |f'(z)|^2$.

(*G.B.T.U. 2012*)

17. Find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$.

18. If $f(z) = u + iv$ is an analytic function, find $f(z)$ in terms of z if

(*i*) $u - v = e^x(\cos y - \sin y)$ (*ii*) $u + v = \dfrac{x}{x^2 + y^2}$, when $f(1) = 1$

[*U.P.T.U. (C.O.) 2008*]

(*iii*) $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ when $f\left(\dfrac{\pi}{2}\right) = \dfrac{3 - i}{2}$.

19. (*i*) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u + v = (x + y)(2 - 4xy + x^2 + y^2)$, then construct $f(z)$ in terms of z.

(*ii*) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = e^{-x}[(x - y)\sin y - (x + y)\cos y]$, then construct $f(z)$ in terms of z. [*U.P.T.U. (C.O.) 2009*]

20. If $f = u + iv$ is analytic show that $g = -v + iu$ is also analytic. Also show that u and $-v$ are conjugate harmonic.

21. Show that the function

(*i*) $f(z) = \dfrac{z}{z+1}$ is analytic at $z = \infty$. (*ii*) $f(z) = z$ is not analytic at $z = \infty$.

22. If $f(z) = u(x, y) + iv(x, y)$ where $x = \dfrac{z + \bar{z}}{2}$, $y = \dfrac{z - \bar{z}}{2i}$ is continuous as a function of two variables z and $\bar{z}$ then show that $\dfrac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the Cauchy-Riemann equations.

$$\left[\textbf{Hint. } \frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)\right]$$

23. (*i*) Show that a harmonic function satisfies the formal differential equation $\dfrac{\partial^2 u}{\partial z \partial \bar{z}} = 0$

(*ii*) If $w = f(z)$ is a regular function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) \log |f'(z)| = 0$. Further, if $|f'(z)|$ is the product of a function of x and function of y, show that $f'(z) = \exp.(\alpha z^2 + \beta z + \gamma)$ where α is real and β, γ are complex constants.

24. If $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z if

(*i*) $3u + v = 3 \sin x \cosh y + \cos x \sinh y$ (*ii*) $u - 2v = \cos x \cosh y + 2 \sin x \sinh y$

(*iii*) $2u - v = e^x (2 \cos y - \sin y)$

25. (*i*) If $f'(z) = f(z)$ for all z, then show that $f(z) = ke^z$, where k is an arbitrary constant.

(*ii*) Find an analytic function $f(z)$ such that Re $[f'(z)] = 3x^2 - 4y - 3y^2$ and $f(1 + i) = 0$

(*iii*) Let $f(z) = u + iv$ and $g(z) = v + iu$ be analytic functions for all z. Let $f(0) = 1$ and $g(0) = i$. Obtain the value of $h(z)$ at $z = 1 + i$ where $h(z) = f'(z) + g'(z) + 2f(z)\,g(z)$.

(*iv*) If $f(z) = u + iv$ is an analytic function of z and ϕ is a function of u and v, then show that

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = \left[\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right] |f'(z)|^2$$

Answers

1. (*i*) $a = 2, b = -1, c = -1, d = 2$ (*ii*) $a = -\dfrac{1}{2}, b = -2, c = \dfrac{1}{2}$; $f(z) = -\dfrac{1}{2}(2 + i) z^2$

(*iii*) $a = -1, b = -1$ (*iv*) $p = -1, f'(z) = \dfrac{1}{z}$

6. (*ii*) constant function **7.** (*ii*) No

9. (*i*) $v = 2y - 3xy^2 + x^3 + c$; $f(z) = 2z + iz^3 + ic$ (*ii*) $u = e^x \cos y + c$; $f(z) = e^z + c$

(*iii*) $v(x, y) = x^2 - y^2 + 2y + c$ (*v*) $v(x, y) = 2x^2 - 2y^2 + 3x^2y - y^3 + c, f(z) = z^3 + 2iz^2 + c$

10. (*i*) $v = r^2 \sin 2\theta + c$; $f(z) = z^2 + ic$ (*ii*) $b = \pm 5$

(*iii*) $f(z) = -2z + i(2 \log z + z) + c$ (*iv*) $u(x, y) = e^{-x}(x \sin y - y \cos y) + c$

11. (*i*) $\log z + c$ (*ii*) $\cos z + c$ (*iii*) $1 + ze^{-z}$

(*iv*) $(1 - i) z^3 + c$ (*v*) $\cot z + c$ (*vi*) $\tan z + c$

12. (*i*) $\dfrac{1+i}{z} + c$ (*ii*) $i \cos z + c$ (*iii*) $i \sinh z + c$

(*iv*) $3z^2 - 5iz + c$ (*v*) $\dfrac{i}{z} + i \cosh z + c$ (*vi*) $z\,e^z + c$

13. $v = x^2 - y^2 + 2xy - 2y - 3x + c$, $f(z) = (1 + i)z^2 - (2 + 3i)z + ic$

14. (*i*) $\psi = 2xy + c$ (*ii*) $2 \tan^{-1}\left(\frac{y}{x}\right) + c$, $2 \log z + ic$

15. (*i*) $\frac{1}{2} \log (x^2 + y^2) + c$ (*ii*) $-2xy + \frac{y}{x^2 + y^2} + c$ (*iii*) $v = 3y (1 + x^2) - y^3$

17. $i(z^2 - z + 2) + c$

18. (*i*) $e^z + c$ (*ii*) $\frac{1}{1+i}\left(\frac{i}{z} + 1\right)$ (*iii*) $\cot \frac{z}{2} + \frac{1}{2} (1 - i)$

21. (*i*) $2z + iz^3 + c$ (*ii*) $ize^{-z} + c$

24. (*i*) $f(z) = \sin z + c$ (*ii*) $f(z) = \cos z + c$ (*iii*) $f(z) = e^z + c$

25. (*ii*) $f(z) = z^3 + 2iz^2 + 6 - 2i$ (*iii*) $2i$

1.19 LINE INTEGRAL IN THE COMPLEX PLANE

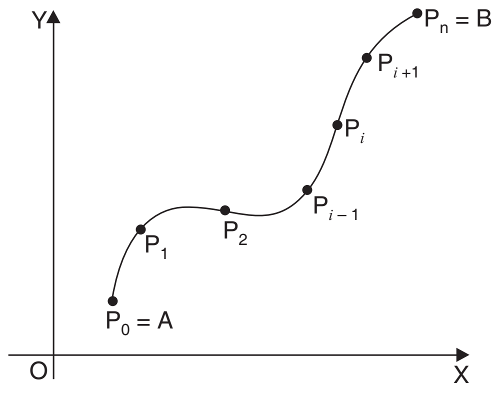
In case of real variable, the path of integration of $\int_a^b f(x)\, dx$ is always along the x-axis from $x = a$ to $x = b$. But in case of a complex function $f(z)$, the path of the definite integral $\int_a^b f(z)\, dz$ can be along any curve from $z = a$ to $z = b$.

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide the curve C into n parts at the points

$$A = P_0(z_0),\ P_1(z_1),\ \ldots\ldots,\ P_i(z_i),\ \ldots\ldots,\ P_n(z_n) = B.$$

Let $\delta z_i = z_i - z_{i-1}$ and ξ_i be any point on the arc $P_{i-1} P_i$. Then the limit of the sum $\sum_{i=1}^{n} f(\xi_i)\, \delta z_i$ as $n \to \infty$ and each $\delta z_i \to 0$, if it exists, is called the **line integral of f(z)** along the curve C. It is denoted by $\int_C f(z)\, dz$.

In case the points P_0 and P_n coincide so that C is a closed curve, then this integral is called **contour integral** and is denoted by $\oint_C f(z)\, dz$.



If $f(z) = u(x, y) + iv(x, y)$, then since $dz = dx + i\, dy$, we have

$$\int_C f(z)\, dz = \int_C (u + iv)(dx + i\, dy)$$

$$= \int_C (udx - vdy) + i \int_C (vdx + udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Moreover, the value of the integral depends on the path of integration unless the integrand is analytic.

When the same path of integration is used in each integral, then

$$\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$$

If c is a point on the arc joining a and b, then

$$\int_a^b f(z)\,dz = \int_a^c f(z)\,dz + \int_c^b f(z)\,dz.$$

EXAMPLES

Example 1. *Evaluate* $\int_0^{1+i} (x - y + ix^2)\,dz$.

(a) along the straight line from z = 0 to z = 1 + i

(b) along the real axis from z = 0 to z = 1 and then along a line parallel to imaginary axis from z = 1 to z = 1 + i.

(c) along the imaginary axis from z = 0 to z = i and then along a line parallel to real axis from z = i to z = 1 + i.

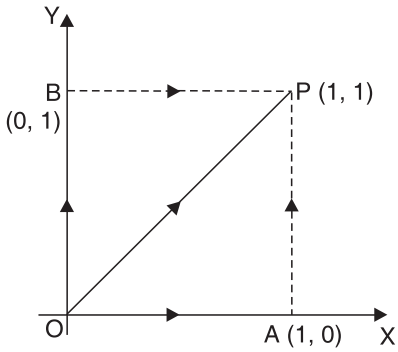
Sol. (a) Along the straight line OP joining O(z = 0) and P(z = 1 + i), $y = x$, $dy = dx$ and x varies from 0 to 1.

$$\therefore \quad \int_0^{1+i} (x - y + ix^2)\,dz = \int_0^{1+i} (x - y + ix^2)(dx + i\,dy)$$

$$= \int_0^1 (x - x + ix^2)(dx + idx) = \int_0^1 ix^2(1+i)\,dx$$

$$= (i - 1)\left(\frac{x^3}{3}\right)_0^1 = -\frac{1}{3} + \frac{1}{3}i.$$

(b) Along the path OAP where A is $z = 1$



$$\int_0^{1+i} (x - y + ix^2)\,dz$$

$$= \int_{\text{OA}} (x - y + ix^2)\,dz + \int_{\text{AP}} (x - y + ix^2)\,dz \qquad \ldots(1)$$

Now along OA, $y = 0$, $dz = dx$ and x varies from 0 to 1.

$$\therefore \quad \int_{\text{OA}} (x - y + ix^2)\,dz = \int_0^1 (x + ix^2)\,dx = \left[\frac{x^2}{2} + i\,\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + \frac{1}{3}i$$

Also along AP, $x = 1$, $dz = idy$ and y varies from 0 to 1

$$\therefore \quad \int_{\text{AP}} (x - y + ix^2)\,dz = \int_0^1 (1 - y + i)\,idy = \left[(-1 + i)y - i\,\frac{y^2}{2}\right]_0^1 = -1 + i - \frac{1}{2}i = -1 + \frac{1}{2}i$$

Hence from (1), $\displaystyle\int_0^{1+i} (x - y + ix^2)\,dz = \left(\frac{1}{2} + \frac{1}{3}i\right) + \left(-1 + \frac{1}{2}i\right) = -\frac{1}{2} + \frac{5}{6}i.$

(*c*) Along the path OBP where B is $z = i$

$$\int_0^{1+i} (x - y + ix^2)\, dz = \int_{OB} (x - y + ix^2)\, dz + \int_{BP} (x - y + ix^2)\, dz \qquad \ldots(2)$$

Now along OB, $x = 0$, $dz = idy$ and y varies from 0 to 1

$$\therefore \qquad \int_{OB} (x - y + ix^2)\, dz = \int_0^1 (-y)\, idy = -i\left[\frac{y^2}{2}\right]_0^1 = -\frac{1}{2}i$$

Also, along BP, $y = 1$, $dz = dx$ and x varies from 0 to 1

$$\therefore \qquad \int_{BP} (x - y + ix^2)\, dz = \int_0^1 (x - 1 + ix^2)\, dx = \left[\frac{x^2}{2} - x + i\frac{x^3}{3}\right]_0^1 = -\frac{1}{2} + \frac{1}{3}i$$

Hence from (2), $\displaystyle\int_0^{1+i} (x - y + ix^2)\, dz = -\frac{1}{2}i + \left(-\frac{1}{2} + \frac{1}{3}i\right) = -\frac{1}{2} - \frac{1}{6}i.$

Note. The values of the integral are different along the three different paths.

Example 2. *Evaluate* $\displaystyle\int_0^{1+i} (x^2 - iy)\, dz$ *along the paths*

(a) $y = x$ *(b)* $y = x^2$. (G.B.T.U. 2010)

Sol. (*a*) Along the line $y = x$,

$$dy = dx \text{ so that } dz = dx + idx = (1 + i)\, dx$$

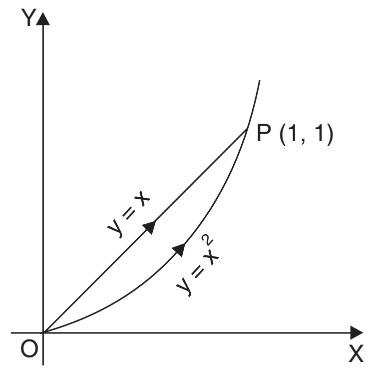
$$\int_0^{1+i} (x^2 - iy)\, dz = \int_0^1 (x^2 - ix)(1 + i)\, dx$$

$$= (1 + i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1 = (1 + i)\left(\frac{1}{3} - \frac{1}{2}i\right) = \frac{5}{6} - \frac{1}{6}i.$$

(*b*) Along the parabola $y = x^2$, $dy = 2x\, dx$ so that

$$dz = dx + 2ix\, dx = (1 + 2ix)\, dx$$

and x varies from 0 to 1.



$$\therefore \qquad \int_0^{1+i} (x^2 - iy)\, dz = \int_0^1 (x^2 - ix^2)(1 + 2ix)\, dx$$

$$= (1 - i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right]_0^1$$

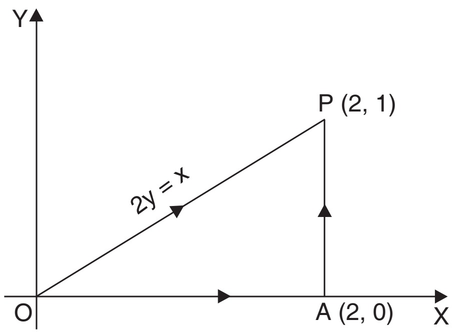
$$= (1 - i)\left(\frac{1}{3} + \frac{1}{2}i\right) = \frac{5}{6} + \frac{1}{6}i.$$

Example 3. *Evaluate* $\displaystyle\int_0^{2+i} (\bar{z})^2\, dz$*, along*

(a) the real axis from $z = 0$ *to* $z = 2$ *and then along a line parallel to y-axis from* $z = 2$ *to* $z = 2 + i$. (U.P.T.U. 2009, U.K.T.U. 2011)

(b) along the line $2y = x$. (U.P.T.U. 2009)

Sol. $(\bar{z})^2 = (x - iy)^2 = (x^2 - y^2) - 2ixy$



(a) Along the path OAP where A is (2, 0) and P is (2, 1).

$$\int_0^{2+i} (\bar{z})^2\, dz$$

$$= \int_{OA} (x^2 - y^2 - 2ixy)\, dz + \int_{AP} (x^2 - y^2 - 2ixy)\, dz \quad ...(1)$$

Now, along OA, $y = 0$, $dz = dx$ and x varies from 0 to 2

$$\therefore \qquad \int_{OA} (x^2 - y^2 - 2ixy)\, dz = \int_0^2 x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

Also, along AP, $x = 2$, $dz = idy$ and y varies from 0 to 1

$$\therefore \int_{AP} (x^2 - y^2 - 2ixy)\, dz = \int_0^1 (4 - y^2 - 4iy)\, idy$$

$$= \left[4iy - i\frac{y^3}{3} + 2y^2\right]_0^1 = 4i - \frac{1}{3}i + 2 = 2 + \frac{11}{3}i$$

Hence from (1), we have $\displaystyle\int_0^{2+i} (\bar{z})^2\, dz = \frac{8}{3} + 2 + \frac{11}{3}i = \frac{14}{3} + \frac{11}{3}i.$

(b) Along the line OP, $2y = x$, $dx = 2dy$

so that $dz = 2dy + i\, dy = (2 + i)\, dy$

and y varies from 0 to 1.

$$\therefore \qquad \int_0^{2+i} (\bar{z})^2\, dz = \int_0^{2+i} (x^2 - y^2 - 2ixy)\, dz = \int_0^1 (4y^2 - y^2 - 4iy^2)(2 + i)\, dy$$

$$= (2 + i)(3 - 4i) \int_0^1 y^2\, dy = (10 - 5i)\left[\frac{y^3}{3}\right]_0^1 = \frac{10}{3} - \frac{5}{3}i.$$

Example 4. *Integrate $f(z) = x^2 + ixy$ from A(1, 1) to B(2, 4) along the curve $x = t$, $y = t^2$.*

Sol. Equations of the path of integration are $x = t$, $y = t^2$

$\therefore \qquad dx = dt, \quad dy = 2t\, dt$

At A(1, 1), $t = 1$ and at B(2, 4), $t = 2$

$$\therefore \qquad \int_A^B f(z)\, dz = \int_A^B (x^2 + ixy)(dx + idy) = \int_1^2 (t^2 + it^3)(dt + 2it\, dt)$$

$$= \int_1^2 (t^2 - 2t^4)\, dt + i\int_1^2 3t^3\, dt = \left[\frac{t^3}{3} - \frac{2t^5}{5}\right]_1^2 + i\left[\frac{3t^4}{4}\right]_1^2$$

$$= \left(\frac{8}{3} - \frac{64}{5}\right) - \left(\frac{1}{3} - \frac{2}{5}\right) + i\left(12 - \frac{3}{4}\right) = -\frac{151}{5} + \frac{45}{4}i.$$

Example 5. *Prove that*

(i) $\displaystyle\oint_C \frac{dz}{z - a} = 2\pi i$

(ii) $\displaystyle\oint_C (z - a)^n\, dz = 0$ *[n is an integer $\neq -1$] where C is the circle $|\ z - a\ | = r$.*

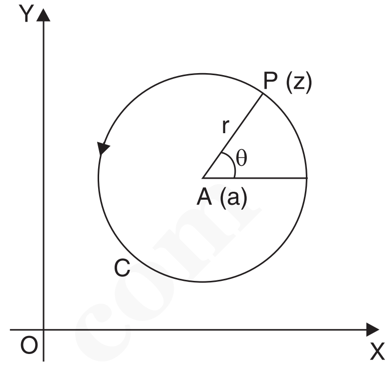
(G.B.T.U. 2011)

Sol. The equation of the circle C is

$$|z-a| = r \quad \text{or} \quad z-a = re^{i\theta}$$

where θ varies from 0 to 2π as z describes C once in the anti-clockwise direction.

Also $\qquad dz = ire^{i\theta}\, d\theta.$



$$(i) \qquad \oint_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}\, d\theta}{re^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

$$(ii) \quad \oint_C (z-a)^n\, dz = \int_0^{2\pi} r^n\, e^{ni\theta} \,.\, ire^{i\theta}\, d\theta$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta}\, d\theta$$

$$= ir^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi} \qquad [\because \quad n \neq -1]$$

$$= \frac{r^{n+1}}{n+1}\, [e^{i2(n+1)\pi} - 1]$$

$$= 0. \qquad [\because \quad e^{i2(n+1)\pi} = \cos 2(n+1)\pi + i \sin 2(n+1)\pi = 1]$$

Example 6. *Evaluate the integral* $\int_c |z|\, dz$, *where c is the contour*

(i) *The straight line from* $z = -i$ *to* $z = i$

(ii) *The left half of the unit circle* $|z| = 1$ *from* $z = -i$ *to* $z = i$.

Sol. (i) The straight line from $z = -i$ to $z = i$ is $x = 0$

i.e., $\qquad z = iy$ so that $dz = idy$

$$\therefore \qquad \int_c |z|\, dz = \int_{-1}^{1} |iy|\, i\, dy = i\int_{-1}^{0} (-y)\, dy + i\int_0^1 y\, dy$$

$$= -i\left(\frac{y^2}{2}\right)_{-1}^{0} + i\left(\frac{y^2}{2}\right)_0^1 = -i\left(-\frac{1}{2}\right) + i\left(\frac{1}{2}\right) = i.$$

(ii) For a point on the unit circle $|z| = 1$,

$$z = e^{i\theta}$$

$$\therefore \qquad dz = ie^{i\theta}\, d\theta.$$

The points $z = -i$ and i correspond to $\theta = \frac{3\pi}{2}$ and $\theta = \frac{\pi}{2}$ respectively.

$$\therefore \qquad \int_c |z|\, dz = \int_{3\pi/2}^{\pi/2} 1\,.\, e^{i\theta}\, i d\theta = \left(e^{i\theta}\right)_{3\pi/2}^{\pi/2} = e^{i\pi/2} - e^{3i\pi/2}$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2} = 0 + i - 0 - i(-1) = 2i.$$

Example 7. *Evaluate the integral* $\int_c \log z\, dz$, *where c is the unit circle* $|z| = 1$.

Sol. Here, $\qquad c \equiv |z| = 1 \qquad$ ...(1)

$$\int_c \log z dz = \int_c \log(x+iy)\, dz$$

$$= \int_c \left[\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}\right] dz$$

$$= i\int_c \tan^{-1}\left(\frac{y}{x}\right) dz. \qquad \text{...(2)} \quad (\because \; x^2+y^2=1)$$

On the unit circle, $z = e^{i\theta}$

$\therefore$ $dz = ie^{i\theta}\, d\theta$.

Now (2) becomes,

$$\int_c \log z\, dz = i\int_0^{2\pi} \tan^{-1}(\tan\theta)\, ie^{i\theta}\, d\theta = -\int_0^{2\pi} \theta e^{i\theta}\, d\theta$$

$$= -\left[\left(\theta\frac{e^{i\theta}}{i}\right)_0^{2\pi} - \int_0^{2\pi} 1.\frac{e^{i\theta}}{i}\, d\theta\right] = -\left[\frac{2\pi}{i}e^{2\pi i} - \frac{1}{i}\left(\frac{e^{i\theta}}{i}\right)_0^{2\pi}\right]$$

$$= -\left[\frac{2\pi}{i}e^{2\pi i} + e^{2\pi i} - 1\right] = 2\pi i e^{2\pi i} + 1 - e^{2\pi i} = 2\pi i \qquad | \because \; e^{2\pi i} = 1$$

ASSIGNMENT

1. Evaluate $\int_0^{3+i} z^2\, dz$, along

   (*a*) the line $y = \dfrac{x}{3}$ (*b*) the real axis to 3 and then vertically to $3 + i$

   (*c*) the parabola $x = 3y^2$.

2. Find the value of the integral $\int_0^{1+i} (x - y - ix^2)\, dz$, along real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + i$. [*G.B.T.U. (C.O.) 2011*]

3. Evaluate $\int_0^{4+2i} \bar{z}\, dz$ along the curve given by $z = t^2 + it$.

4. (*i*) Evaluate $\oint_C |z|^2\, dz$ around the square with vertices at (0, 0), (1, 0), (1, 1) and (0, 1).

   (*ii*) Show that $\oint_C (z+1)\, dz = 0$ where C is the boundary of the square whose vertices are at the points $z = 0, z = 1, z = 1 + i$ and $z = i$.

5. (*a*) Evaluate $\int_C \left[(x+y)\, dx + x^2 y\, dy\right]$

   (*i*) along $y = x^2$ having (0, 0), (3, 9) as end points.

   (*ii*) along $y = 3x$ between the same points.

   (*b*) Evaluate $\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + 3y^2)\, dx + 2(x^2 + 3xy + 4y^2) dy$

   (*i*) along $y = x^2$ (*ii*) along $y = x$

   Does the value of the integral depend upon the path?

6. (*i*) Evaluate $\oint_C (y - x - 3x^2 i)\, dz$ where C is the straight line from $z = 0$ to $z = 1 + i$.

(*ii*) Evaluate $\int_{1-i}^{2+3i} (z^2 + z)\, dz$ along the line joining the points (1, – 1) and (2, 3).

7. (*i*) Evaluate $\oint_C (z - z^2)\, dz$ where C is the upper half of the circle $|\, z\, | = 1$. What is the value of this integral if C is lower half of the given circle?

(*ii*) Evaluate the integral $\int_C (z - z^2)\, dz$ where C is the upper half of the circle $|z - 2| = 3$. What is the value of the integral if C is the lower half of the circle? (*M.T.U. 2013*)
[**Hint:** $z = 2 + 3e^{i\theta}$]

8. Prove that $\int_C \frac{1}{z}\, dz = -\pi i$ or πi according as C is the semi-circular arc $|\, z\, | = 1$ from $z = -1$ to $z = 1$ above or below the real axis.

9. Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\, dz$ along

(*a*) the straight line joining $(1 - i)$ to $(2 + i)$ (*b*) the curve $x = t + 1$, $y = 2t^2 - 1$.

10. Evaluate the line integral $\int_C (3y^2 dx + 2y\, dy)$ where C is the circle $x^2 + y^2 = 1$ counter clockwise from (1, 0) to (0, 1).

11. Evaluate the integral $I = \int_C \left(\frac{z}{\bar{z}}\right) dz$ where C is the boundary of the half annulus as given in figure 1.

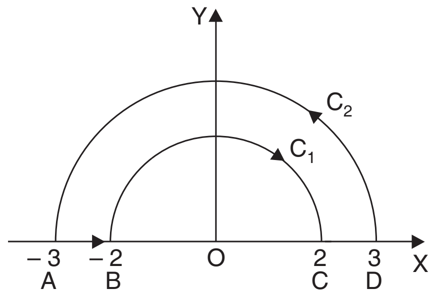


Fig. 1

12. Evaluate the integral $\int_C z^2 dz$ where C is the arc of the circle $|z| = 2$ from $\theta = 0$ to $\theta = \frac{\pi}{3}$.

13. Evaluate $\int_C \frac{2z + 3}{z}\, dz$ where C is

(*i*) upper half of the circle $|z| = 2$ in clockwise direction.
(*ii*) lower half of the circle $|z| = 2$ in anticlockwise direction.
(*iii*) the circle $|z| = 2$ in anticlockwise direction.

14. Evaluate the integral $\int_C \text{Re}\,(z^2)\, dz$ from 0 to $2 + 4i$ along the line segment joining the points (0, 0) and (2, 4).

15. Evaluate $\int_0^{3+i} (\bar{z})^2\, dz$ along the real axis from $z = 0$ to $z = 3$ and then along a line parallel to imaginary axis from $z = 3$ to $z = 3 + i$. (*G.B.T.U. 2013*)

16. Integrate $f(z) = \text{Re}\,(z)$ from $z = 0$ to $z = 1 + 2i$.
(*i*) along straight line joining $z = 0$ to $z = 1 + 2i$.
(*ii*) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + 2i$. (*U.P.T.U. 2014*)

Answers

1. (*a*) $6 + \frac{26}{3} i$	(*b*) $6 + \frac{26}{3} i$	(*c*) $6 + \frac{26}{3} i$	**2.** $\frac{3}{2} + \frac{i}{6}$
3. $10 - \frac{8}{3} i$	**4.** (*i*) $-1 + i$		
5. (*a*) (*i*) 256.5 (*ii*) 200.25		(*b*) (*i*) 26/3	(*ii*) 26/3 ; No
6. (*i*) $1 - i$	(*ii*) $\frac{1}{6}(64i - 103)$	**7.** (*i*) $\frac{2}{3}$; $-\frac{2}{3}$	(*ii*) 66, – 66
9. (*a*) $4 + 8i$	(*b*) $4 + \frac{25}{3} i$	**10.** – 1	**11.** $\frac{4}{3}$
12. $\frac{-16}{3}$	**13.** (*i*) $8 - 3\pi i$	(*ii*) $8 + 3\pi i$	(*iii*) $6\pi i$
14. $-8(1 + 2i)$	**15.** $12 + \frac{26}{3} i$	**16.** (*i*) $\frac{1 + 2i}{2}$	(*ii*) $\frac{1}{2} + 2i$.

1.20 SIMPLY AND MULTIPLY CONNECTED DOMAINS

A domain in which every closed curve can be shrunk to a point without passing out of the region is called a *simply connected domain.* If a domain is not simply connected, then it is called *multiply connected domain.*

1.21 SIMPLY AND MULTIPLY CONNECTED REGIONS

A curve is called *simple closed curve* if it does not cross itself (Fig. 1). A curve which crosses itself is called a *multiple curve* (Fig. 2).

A region is called *simply connected* if every closed curve in the region encloses points of the region only, *i.e.*, every closed curve lying in it can be contracted indefinitely without passing out of it. A region which is not simply connected is called a *multiply connected* region. In plain terms, a simply connected region is one which has no holes. Figure 3 shows a multiply connected region R enclosed between two separate curves C_1 and C_2. (There can be more than two separate curves). We can convert a multiply connected region into a simply connected one, by giving it one or more cuts (*e.g.* along the dotted line AB).

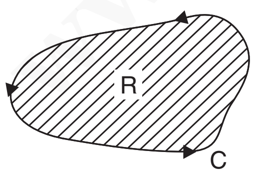


Fig. 1

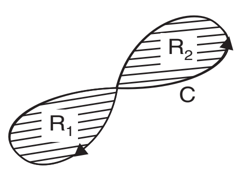


Fig. 2

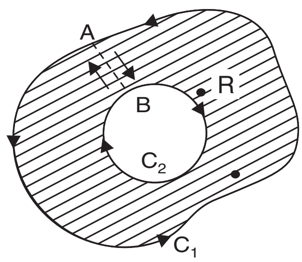


Fig. 3

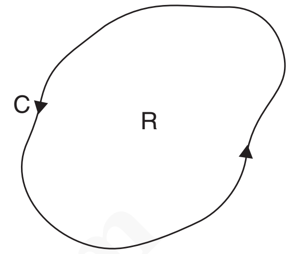
Remark 1. Jordan arc is a continuous arc without multiple points.

Remark 2. Contour is a Jordan curve consisting of continuous chain of a finite number of regular arcs.

1.22 CAUCHY'S INTEGRAL THEOREM [*M.T.U. 2013, 2014; G.B.T.U. (C.O.) 2011*]

Statement. *If f(z) is an analytic function and f′(z) is continuous at each point within and on a simple closed curve C, then*

$$\boxed{\oint_C f(z)\,dz = 0.}$$



Proof. Let R be the region bounded by the curve C.

Let $f(z) = u(x, y) + iv(x, y)$, then

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + idy)$$

$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \qquad \text{...(1)}$$

Since $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in R. Hence by Green's Theorem, we have

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \qquad \text{...(2)}$$

Now $f(z)$ being analytic at each point of the region R, by Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, the two double integrals in (2) vanish.

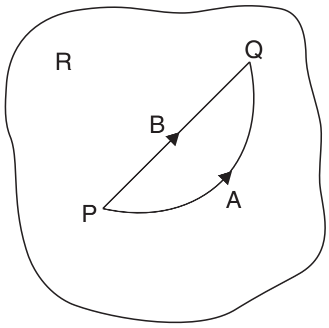
Hence $\oint_C f(z)\,dz = 0.$

However Cauchy with the help of Goursat developed the revised form of Cauchy's fundamental theorem which states that

"If $f(z)$ is analytic and one valued within and on a simple closed contour C then $\int_C f(z)\,dz = 0$."

Goursat showed that for the truth of the original theorem, the assumption of continuity of $f'(z)$ is unnecessary and Cauchy's theorem holds if $f(z)$ is analytic within and on C.

Corollary. *If f(z) is analytic in a region R and P, Q are two points in R, then* $\int_P^Q f(z)\,dz$ *is independent of the path joining P and Q and lying entirely in R.*



Let PAQ and PBQ be any two paths joining P and Q.

By Cauchy's theorem,

$$\int_{PAQBP} f(z)\,dz = 0$$

$$\Rightarrow \quad \int_{PAQ} f(z)\,dz + \int_{QBP} f(z)\,dz = 0$$

$$\Rightarrow \quad \int_{PAQ} f(z)\,dz - \int_{PBQ} f(z)\,dz = 0$$

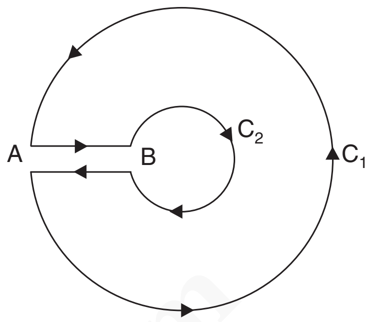
Hence $$\int_{PAQ} f(z)\,dz = \int_{PBQ} f(z)\,dz.$$

1.23 EXTENSION OF CAUCHY'S THEOREM TO MULTIPLY CONNECTED REGION

If f(z) is analytic in the region R between two simple closed curves C_1 and C_2, then

$$\boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz}$$

when integral along each curve is taken in anti-clockwise direction.



Proof. $\int f(z)\,dz = 0$

where the path of integration is along AB and curve C_2 in clockwise direction and along BA and along C_1 in anti-clockwise direction.

$$\int_{AB} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

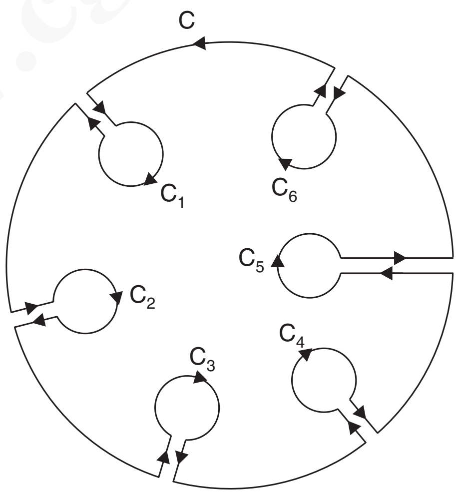
or $$\int_{C_2} f(z)\,dz + \int_{C_1} f(z)\,dz = 0 \qquad \Big|\because \int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz$$

Reversing the direction of the integral around C_2, we get

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

However if a closed curve C contains non-intersecting closed curves $C_1, C_2, \ldots, C_n$, then by introducing cross-cuts, it can be shown that

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots\cdots + \oint_{C_n} f(z)\,dz \cdot$$



EXAMPLES

Example 1. *Evaluate* $\oint_C (x^2 - y^2 + 2ixy)\,dz$, *where C is the contour* $|\,z\,| = 1$.

Sol. $f(z) = x^2 - y^2 + 2ixy = (x + iy)^2 = z^2$ is analytic everywhere within and on $|\,z\,| = 1$.

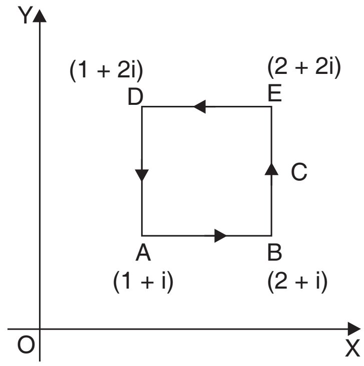
$\therefore$ By Cauchy's integral theorem, $\oint_C f(z)\,dz = 0$.

Example 2. *Evaluate* $\oint_C (3z^2 + 4z + 1)\,dz$ *where C is the arc of the cycloid* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ *between (0, 0) and* $(2\pi a, 0)$.

Sol. Here, $f(z) = 3z^2 + 4z + 1$ is analytic everywhere so that the integral is independent of the path of integration and depends only on the end points $z_1 = 0 + i0$ and $z_2 = 2\pi a + i0$.

$$\therefore \quad \int_C (3z^2 + 4z + 1)\,dz = \int_0^{2\pi a} (3z^2 + 4z + 1)\,dz = \left[z^3 + 2z^2 + z \right]_0^{2\pi a} = 2\pi a\,(4\pi^2 a^2 + 4\pi a + 1).$$

Example 3. *Evaluate:* $\oint_C \dfrac{2z^2 + 5}{(z + 2)^3\,(z^2 + 4)}\,dz$, *where C is the square with vertices at* $1 + i,\ 2 + i,\ 2 + 2i,\ 1 + 2i$.



Sol. Here, $$f(z) = \frac{2z^2 + 5}{(z + 2)^3\,(z^2 + 4)}$$

Singularities are given by

$(z + 2)^3\,(z^2 + 4) = 0$

$z = -\,2$ (order 3), $\pm\, 2i$ (simple poles)

Since the singularities donot lie inside the contour C hence by Cauchy's integral theorem,

$$\oint_C \frac{2z^2 + 5}{(z + 2)^3\,(z^2 + 4)}\,dz = 0.$$

Example 4. *Evaluate* $\oint_C (5z^4 - z^3 + 2)\,dz$ *around*

(*i*) *unit circle* $|z| = 1$

(*ii*) *square with vertices (0, 0), (1, 0), (1, 1), (1, 0)*

(*iii*) *curve consisting of the parabola* $y = x^2$ *from (0, 0) to (1, 1) and* $y^2 = x$ *from (1, 1) to (0, 0).*

Sol. $f(z) = 5z^4 - z^3 + 2$ is analytic everywhere. So by Cauchy integral theorem,

$$\oint_C f(z)\,dz = 0$$

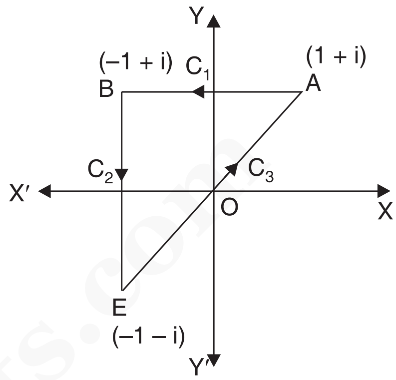
$\therefore$ For all given curves, $\oint_C (5z^4 - z^3 + 2)\,dz = 0.$

Example 5. *Verify Cauchy theorem by integrating* e^{iz} *along the boundary of the triangle with the vertices at the points* $1 + i$, $-\,1 + i$ *and* $-\,1 - i$. [G.B.T.U. 2012, 2013; M.T.U. 2012)

Sol. The boundary of triangle C consists of three lines C_1, C_2 and C_3. So,

$$I = \oint_C e^{iz}\, dz$$

$$= \int_{C_1} e^{iz}\, dz + \int_{C_2} e^{iz}\, dz + \int_{C_3} e^{iz}\, dz = I_1 + I_2 + I_3 \qquad \text{...(1)}$$



Along C_1: AB: $\quad y = 1,\ z = x + iy = x + i$

$\therefore \quad dz = dx$

$$I_1 = \int_{C_1} e^{iz}\, dz = \int_1^{-1} e^{i(x+i)}\, dx$$

$$= \int_1^{-1} e^{(ix-1)}\, dx$$

$$= \frac{1}{e}\left(\frac{e^{ix}}{i}\right)_1^{-1} = \frac{e^{-i-1} - e^{i-1}}{i}$$

Along C_2 : BE: $\quad x = -1$

$$z = x + iy = -1 + iy$$

$\therefore \quad dz = i\, dy$

$$I_2 = \int_{C_2} e^{iz}\, dz = \int_1^{-1} e^{i(-1+iy)}\, i\, dy$$

$$= ie^{-i}\int_1^{-1} e^{-y}\, dy = i\, e^{-i}\left(-e^{-y}\right)_1^{-1}$$

$$= -i\,(e^{-i+1} - e^{-1-i}) = \frac{1}{i}\left(e^{-i+1} - e^{-1-i}\right)$$

Along C_3 : EA: $\quad y = x,\ z = x + iy = (1+i)\,x$

$\therefore \quad dz = (1+i)\, dx$

$$I_3 = \int_{C_3} e^{iz}\, dz = \int_{-1}^{1} e^{i(1+i)x}\,(1+i)\, dx$$

$$= (1+i)\left[\frac{e^{i(1+i)x}}{i(1+i)}\right]_{-1}^{1} = \left(\frac{e^{i-1} - e^{-i+1}}{i}\right)$$

From (1), $\quad I = I_1 + I_2 + I_3 = \dfrac{1}{i}\left[e^{-i-1} - e^{i-1} + e^{-i+1} - e^{-1-i} + e^{i-1} - e^{-i+1}\right] = 0$

Hence Cauchy's theorem is verified.

Example 6. *Can the Cauchy-integral theorem be applied for evaluating the following integrals? Hence evaluate these integrals.*

(*i*) $\oint_C e^{\sin z^2}\, dz;\ C \equiv |z| = 1$ $\qquad$ (*ii*) $\oint_C \tan z\, dz;\ C \equiv |z| = 1$

(*iii*) $\oint_C \dfrac{e^z}{z^2+9}\, dz;\ C \equiv |z| = 2$

Sol. (*i*) Let $$I = \oint_C e^{\sin z^2}\, dz$$

The integrand $f(z) = e^{\sin z^2}$ is analytic for all z and $f'(z)$ is continuous inside C. Hence, Cauchy integral theorem can be applied.

$$\therefore \qquad I = 0$$

(*ii*) Let $$I = \oint_C \tan z\, dz$$

The integrand $f(z) = \tan z = \dfrac{\sin z}{\cos z}$ is analytic for all z except at the points $z = \pm\dfrac{\pi}{2}$, $\pm\dfrac{3\pi}{2}$, ... All these points lie outside C. Also $f'(z)$ is continuous inside C. Hence Cauchy integral theorem is applicable.

$$\therefore \qquad I = 0$$

(*iii*) Let $$I = \oint_C \frac{e^z}{z^2+9}\, dz$$

The integrand $f(z) = \dfrac{e^z}{z^2+9}$ is analytic everywhere except at the points $z = \pm 3i$. These points lie outside c and $f'(z)$ is continuous inside C. Hence Cauchy integral theorem is applicable and I = 0.

ASSIGNMENT

1. (*i*) State Cauchy-integral theorem for an analytic function. Verify this theorem by integrating the function $z^3 + iz$ along the boundary of the rectangle with vertices +1, –1, i, – i. (*U.P.T.U. 2015*)

(*ii*) Verify Cauchy's integral theorem for $f(z) = z^2$ taken over the boundary of a square with vertices *at* $\pm 1 \pm i$ in counter-clockwise direction.

2. Using Cauchy's integral theorem, evaluate $\oint_C f(z)\, dz$, where $f(z)$ is

(*i*) e^z (*ii*) $\sin z$ (*iii*) $\cos z$

(*iv*) z^n ; $n = 0, 1, 2, 3$, ... and C is any simple closed path.

3. Evaluate: (*i*) $\oint_C \dfrac{z^2 - z + 1}{z-2}\, dz$; $C \equiv |z-1| = \dfrac{1}{2}$ (*ii*) $\oint_C \dfrac{1}{z^2(z^2+9)}\, dz$; $C \equiv 1 < |z| < 2$

4. (*i*) Verify Cauchy's theorem for $f(z) = z^3$ taken over the boundary of the rectangle with vertices at – 1, 1, 1 + i, – 1 + i.

(*ii*) Verify Cauchy's theorem by integrating z^3 along the boundary of a square with vertices at 1 + i, 1 – i, –1 + i and –1 – i. (*U.P.T.U. 2014*)

5. Evaluate:

(*i*) $\oint_C \dfrac{e^{-z}}{z+1}\, dz$, where C is the circle $|z| = \dfrac{1}{2}$

(*ii*) $\oint_C \dfrac{z^2+5}{z-3}\, dz$, where C is the circle $|z| = 1$

(*iii*) $\oint_C \dfrac{3z^2+7z+1}{z+1}\, dz$, where C is the circle $|z+i| = 1$

6. Evaluate the following integrals:

(i) $\oint_C \frac{z^3 + z + 1}{z^2 - 3z + 2}\, dz$, where C is the ellipse $4x^2 + 9y^2 = 1$

(ii) $\oint_C \frac{z + 4}{z^2 + 2z + 5}\, dz$, where C is the circle $|z + 1| = 1$

(iii) $\oint_C \frac{z^2 - z + 1}{z - 1}\, dz$, where C is the circle $|z| = \frac{1}{2}$

7. Evaluate $I = \oint_C \frac{dz}{z - 2}$ around a triangle with vertices at (0, 0), (1, 0) and (0, 1).

8. State and prove Cauchy's integral theorem. Hence evaluate $\int_C \frac{z^2 + 5z + 6}{z - 2} dz$ where C: $|z| = \frac{3}{2}$
[*M.T.U. 2014, G.B.T.U. (C.O.) 2011*]

9. Evaluate $\oint_C \frac{e^{3iz}}{(z + \pi)^3}\, dz$, where C is the circle $|z - \pi| = 3.2$. (*U.P.T.U. 2007*)

10. (i) Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ along the perimeter of square with vertices $1 \pm i, -1 \pm i$. (*G.B.T.U. 2011*)

(ii) Verify Cauchy's theorem for the function $f(z) = 4z^2 + iz - 3$ along the positively oriented square with vertices (1, 0), (– 1, 0), (0, 1) and (0, – 1). (*M.T.U. 2012*)

(iii) Verify Cauchy's theorem for $f(z) = z^2 + 3z + 2$ where c is the perimeter of square with vertices $1 \pm i, - 1 \pm i$. (*G.B.T.U. 2012*)

Answers

2. 0 in all cases	**3.** (i) 0	(ii) 0
5. (i) 0	(ii) 0	(iii) 0
6. (i) 0	(ii) 0	(iii) 0
7. 0	**8.** 0	**9.** 0.

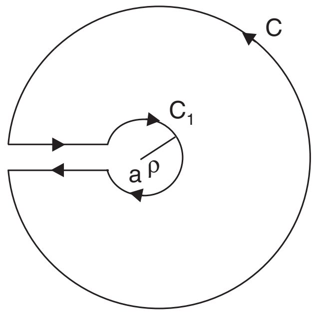
1.24 CAUCHY'S INTEGRAL FORMULA

(*M.T.U. 2012, U.P.T.U. 2006, 2007, 2009, 2014; G.B.T.U. 2011, 2013*)

Statement. *If f(z) is analytic within and on a closed curve C and a is any point within C, then*

$$\boxed{f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a}\, dz.}$$

Proof. Consider the function $\frac{f(z)}{z - a}$, which is analytic at every point within C except at $z = a$. Draw a circle C_1 with a as centre and radius ρ such that C_1 lies entirely inside C. Thus $\frac{f(z)}{z - a}$ is analytic in the region between C and C_1.



$\therefore$ By Cauchy's theorem, we have

$$\oint_C \frac{f(z)}{z - a}\, dz = \oint_{C_1} \frac{f(z)}{z - a}\, dz \qquad ...(1)$$

Now, the equation of circle C_1 is

$$|z-a| = \rho \quad \text{or} \quad z-a=\rho e^{i\theta}$$

so that $$dz = i\rho e^{i\theta}\, d\theta$$

$$\therefore \qquad \oint_{C_1} \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}} . i\rho e^{i\theta}\, d\theta = i\int_0^{2\pi} f(a+\rho e^{i\theta})\, d\theta$$

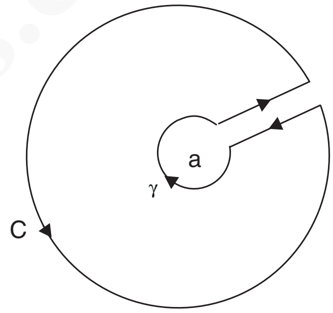
Hence by (1), we have $\oint_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a+\rho e^{i\theta})\, d\theta$ ...(2)

In the limiting form, as the circle C_1 shrinks to the point a, *i.e.*, $\rho \to 0$, then from (2),

$$\oint_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a)\, d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a)$$

Hence $$\boxed{f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\, dz}$$

Aliter: About the point $z = a$, describe a small circle γ of radius r lying entirely within C. Consider the function $\frac{f(z)}{z-a}$.



This function is analytic in the region between C and γ. Hence by Cauchy's theorem for multiply connected region, we have

$$\int_C \frac{f(z)}{z-a}\, dz = \int_\gamma \frac{f(z)}{z-a}\, dz$$

$$\Rightarrow \quad \int_C \frac{f(z)}{z-a}\, dz - \int_\gamma \frac{f(a)}{z-a}\, dz = \int_\gamma \frac{f(z)-f(a)}{z-a}\, dz$$

$$\Rightarrow \quad \int_C \frac{f(z)}{z-a}\, dz - f(a)\int_\gamma \frac{dz}{z-a} = \int_\gamma \frac{f(z)-f(a)}{z-a}\, dz$$

$$\Rightarrow \quad \int_C \frac{f(z)}{z-a}\, dz - 2\pi i f(a) = \int_\gamma \frac{f(z)-f(a)}{z-a}\, dz \qquad \left|\; \because \int_\gamma \frac{dz}{z-a} = 2\pi i \text{ since } |z-a| = r \text{ on } \gamma\right.$$

$$\Rightarrow \quad \left| \int_C \frac{f(z)}{z-a}\, dz - 2\pi i f(a)\right| = \left|\int_\gamma \frac{f(z)-f(a)}{z-a}\, dz\right|$$

$$\leq \int_\gamma \frac{|f(z)-f(a)|}{|z-a|}|dz|$$

$$\leq \frac{\varepsilon}{r}\int_\gamma |dz| \qquad \left|\; \begin{array}{l} \because f(z) \text{ is continuous at } z=a \\ \therefore |f(z)-f(a)| < \varepsilon \\ \text{and } |z-a| = r \text{ for } z \text{ on } \gamma \end{array}\right.$$

$$\leq \frac{\varepsilon}{r} . 2\pi r$$

$$\leq 2\pi\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

$$\Rightarrow \quad \int_C \frac{f(z)}{z-a}\, dz - 2\pi i f(a) = 0$$

$$\Rightarrow \quad \boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\, dz}$$

1.25 CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION [U.P.T.U. (C.O.) 2009, 2010]

If a function f(z) is analytic in a region D, then its derivative at any point z = a of D is also analytic in D and is given by

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

where C is any closed contour in D surrounding the point z = a.

Proof. Let $a + h$ be a point in the neighbourhood of the point a. Then by Cauchy's Integral Formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a-h}\,dz$$

$$\therefore \quad f(a+h) - f(a) = \frac{1}{2\pi i}\int_C \left\{\frac{1}{z-a-h} - \frac{1}{z-a}\right\} f(z)\,dz = \frac{h}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a-h)(z-a)}$$

$$\Rightarrow \quad \frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a-h)(z-a)}$$

Take limit as $h \to 0$

$$\underset{h\to 0}{\text{Lt}}\ \frac{f(a+h)-f(a)}{h} = \underset{h\to 0}{\text{Lt}}\ \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a-h)(z-a)}$$

$$\Rightarrow \quad \boxed{f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz} \qquad \text{...(1)}$$

Since a is any point of the region D, so by (1) it is clear that $f'(a)$ is analytic in D. Thus, *the derivative of an analytic function is also analytic.*

1.26 THEOREM

If a function f(z) is analytic in a domain D, then at any point z = a of D, f(z) has derivatives of all orders, all of which are again analytic functions in D, their values are given by

$$\boxed{f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

where C is any closed contour in D surrounding the point z = a.

Proof. We shall prove this theorem by Mathematical Induction.

Let the theorem be true for $n = m$. Then

$$f^m(a) = \frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+1}}\,dz \text{ is true.}$$

$$\Rightarrow \quad \frac{f^m(a+h) - f^m(a)}{h} = \frac{m!}{2\pi i}\,\frac{1}{h}\left[\int_C \frac{f(z)\,dz}{(z-a-h)^{m+1}} - \int_C \frac{f(z)\,dz}{(z-a)^{m+1}}\right\}$$

$$= \frac{m!}{2\pi i} \cdot \frac{1}{h} \cdot \int_C \frac{1}{(z-a)^{m+1}} \left\{ \left(1 - \frac{h}{z-a}\right)^{-(m+1)} - 1 \right\} f(z)\, dz$$

$$= \frac{m!}{2\pi i} \cdot \frac{1}{h} \cdot \int_C \frac{1}{(z-a)^{m+1}} \left\{ (m+1) \frac{h}{z-a} + \frac{(m+1)(m+2)}{2!} \frac{h^2}{(z-a)^2} + \cdots \right\} f(z)\, dz$$

Take limit as $h \to 0$

$$\lim_{h \to 0} \frac{f^m(a+h) - f^m(a)}{h} = \frac{(m+1)!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{m+2}}\, dz$$

$$\Rightarrow \qquad f^{m+1}(a) = \frac{(m+1)!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{m+2}}\, dz$$

Hence the theorem is true for $n = m + 1$ if the theorem is true for $n = m$. But we know by Cauchy's Integral formula for the derivative of a function that the theorem is true for $n = 1$. Hence the theorem must be true for $n = 2, 3, 4,$ ...... and so on *i.e.,* for all +ve integral values of n. Thus,

$$\boxed{f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz} \qquad ...(1)$$

Since a is any point of the region D, so by (1) it is clear that $f^n(a)$ is analytic in D. Thus the derivatives of $f(z)$ of all orders are analytic if $f(z)$ is analytic.

Thus, *if a function of a complex variable has a first derivative in a simply connected region, all its higher derivatives exist in that region. This property is not exhibited by the functions of real variables.*

1.27 CAUCHY'S INEQUALITY

If $f(z)$ is analytic within a circle C given by $|z - a| = R$ and if $|f(z)| \le M$ on C, then

$$|f^n(a)| \le \frac{Mn!}{R^n}.$$

Proof.
$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$\Rightarrow \qquad |f^n(a)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{n+1}} \right|$$

$$\le \frac{n!}{|2\pi i|} \int_C \frac{|f(z)|\,|dz|}{|(z-a)^{n+1}|}$$

$$\le \frac{n!}{2\pi} \frac{M}{R^{n+1}} \int_0^{2\pi} R\, d\theta$$

$$\le \frac{n!}{2\pi} \frac{M}{R^{n+1}}\, 2\pi R \le \frac{Mn!}{R^n}.$$

$\because \quad z - a = R\, e^{i\theta}$

$\therefore \quad dz = i\, R\, e^{i\theta}\, d\theta$

$\therefore \quad |dz| = |i\, R\, e^{i\theta}\, d\theta| = R\, d\theta$

EXAMPLES

Example 1. *Evaluate* $\oint_C \frac{e^{-z}}{z+1} dz$, *where C is the circle* $|z| = 2$

Sol. $f(z) = e^{-z}$ is an analytic function.

The point $a = -1$ lies inside the circle $|z| = 2$.

$\therefore$ By Cauchy's integral formula,

$$\oint_C \frac{e^{-z}}{z+1} dz = 2\pi i (e^{-z})_{z=-1} = 2\pi i e.$$

Example 2. *Evaluate the following integral:*

$$\int_C \frac{1}{z} \cos z \, dz$$

where C is the ellipse $9x^2 + 4y^2 = 1$.

Sol. Pole is given by $z = 0$. The given ellipse encloses the simple pole.

$\therefore$ By Cauchy's integral formula,

$$\int_C \frac{\cos z}{z} dz = 2\pi i (\cos z)_{z=0} = 2\pi i.$$

Example 3. (*i*) *Use Cauchy Integral formula to evaluate*

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

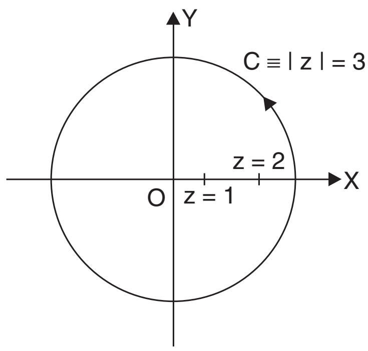
where C is the circle $|z| = 3$. [G.B.T.U. 2010; G.B.T.U. (C.O.) 2011]

(*ii*) *Evaluate:* $\int_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz$, *where C is the circle* $|z| = 4$. (U.P.T.U. 2008)

Sol. (*i*) The integrand has singularities given by

$$(z-1)(z-2) = 0 \quad \Rightarrow \quad z = 1, 2$$

The given circle $|z| = 3$ with centre at $z = 0$ and radius 3 encloses both the singularities.



$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_{C_1} \frac{\left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right)}{z-1} dz + \int_{C_2} \frac{\left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right)}{z-2} dz$$

$$= 2\pi i \left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right]_{z=1} + 2\pi i \left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right]_{z=2}$$

$$= 2\pi i \left(\frac{0-1}{-1}\right) + 2\pi i \left(\frac{0+1}{1}\right) = 2\pi i + 2\pi i = 4\pi i.$$

(*ii*) Singularities are given by

$$(z-1)(z-2)=0 \quad \Rightarrow \quad z = 1, 2$$

The given circle $|z| = 4$ with centre at $z = 0$ and radius 4 encloses both the singularities.

$$\therefore \quad \int_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)}\, dz = \int_{C_1} \frac{\left(\dfrac{\sin \pi z + \cos \pi z}{z-2}\right)}{z-1}\, dz + \int_{C_2} \frac{\left(\dfrac{\sin \pi z + \cos \pi z}{z-1}\right)}{z-2}\, dz$$

$$= 2\pi i \left[\frac{\sin \pi z + \cos \pi z}{z-2}\right]_{z=1} + 2\pi i \left[\frac{\sin \pi z + \cos \pi z}{z-1}\right]_{z=2}$$

$$= 2\pi i \left[\frac{-1}{-1}\right] + 2\pi i \left[\frac{1}{1}\right] = 2\pi i + 2\pi i = 4\pi i$$

Example 4. (*i*) *Evaluate the following integral using Cauchy Integral formula*

$$\int_C \frac{4-3z}{z(z-1)(z-2)} dz, \quad \textit{where C is the circle } |z| = 3/2.$$

(U.P.T.U. 2015)

(*ii*) *Use Cauchy-integral formula to evaluate*

$$\int_C \frac{z}{z^2-3z+2}\, dz, \quad \textit{where C is the circle } |z-2| = \frac{1}{2}. \qquad \text{(U.P.T.U. 2009)}$$

Sol. (*i*) Poles of the integrand are $z = 0, 1, 2$. These are simple poles.

Given circle $|z| = \frac{3}{2}$ with centre at $z = 0$ and radius $\frac{3}{2}$ encloses two poles $z = 0$ and $z = 1$.

$$\therefore \quad \int_C \frac{4-3z}{z(z-1)(z-2)}\, dz = \int_{C_1} \frac{\dfrac{4-3z}{(z-1)(z-2)}}{z}\, dz + \int_{C_2} \frac{\dfrac{4-3z}{z(z-2)}}{(z-1)}\, dz$$

$$= 2\pi i \left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i \left[\frac{4-3z}{z(z-2)}\right]_{z=1} = 2\pi i.$$

(*ii*) Poles of the integrand are given by

$$z^2 - 3z + 2 = 0 \quad \Rightarrow \quad z = 1, 2$$

Both are simple poles. The given circle $|z-2| = \frac{1}{2}$ with centre at $z = 2$ and radius $\frac{1}{2}$ encloses only one of the poles at $z = 2$.

$\therefore$ By Cauchy's integral formula,

$$\int_C \frac{z}{z^2-3z+2}\, dz = \int_C \frac{\left(\dfrac{z}{z-1}\right)}{z-2}\, dz = 2\pi i \left[\frac{z}{z-1}\right]_{z=2} = 2\pi i \left(\frac{2}{1}\right) = 4\pi i$$

Example 5. *Evaluate by Cauchy's integral formula*

$$\int_C \frac{dz}{z(z+\pi i)}, \quad \textit{where C is } |z+3i| = 1$$

Sol. Poles of the integrand are $z = 0, -\pi i$ (simple poles)

The given curve C is a circle with centre at $z = -3i$, *i.e.,* at (0, – 3) and radius 1.

Clearly, only the pole $z = -\pi i$ lies inside the circle.

$$\therefore \quad \int_C \frac{dz}{z(z+\pi i)} = \int_C \frac{\left(\frac{1}{z}\right)}{z+\pi i}\, dz$$

$$= 2\pi i \left(\frac{1}{z}\right)_{z=-\pi i} \qquad \text{| By Cauchy's Integral formula}$$

$$= \frac{2\pi i}{-\pi i} = -2$$

Example 6. *Evaluate* $\oint_C \frac{z^2+1}{z^2-1}\, dz$ *where C is circle,*

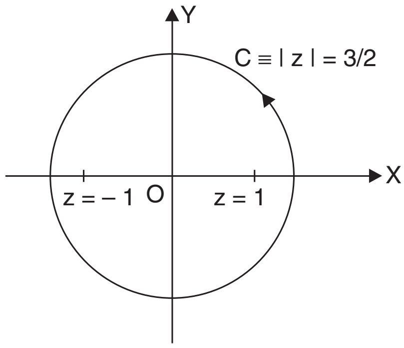
(*i*) $|\, z \,| = 3/2$ (*ii*) $|\, z - 1 \,| = 1$ (U.P.T.U. 2014) (*iii*) $|\, z \,| = 1/2.$ (U.P.T.U. 2014)

Sol. The integrand has singularities given by

$$z^2 - 1 = 0 \quad \Rightarrow \quad z = \pm 1$$

(*i*) The given curve *C* is a circle with centre at origin (0, 0) and radius 3/2.

Both the singularities $z = 1$ and $z = -1$ lie inside the circle $|\, z \,| = 3/2$.



$$\therefore \quad \oint_C \frac{z^2+1}{z^2-1}\, dz = \oint_{C_1} \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\, dz + \oint_{C_2} \frac{\left(\frac{z^2+1}{z-1}\right)}{z+1}\, dz$$

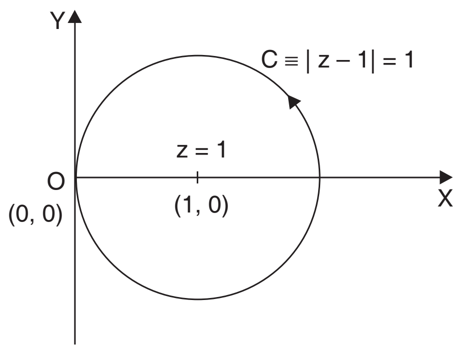
$$= 2\pi i \left(\frac{z^2+1}{z+1}\right)_{z=1} + 2\pi i \left(\frac{z^2+1}{z-1}\right)_{z=-1}$$

| By Cauchy's Integral formula

$$= 2\pi i\,(1) + 2\pi i\,(-1) = 0$$

(*ii*) The given curve C is a circle with centre at (1, 0) and radius 1.

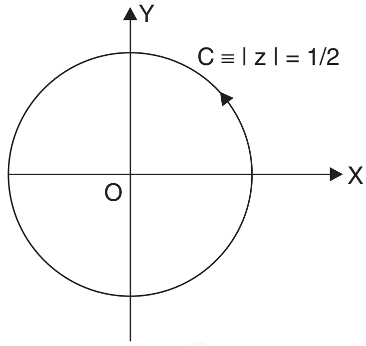
Only the singularity $z = 1$ lie inside the given circle $|\, z - 1 \,| = 1$.



$$\therefore \quad \oint_C \frac{z^2+1}{z^2-1}\, dz = \oint_C \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\, dz$$

$$= 2\pi i \left(\frac{z^2+1}{z+1}\right)_{z=1} = 2\pi i \qquad \text{| By Cauchy's Integral formula}$$

(*iii*) The given curve C is a circle with centre at origin (0, 0) and radius $\frac{1}{2}$. Clearly both the singularities $z = 1$ and $z = -1$ lie outside the given circle $|z| = \frac{1}{2}$.



Hence, by Cauchy's Integral theorem

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 0.$$

Example 7.(*i*) *Use Cauchy's integral formula to show that*

$$\int_C \frac{e^{zt}}{z^2+1}\,dz = 2\pi i \sin t \text{ if } t > 0 \text{ and C is the circle } |z| = 3. \qquad \text{(U.P.T.U. 2009)}$$

(*ii*) *Evaluate the following complex integration using Cauchy's integral formula*

$$\int_C \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz \quad \text{where C is the circle } |z| = 2$$

Sol. (*i*) Singularities of the integrand are given by

$$z^2 + 1 = 0 \quad \Rightarrow \quad z = \pm i \quad \text{(order 1)}$$

The circle $|z| = 3$ has centre at $z = 0$ and radius 3. It encloses both the singularities $z = i$ and $z = -i$.

Now,
$$\int_C \frac{e^{zt}}{z^2+1}\,dz = \int_C \frac{e^{zt}}{(z-i)(z+i)}\,dz = \int_{C_1} \frac{\left(\frac{e^{zt}}{z+i}\right)}{z-i}\,dz + \int_{C_2} \frac{\left(\frac{e^{zt}}{z-i}\right)}{z+i}\,dz$$

$$= 2\pi i\left(\frac{e^{zt}}{z+i}\right)_{z=i} + 2\pi i\left(\frac{e^{zt}}{z-i}\right)_{z=-i} = \pi\,(e^{it} - e^{-it}) = 2\pi i \sin t$$

(*ii*) Poles of the integrand are given by

$$(z^2 - 1)(z + 3) = 0 \quad \Rightarrow \quad z = 1, -1, -3 \text{ (simple poles)}$$

The circle $|z| = 2$ has centre at $z = 0$ and radius 2. clearly the poles $z = 1$ and $z = -1$ lie inside the given circle while the pole $z = -3$ lie outside it.

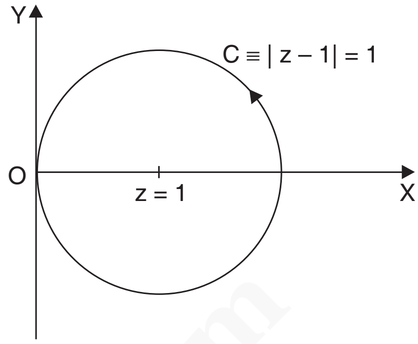
$$\therefore \quad \int_C \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz = \int_{C_1} \frac{\left\{\frac{3z^2+z+1}{(z+1)(z+3)}\right\}}{z-1}\,dz + \int_{C_2} \frac{\left\{\frac{3z^2+z+1}{(z-1)(z+3)}\right\}}{z+1}\,dz$$

$$= 2\pi i\left[\frac{3z^2+z+1}{(z+1)(z+3)}\right]_{z=1} + 2\pi i\left[\frac{3z^2+z+1}{(z-1)(z+3)}\right]_{z=-1}$$

| Using Cauchy's Integral formula

$$= 2\pi i\left(\frac{5}{8}\right) + 2\pi i\left(-\frac{3}{4}\right) = 2\pi i\left(\frac{-1}{8}\right) = -\frac{\pi i}{4}$$

Example 8. *Integrate $(z^3 - 1)^{-2}$ the counterclockwise sense around the circle $|z - 1| = 1$*



Sol. Singularities of integrand are given by

$$(z^3 - 1)^2 = 0$$

$$\Rightarrow \quad (z - 1)^2 (z^2 + z + 1)^2 = 0$$

$$\Rightarrow \quad z = 1, \frac{-1 \pm i\sqrt{3}}{2}$$

Singularities are of second order.

The circle $|z - 1| = 1$ has centre at $z = 1$ and radius 1. Clearly, only $z = 1$ lies inside the circle $|z - 1| = 1$

Now,
$$\int_C \frac{dz}{(z^3 - 1)^2} = \int_C \frac{\left\{\dfrac{1}{(z^2 + z + 1)^2}\right\}}{(z - 1)^2} dz$$

$$= \frac{2\pi i}{1!}\left[\frac{d}{dz}\left\{\frac{1}{(z^2 + z + 1)^2}\right\}\right]_{z = 1}$$ | Using Cauchy's Integral formula for derivatives

$$= 2\pi i \left[\frac{-2(2z + 1)}{(z^2 + z + 1)^3}\right]_{z = 1} = -4\pi i\left(\frac{3}{27}\right) = -\frac{4\pi i}{9}$$

Example 9. *Evaluate:* $\int_C \frac{e^z}{(z^2 + \pi^2)^2} dz$, *where C is* $|z| = 4$. (U.P.T.U. 2008)

Sol. Singularities of the integrand are given by

$$(z^2 + \pi^2)^2 = 0 \quad \Rightarrow \quad z = \pm \pi i \qquad \text{(order 2)}$$

The given curve C is a circle with centre at origin and radius 4. The circle encloses both the singularities.

$$\therefore \quad \oint_C \frac{e^z}{(z^2 + \pi^2)^2} dz = \oint_{C_1} \frac{\left\{\dfrac{e^z}{(z + \pi i)^2}\right\}}{(z - \pi i)^2} dz + \int_{C_2} \frac{\left\{\dfrac{e^z}{(z - \pi i)^2}\right\}}{(z + \pi i)^2} dz$$

$$= 2\pi i\left[\frac{d}{dz}\left\{\frac{e^z}{(z + \pi i)^2}\right\}\right]_{z = \pi i} + 2\pi i\left[\frac{d}{dz}\left\{\frac{e^z}{(z - \pi i)^2}\right\}\right]_{z = -\pi i}$$

| By C–I formula for derivatives

$$= 2\pi i\left[\frac{e^z (z + \pi i - 2)}{(z + \pi i)^3}\right]_{z = \pi i} + 2\pi i\left[\frac{e^z (z - \pi i - 2)}{(z - \pi i)^3}\right]_{z = -\pi i}$$

$$= \left(\frac{\pi i - 1}{2\pi^2}\right) + \left(\frac{\pi i + 1}{2\pi^2}\right) = \frac{i}{\pi}.$$

Example 10. *Use Cauchy's integral formula to evaluate*

$$\oint_C \frac{e^{2z}}{(z+1)^4}\, dz \text{ where } C \text{ is the circle } |z| = 2. \qquad \text{[U.K.T.U. 2011]}$$

Sol. The integrand has a singularity at $z = -1$ which lies within the circle $|z| = 2$.

$$\oint_C \frac{e^{2z}}{(z+1)^4}\, dz = \frac{2\pi i}{3!}\left\{\frac{d^3}{dz^3}(e^{2z})\right\}_{z=-1} = \frac{\pi i}{3}\,(8e^{2z})_{z=-1} = \frac{8\pi i}{3e^2}.$$

Example 11. *Evaluate* $\int_C \frac{z}{z^2+1}\, dz,$ *where*

(*i*) $C \equiv \left|z + \frac{1}{z}\right| = 2$ (*ii*) $C \equiv |z + i| = 1.$

Sol. Poles of the integrand are given by

$$z^2 + 1 = 0 \quad \Rightarrow \quad z = \pm i$$

Integrand has two simple poles $z = i$ and $z = -i$

(*i*) The given curve is

$$\left|z + \frac{1}{z}\right| = 2$$

$$\Rightarrow \quad \left|x + iy + \frac{1}{x+iy}\right| = 2$$

$$\Rightarrow \quad \left|\frac{x^2 - y^2 + 2ixy + 1}{x+iy}\right| = 2$$

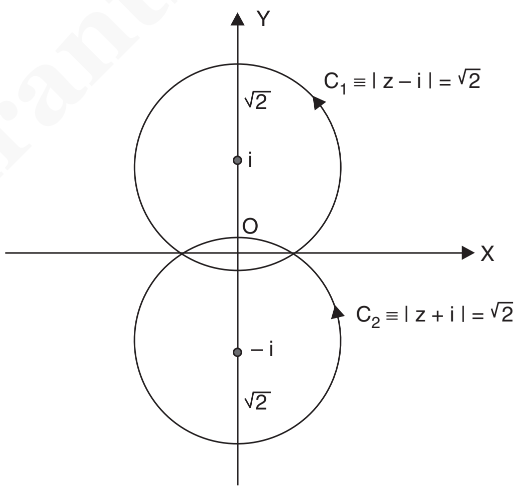
$$\Rightarrow \quad (x^2 - y^2 + 1)^2 + 4x^2y^2 = 4x^2 + 4y^2$$

$$\Rightarrow \quad x^4 + y^4 - 2x^2y^2 + 1 + 2x^2 - 2y^2 + 4x^2y^2 = 4x^2 + 4y^2$$

$$\Rightarrow \quad (x^2 + y^2)^2 - 2(x^2 + y^2) + 1 = 4y^2$$

$$\Rightarrow \quad x^2 + y^2 - 1 = \pm 2y$$

$$\Rightarrow \quad x^2 + (y \pm 1)^2 = 2$$



Above eqn. represents two circles with centres (0, 1), (0, – 1) and radius $\sqrt{2}$.

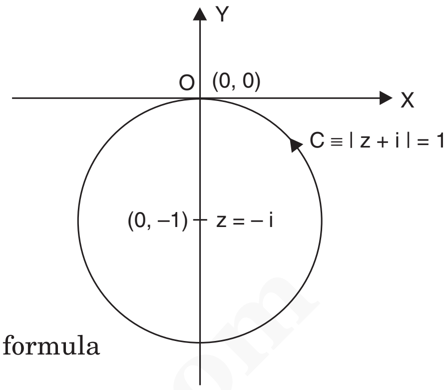
$$\int_C \frac{z}{z^2+1}\, dz = \int_{C_1} \frac{z}{z^2+1}\, dz + \int_{C_2} \frac{z}{z^2+1}\, dz$$

$$= \int_{C_1} \frac{\left(\frac{z}{z+i}\right)}{z-i}\, dz + \int_{C_2} \frac{\left(\frac{z}{z-i}\right)}{z+i}\, dz$$

$$= 2\pi i\left(\frac{z}{z+i}\right)_{z=i} + 2\pi i\left(\frac{z}{z+i}\right)_{z=-i}$$

$$= 2\pi i \left(\frac{1}{2}\right) + 2\pi i \left(\frac{1}{2}\right) = 2\pi i.$$

(*ii*) The given curve $| z + i | = 1$ is a circle with centre at $z = -i$ and radius 1. Clearly only the pole $z = -i$ lies inside the circle $| z + i | = 1$



$$\int_C \frac{z}{z^2+1}\,dz = \int_C \frac{\left(\dfrac{z}{z-i}\right)}{z+i}\,dz$$

$$= 2\pi i \left(\frac{z}{z-i}\right)_{z=-i}$$

$= \pi i$ | By Cauchy Integral formula

Example 12. *Evaluate by using Cauchy Integral formula*

$$\int_C \frac{z-1}{(z+1)^2\,(z-2)}\,dz, \quad \text{where C is } |z - i| = 2.$$

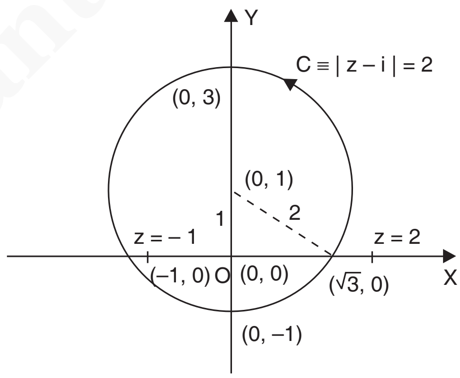
Sol. Poles of the integrand are given by

$$(z + 1)^2\,(z - 2) = 0$$

$$\Rightarrow \qquad z = -1, 2$$

$z = -1$ is a double pole while $z = 2$ is a simple pole.

The given curve C is a circle with centre at (0, 1) and radius 2. Clearly, the pole $z = -1$ lies inside the given circle while the pole $z = 2$ lies outside it.



Hence,

$$\oint_C \frac{z-1}{(z+1)^2\,(z-2)}\,dz = \oint_C \frac{\left(\dfrac{z-1}{z-2}\right)}{(z+1)^2}\,dz$$

$$= \frac{2\pi i}{1!}\left\{\frac{d}{dz}\left(\frac{z-1}{z-2}\right)\right\}_{z=-1}$$

$$= 2\pi i \left\{\frac{-1}{(z-2)^2}\right\}_{z=-1} = -\frac{2\pi i}{9}.$$

ASSIGNMENT

1. Evaluate $\oint_C \dfrac{z^2+5}{z-3}\,dz$, where C is the circle $| z | = 4$.

2. Evaluate $\int_C \dfrac{e^z}{z^2+1}\,dz$ over the circular path $| z | = 2$.

3. Evaluate $\oint_C \dfrac{3z^2+7z+1}{z+1}\,dz$, where C is the circle $| z | = 1.5$.

4. Evaluate $\oint_C \frac{\cos z}{z - \pi}\, dz$, where C is the circle $|z - 1| = 3$.

5. Evaluate the complex integration

(*i*) $\int_C \left\{\frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)}\right\} dz$ where C is the circle $|z| = 3$.

(*ii*) $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-3)}\, dz$ where C: $|z| = 2$ (*M.T.U. 2013*)

6. (*i*) Evaluate $\oint_C \frac{z\, dz}{(z-1)(z-3)}$, where C is the circle

(*a*) $|z| = 3$ (*b*) $|z| = 3/2$.

(*ii*) Evaluate $\oint_C \frac{e^z}{(z-1)(z-4)}\, dz$, where C is the circle $|z| = 2$.

(*iii*) Evaluate using Cauchy's integral formula:

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}\, dz \quad \text{where C is the circle } |z| = 3.$$

(*iv*) State Cauchy's integral formula. Hence evaluate: (*G.B.T.U. 2011, 2012*)

$$\int_C \frac{\exp(i\pi z)}{(2z^2 - 5z + 2)}\, dz$$

where C is the unit circle with centre at origin and having positive orientation.

7. (*i*) Evaluate $\oint_C \frac{e^z}{z(z+1)}\, dz$, where C is the circle $|z| = \frac{1}{4}$.

(*ii*) Using Cauchy Integral formula, evaluate $\int_C \frac{dz}{z^2 - 1}$ where C ≡ $|z| = 2$.

(*iii*) Evaluate $\int_C \frac{2z+1}{z^2 + z}\, dz$ where C is $|z| = \frac{1}{2}$.

8. Evaluate $\oint_C \frac{\cos \pi z}{z^2 - 1}\, dz$ around a rectangle with vertices

(*a*) $2 \pm i, -2 \pm i$ (*b*) $-i, 2 - i, 2 + i, i$.

9. Integrate $\frac{e^z}{z^2 + 1}$ around the contour C, where C is

(*i*) $|z - i| = 1$ (*ii*) $|z + i| = 1$

10. Show that $\oint_C \frac{e^z}{z}\, dz = 2\pi i$, C ≡ $|z| = 1$. Hence show that

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = 2\pi \quad \text{and} \quad \int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta = 0$$

11. Evaluate $\oint_C \frac{dz}{z^2 + 9}$, where C is

(*i*) $|z - 3i| = 4$ (*ii*) $|z + 3i| = 2$ (*iii*) $|z| = 5$

12. Evaluate:

(*i*) $\int_C \frac{z+4}{z^2+2z+5}\,dz$; $C \equiv |\, z+1-i \,| = 2$ (*ii*) $\int_C \frac{z^3-6}{2z-i}\,dz$; $C \equiv |\, z \,| = 1$

(*iii*) $\int_C \frac{\tan z}{z^2-1}\,dz$; $C \equiv |\, z \,| = 3/2$ (*iv*) $\int_C \frac{2z^2+z}{z^2-1}\,dz$; $C \equiv |\, z-1 \,| = 1$.

13. Evaluate by Cauchy-Integral formula: $\oint_C \frac{z^2+1}{z^2-1}\,dz$, where C is

(*i*) $|\, z-1 \,| = 1$ (*ii*) $|\, z+1 \,| = 1$ (*iii*) $|\, z-i \,| = 1$.

14. Evaluate the following integrals:

(*i*) $\oint_C \frac{\cos 2\pi z}{(2z-1)(z-3)}\,dz$; $C \equiv |\, z \,| = 1$ (*ii*) $\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\,dz$; $C \equiv |\, z \,| = 2$

(*iii*) $\oint_C \frac{\cosh z}{z^4}\,dz$; $C \equiv |\, z \,| = 1/2$

15. Evaluate $\oint_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}\,dz$, where C is the circle $|\, z \,| = 1$.

16. (*i*) Evaluate $\oint_C \frac{e^{-2z}\,dz}{(z+1)^3}$, where C is the circle $|\, z \,| = 2$.

(*ii*) Evaluate the integral $\int \frac{e^{2z}}{(z+1)^5}\,dz$ around the boundary of the circle $|z| = 2$.

(*U.P.T.U. 2015*)

17. (*i*) Evaluate: $\oint_C \frac{e^{3z}}{(z-\log 2)^4}\,dz$, where C is the square with vertices at $\pm 1 \pm i$.

(*ii*) Evaluate: $\oint_C \frac{dz}{z^2(z^2-4)\,e^z}$, where $C \equiv |\, z \,| = 1$. (*G.B.T.U. 2013*)

18. Evaluate $\oint_C \frac{e^z}{z(1-z)^3}\,dz$, where C is

(*i*) $|\, z \,| = \frac{1}{2}$ (*ii*) $|\, z-1 \,| = \frac{1}{2}$ (*iii*) $|\, z \,| = 2$

19. Integrate $\frac{\sin 2z}{(z+3)(z+1)^2}$ around the contour C, where C is a rectangle with vertices at $3 \pm i, -2 \pm i$.

20. Evaluate: $\oint_C \frac{z^3-z}{(z-2)^3}\,dz$, where C is

(*i*) $|\, z \,| = 3$ (*ii*) $|\, z-2 \,| = 1$ (*iii*) $|\, z \,| = 1$

21. Using Cauchy-integral formula, evaluate:

(*i*) $\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz$; $C \equiv |\, z \,| = 5$ (*ii*) $\oint_C \frac{e^z}{z^3}\,dz$; $C \equiv |\, z \,| = 1$

(*iii*) $\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz$; $C \equiv |\, z-1 \,| = \frac{1}{2}$

(*iv*) $\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz$; $C \equiv |\, z \,| = 3$, $t > 0$.

22. Evaluate $\int_C \frac{\sin z}{z^2 - iz + 2}\, dz$, where C is

(*i*) $|\, z + 2 \,| = 2$

(*ii*) A rectangle with vertices at (1, 0), (1, 3), (– 1, 3) and (– 1, 0)

(*iii*) A rectangle with vertices at (2, 0), (2, 3), (– 2, 3) and (– 2, – 3).

23. Evaluate the integrals

(*i*) $\oint_C \frac{e^z + \sin \pi z}{(z-1)(z-3)^2 (z+4)}\, dz$, $\quad C \equiv |\, z \,| = 2$ $\qquad$ (*ii*) $\oint_C \frac{z+1}{z^2 - 9}\, dz$; $\quad C \equiv |\, z + 3 \,| = 1$

24. Show that

(*i*) $\oint_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}$; $C \equiv |\, z - i \,| = 2$

(*ii*) $\oint_C \frac{e^z}{z^2 (z+1)^3}\, dz = \left(\frac{11}{e} - 4\right)\pi i$; $C \equiv |\, z \,| = 2$

(*iii*) $\oint_C \frac{dz}{(z^2+4)^3} = 0$; $C \equiv |\, z - 1 \,| = 4$

25. Evaluate $\oint_C \frac{z}{(z^2 - 6z + 25)^2}\, dz$ by Cauchy integral formula, where C is $|\, z - 3 - 4i \,| = 4$.

26. Let $P(z) = a + bz + cz^2$ and $\oint \frac{P(z)}{z}\, dz = \oint_C \frac{P(z)}{z^2}\, dz = \oint_C \frac{P(z)}{z^3}\, dz = 2\pi i$ where C is the circle $|\, z \,| = 1$. Evaluate $P(z)$.

27. If $f(\xi) = \int_C \frac{3z^2 + 7z + 1}{z - \xi}\, dz$, where C is the circle $x^2 + y^2 = 4$, find the values of $f(3)$, $f'(1-i)$ and $f''(1-i)$.

28. Evaluate: $\int_C \frac{(1+z)\sin z}{(2z-3)^2}\, dz$, where $C \equiv |\, z - i \,| = 2$ counter-clockwise. (*U.P.T.U. 2014*)

Answers

1.	$28\pi i$	**2.** $2\pi i \sin 1$	**3.** $-6\pi i$	**4.** $-2\pi i$
5.	(*i*) $-4\pi i$	(*ii*) πi		
6.	(*i*) (*a*) $2\pi i$ (*b*) $-\pi i$	(*ii*) $-\frac{2}{3}\pi i e$	(*iii*) $2\pi i\,(e^4 - e^2)$	(*iv*) $\frac{2\pi}{3}$
7.	(*i*) $2\pi i$	(*ii*) 0	(*iii*) $2\pi i$	
8.	(*a*) 0	(*b*) $-\pi i$		
9.	(*i*) $\pi(\cos 1 + i \sin 1)$	(*ii*) $-\pi(\cos 1 - i \sin 1)$		
11.	(*i*) $\frac{\pi}{3}$	(*ii*) $-\frac{\pi}{3}$	(*iii*) 0	
12.	(*i*) $\frac{\pi}{2}(3 + 2i)$	(*ii*) $\frac{\pi}{8} - 6\pi i$	(*iii*) $2\pi i \tan 1$	(*iv*) $3\pi i$
13.	(*i*) $2\pi i$	(*ii*) $-2\pi i$	(*iii*) 0	
14.	(*i*) $\frac{2\pi i}{5}$	(*ii*) $-18\pi i$	(*iii*) 0	**15.** πi
16.	(*i*) $4\pi i e^2$	(*ii*) $\frac{4\pi i}{3e^2}$		

17. (*i*) $72\pi i$ (*ii*) $-\dfrac{\pi i}{2}$

18. (*i*) $2\pi i$ (*ii*) $-\pi ie$ (*iii*) $\pi i\,(2-e)$

19. $\dfrac{\pi i}{2}(4\cos 2+\sin 2)$ **20.** (*i*) $12\pi i$ (*ii*) $12\pi i$ (*iii*) 0

21. (*i*) $2\pi\sinh\pi$ (*ii*) πi (*iii*) $\dfrac{6\pi e}{25}$ (*iv*) $\pi i\,(\sin t - t\cos t)$

22. (*i*) 0 (*ii*) $\dfrac{2\pi i}{3}\sinh 2$ (*iii*) $\dfrac{2\pi i}{3}(\sinh 2+\sinh 1)$

23. (*i*) $\dfrac{\pi ie}{10}$ (*ii*) $\dfrac{2\pi i}{3}$ **25.** $\dfrac{3\pi}{128}$.

26. $P(z) = 1 + z + z^2$ **27.** $f(3)=0,\ f'(1-i)=2\pi(6+13i),\ f''(1-i)=12\pi i$

28. $\dfrac{\pi i}{2}\left(\dfrac{5}{2}\cos\dfrac{3}{2}+\sin\dfrac{3}{2}\right)$

1.28 REPRESENTATION OF A FUNCTION BY POWER SERIES

A series of the form $\sum_{n=0}^{\infty} a_n z^n$ or $\sum_{n=0}^{\infty} a_n (z-a)^n$ whose terms are variable is called a power series, where z is a complex variable and a_n, a are complex constants. The second form can be reduced to first form merely by substitution $z = \zeta + a$ or by changing the origin.

Every complex function $f(z)$ which is analytic in a domain D can be represented by a power series valid in some circular region R about a point z_0. Both the circular region R and the point z_0 lie inside D. Such a power series is **Taylor's series**. If $f(z)$ is not analytic at a point z_0, we can still expand $f(z)$ in an infinite series having both positive and negative powers of $z - z_0$. This series is called the **Laurent's series**.

1.29 TAYLOR'S SERIES [*U.P.T.U. (C.O.) 2008*]

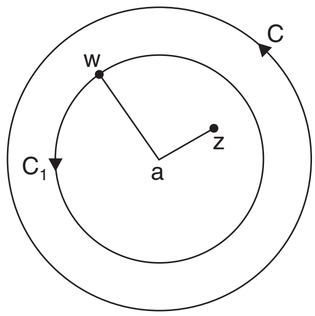
If f(z) is analytic inside a circle C with centre at a, then for all z inside C,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots$$

Or

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad \text{where} \quad a_n = \frac{f^{(n)}(a)}{n!}.$$

Proof. Let z be any point inside the circle C. Draw a circle C_1 with centre at a and radius smaller than that of C such that z is an interior point of C_1. Let w be any point on C_1, then



$$|z-a| < |w-a| \quad i.e., \left|\frac{z-a}{w-a}\right| < 1$$

Now, $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left[1-\frac{z-a}{w-a}\right]^{-1}$$

Expanding RHS by binomial theorem as $\left|\dfrac{z-a}{w-a}\right| < 1$, we get

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^n + \cdots\right] \quad ...(1)$$

This series converges uniformly since $\left|\dfrac{z-a}{w-a}\right| < 1$. Multiplying both sides of eqn. (1) by $\dfrac{1}{2\pi i} f(w)$ and integrating term by term w.r.t. w, over C_1, we get

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw + \frac{(z-a)^2}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^3}\,dw$$
$$+ \cdots + \frac{(z-a)^n}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw + \cdots \quad ...(2)$$

$$\Rightarrow \quad \boxed{f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots} \quad ...(3)$$

which is the required *Taylor's series* for $f(z)$ about $z = a$.

Cor. 1. Putting $z = a + h$ in (3), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots$$

Cor. 2. If $a = 0$, the series (3) becomes

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \cdots + \frac{z^n}{n!}f^n(0) + \cdots$$

This series is called **Maclaurin's series.**

1.30 LAURENT'S SERIES [*U.P.T.U. (C.O.) 2008*]

If $f(z)$ is analytic inside and on the boundary of the annular (ring shaped) region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 $(r_1 > r_2)$ respectively having centre at a, then for all z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

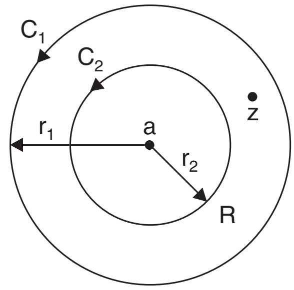
where, $$a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw\ ;\ n = 0, 1, 2, \ldots$$

and $$b_n = \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\,dw\ ;\ n = 1, 2, 3, \ldots$$

Proof. Let z be any point in the region R, then by Cauchy's integral formula for double connected region, we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw \quad ...(1)$$

For the first integral in (1), w lies on C_1



$$\therefore \qquad |z-a| < |w-a| \quad i.e., \quad \left|\frac{z-a}{w-a}\right| < 1$$

Now
$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\cdots\right]$$

Multiplying both sides by $\frac{1}{2\pi i} f(w)$ and integrating term by term w.r.t. w, along the circle C_1, we get

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw + \frac{(z-a)^2}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^3}\,dw + \cdots$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n(z-a)^n \quad \ldots(2) \qquad \left[\because \; a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw, n = 0, 1, 2, \cdots\right]$$

For the second integral in (1), w lies on C_2

$$\therefore \qquad |w-a| < |z-a| \quad i.e., \quad \left|\frac{w-a}{z-a}\right| < 1$$

Now
$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = -\frac{1}{z-a}\left(1-\frac{w-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{z-a}\left[1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\cdots\right]$$

Multiplying both sides by $-\frac{1}{2\pi i} f(w)$ and integrating term by term w.r.t. w, along the circle C_2, we get

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw = \frac{1}{z-a}\cdot\frac{1}{2\pi i}\oint_{C_2} f(w)\,dw + \frac{1}{(z-a)^2}\cdot\frac{1}{2\pi i}\oint_{C_2}(w-a)\,f(w)\,dw$$
$$+ \frac{1}{(z-a)^3}\cdot\frac{1}{2\pi i}\oint_{C_2}(w-a)^2 f(w)\,dw + \cdots$$

$$= b_1(z-a)^{-1} + b_2(z-a)^{-2} + b_3(z-a)^{-3} + \cdots$$

$$= \sum_{n=1}^{\infty} b_n(z-a)^{-n} \quad \ldots(3) \qquad \left[\because \; b_n = \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\,dw, n = 1, 2, 3, \cdots\right]$$

Substituting from (2) and (3) in (1), we get

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}}$$

Note 1. In case $f(z)$ is analytic inside C_1, then $b_n = 0$ and $a_n = \dfrac{1}{2\pi i}\oint_{C_1} \dfrac{f(w)}{(w-a)^{n+1}}\, dw = \dfrac{f^{\,n}(a)}{n\,!}$

and Laurent's series reduces to Taylor's series.

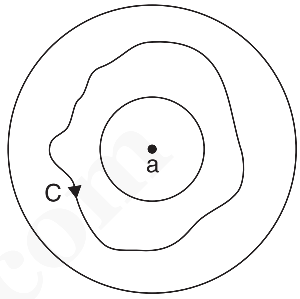
Note 2. If C is any simple closed curve which lies in the ring-shaped region R and encloses the circle C_1, then

$$\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\, dw = \oint_{C} \frac{f(w)}{(w-a)^{n+1}}\, dw$$

and $$\oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\, dw = \oint_{C} \frac{f(w)}{(w-a)^{-n+1}}\, dw$$

$\therefore$ Laurent's series can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, \quad \text{where } a_n = \frac{1}{2\pi i}\oint_{C} \frac{f(w)}{(w-a)^{n+1}}\, dw.$$



EXAMPLES

Example 1. *Expand* $\dfrac{1}{z^2 - 3z + 2}$ *in the region*

(a) $|\,z\,| < 1$ (b) $1 < |\,z\,| < 2$ (U.P.T.U. 2015)

(c) $|\,z\,| > 2$ (d) $0 < |\,z - 1\,| < 1$. (G.B.T.U. 2006, 2008, 2010)

Sol. Here $f(z) = \dfrac{1}{z^2 - 3z + 2} = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$ | Partial Fractions

(a) **When $|\,z\,| < 1$**

$$\therefore \qquad f(z) = \frac{1}{-2\left(1 - \dfrac{z}{2}\right)} + \frac{1}{1-z}$$

[Arranged suitably to make the binomial expansions valid]

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + (1-z)^{-1} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n$$

This is a series in positive powers of z, so it is an expansion of $f(z)$ in Taylor's series within the circle $|\,z\,| = 1$.

(b) **When $1 < |z| < 2$**

$$\therefore \qquad f(z) = \frac{1}{-2\left(1 - \dfrac{z}{2}\right)} - \frac{1}{z\left(1 - \dfrac{1}{z}\right)} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

This is a series in positive and negative powers of z, so it is an expansion of $f(z)$ in Laurent's series within the annulus $1 < |\,z\,| < 2$

(*c*) **When |z| > 2**

$$\therefore \qquad f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

This is Laurent's series within the annulus $2 < |z| < R$, where R is large.

(*d*) **When 0 < |z – 1| < 1**

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1} = -\frac{1}{1-(z-1)} - \frac{1}{z-1} = -(z-1)^{-1} - [1-(z-1)]^{-1}$$

$$= -\frac{1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n.$$

This is also Laurent's series within the annulus $0 < |z-1| < 1$.

Example 2. *Show that when* $|z+1| < 1$, $z^{-2} = 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n$.

Sol. $f(z) = z^{-2} = \dfrac{1}{z^2} = \dfrac{1}{[(z+1)-1]^2} = \dfrac{1}{[1-(z+1)]^2} = [1-(z+1)]^{-2}$

$$= 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + \cdots$$

[By binomial theorem, since $|z+1| < 1$]

$$= 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n.$$

Example 3. *Expand* $\cos z$ *in a Taylor's series about* $z = \dfrac{\pi}{4}$.

Sol. Here $\quad f(z) = \cos z, f'(z) = -\sin z, f''(z) = -\cos z, f'''(z) = \sin z, \ldots$

$$\therefore \qquad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},\quad f'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}},\quad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}},\quad f'''\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \ldots$$

Hence $\quad \cos z = f(z)$

$$= f\left(\frac{\pi}{4}\right) + \left(z-\frac{\pi}{4}\right)f'\left(\frac{\pi}{4}\right) + \frac{\left(z-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right) + \ldots$$

$$= \frac{1}{\sqrt{2}}\left[1-\left(z-\frac{\pi}{4}\right)-\frac{1}{2!}\left(z-\frac{\pi}{4}\right)^2+\frac{1}{3!}\left(z-\frac{\pi}{4}\right)^3+\cdots\right]$$

Example 4. *Expand the function* $\dfrac{\sin z}{z-\pi}$ *about* $z = \pi$.

Sol. Putting $z - \pi = t$, we have

$$\frac{\sin z}{z-\pi} = \frac{\sin(\pi+t)}{t} = \frac{-\sin t}{t} = -\frac{1}{t}\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}\cdots\right)$$

$$= -1+\frac{t^2}{3!}-\frac{t^4}{5!}+\cdots = -1+\frac{(z-\pi)^2}{3!}-\frac{(z-\pi)^4}{5!}+\cdots$$

Example 5. *Expand* $f(z) = \dfrac{z}{(z+1)(z+2)}$ *about* $z = -2$.

Sol. To expand $f(z)$ about $z = -2$, *i.e.*, in powers of $z + 2$, we put $z + 2 = t$.

$$\therefore \quad f(z) = \frac{z}{(z+1)(z+2)} = \frac{t-2}{(t-1)t} = \frac{2-t}{t(1-t)} = \frac{2-t}{t}\,(1-t)^{-1}$$

$$= \frac{2-t}{t}\,(1 + t + t^2 + t^3 + \cdots) \qquad \text{for } 0 < |\,t\,| < 1$$

$$= \frac{1}{t}\,(2 + t + t^2 + t^3 + \cdots) = \frac{2}{t} + 1 + t + t^2 + \cdots$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots \qquad \text{for } 0 < |\,z+2\,| < 1$$

which is Laurent's series.

Example 6. *Expand the following function in a Laurent's series:*

(*i*) $f(z) = \dfrac{e^z}{(z-1)^2}$ *about* $z = 1$. (*ii*) $f(z) = \dfrac{1}{z(z-1)(z-2)}$ *for* $|\,z-1\,| < 1$

Sol. (*i*) $f(z) = \dfrac{e^z}{(z-1)^2}$

Put $z - 1 = t$ then $z = 1 + t$

$$\therefore \quad f(z) = \frac{e^{1+t}}{t^2} = \frac{e}{t^2}\left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right] = e\left[\frac{1}{t^2} + \frac{1}{t} + \frac{1}{2!} + \frac{t}{3!} + \cdots\right]$$

$$= e\left[\frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{2!} + \frac{z-1}{3!} + \cdots\right].$$

(*ii*)

$$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)} \qquad |\text{Partial fractions}$$

$$= \frac{1}{2(z-1+1)} - \frac{1}{z-1} + \frac{1}{2(z-1-1)}$$

$$= \frac{1}{2}\{1 + (z-1)\}^{-1} - \frac{1}{z-1} - \frac{1}{2}\{1 - (z-1)\}^{-1} \qquad |\because\ |z-1| < 1$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n (z-1)^n - \frac{1}{z-1} - \frac{1}{2}\sum_{n=0}^{\infty} (z-1)^n$$

This is a series in positive and negative powers of $(z - 1)$ hence it is an expansion of $f(z)$ in a Laurent's series for $|z - 1| < 1$.

Example 7. *Expand the following function in a Laurent's series about the point* $z = 0$:

$$f(z) = \frac{1 - \cos z}{z^3}.$$

Sol.

$$f(z) = \frac{1-\cos z}{z^3} = \frac{1}{z^3}\left[1 - \left\{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right\}\right]$$

$$= \frac{1}{z^3}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right) = \frac{1}{2!\,z} - \frac{1}{4!}\,z + \frac{1}{6!}\,z^3 - \cdots$$

Example 8. *Find the terms in the Laurent's expansion of* $\dfrac{1}{z(e^z - 1)}$ *for the region* $0 < | z | < 2\pi$.

Sol.
$$f(z) = \frac{1}{z(e^z - 1)} = \frac{1}{z\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots - 1\right]}$$

$$= z^{-1}\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)^{-1} = z^{-2}\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots\right)^{-1}$$

$$= z^{-2}\left[1 - \left(\frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \ldots\right) + \frac{1}{4}z^2\left(1 + \frac{z}{3} + \frac{z^2}{12} + \ldots\right)^2 - \frac{1}{8}z^3\left(1 + \frac{z}{3} + \ldots\right)^3 + \frac{z^4}{16}(1 + \ldots)^4 + \ldots\right]$$

$$= z^{-2}\left[1 - \frac{z}{2} + z^2\left(\frac{1}{4} - \frac{1}{6}\right) - z^3\left(\frac{1}{8} - \frac{1}{6} + \frac{1}{24}\right) + z^4\left(\frac{1}{16} - \frac{1}{8} + \frac{1}{24} + \frac{1}{36} - \frac{1}{120}\right) + \ldots\right]$$

$$= z^{-2}\left[1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \ldots\right] = z^{-2} - \frac{1}{2}z^{-1} + \frac{1}{12} - \frac{z^2}{720} + \ldots$$

The singularities of $\dfrac{1}{z(e^z - 1)}$ are given by $z = 0$, $e^z = 1$ *i.e.,* $z = 0, \pm 2\pi i, \pm 4\pi i, \ldots\ldots$.

Hence the above expansion is valid for the region $0 < | z | < 2\pi$.

Example 9. *Using Taylor's theorem, show that:*

$$\log z = (z - 1) - \frac{(z - 1)^2}{2} + \frac{(z - 1)^3}{3} - \ldots \text{ where } | z - 1 | < 1.$$

Sol. $f(z) = \log z$, $\quad f(1) = 0 \quad | \because \ a = 1$ and $\log 1 = 0$

Now, $f'(z) = \dfrac{1}{z}$, $\quad f'(1) = 1$

$f''(z) = -\dfrac{1}{z^2}$, $\quad f''(1) = -1$

$f'''(z) = \dfrac{2}{z^3}$, $\quad f'''(1) = 2$

$f^{(iv)}(z) = \dfrac{-6}{z^4}$, $\quad f^{(iv)}(1) = -6$ and so on.

We know that,

$$f(z) = f(a) + (z - a)f'(a) + \frac{(z - a)^2}{2!}f''(a) + \frac{(z - a)^3}{3!}f'''(a) + \cdots$$

$$= f(1) + (z - 1)f'(1) + \frac{(z - 1)^2}{2!}f''(1) + \frac{(z - 1)^3}{3!}f'''(1) + \cdots$$

$$= 0 + (z-1)(1) + \frac{(z-1)^2}{2!}(-1) + \frac{(z-1)^3}{3!}(2) + \cdots$$

$$= (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots$$

Example 10. *Find the Taylor's or Laurent's series which represent the function* $\dfrac{1}{(1+z^2)(z+2)}$ *when*

(*i*) $|z| < 1$ (*ii*) $1 < |z| < 2$ (*iii*) $|z| > 2$.

Sol. Let $f(z) = \dfrac{1}{(1+z^2)(z+2)} = \dfrac{1}{5}\left\{\dfrac{1}{z+2} - \dfrac{z-2}{1+z^2}\right\}$

(*i*) $|z| < 1$

$$f(z) = \frac{1}{5}\cdot\frac{1}{2}\left(1+\frac{1}{2}z\right)^{-1} + \frac{2-z}{5}(1+z^2)^{-1}$$

| Binomial expansion of $(1+z)^{-1}$ is valid only when $|z| < 1$

$$= \frac{1}{10}\sum_0^\infty (-1)^n \left(\frac{z}{2}\right)^n + \frac{2-z}{5}\sum_0^\infty (-1)^n z^{2n}$$

This is a series in positive powers of z, so it is an expansion of $f(z)$ in a Taylor's series within the circle $|z| = 1$.

Remark. If $|z| < 1$, $(1+z)^{-1} = \sum_{n=0}^{\infty} (-1)^n z^n$; $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$.

(*ii*) $1 < |z| < 2$

$$f(z) = \frac{1}{5}\cdot\frac{1}{2}\left(1+\frac{1}{2}z\right)^{-1} + \frac{2-z}{5}\cdot\frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{10}\sum_0^\infty (-1)^n \left(\frac{z}{2}\right)^n + \frac{2-z}{5z^2}\sum_0^\infty (-1)^n \left(\frac{1}{z^2}\right)^n$$

This is a series in positive and negative powers of z, so it is an expansion of $f(z)$ in a Laurent's series within the annulus $1 < |z| < 2$.

(*iii*) $|z| > 2$

$$f(z) = \frac{1}{5}\cdot\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{1}{5}(z-2)\frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{5z}\sum_0^\infty (-1)^n \left(\frac{2}{z}\right)^n - \frac{1}{5}\left(\frac{1}{z}-\frac{2}{z^2}\right)\sum_0^\infty (-1)^n \left(\frac{1}{z^2}\right)^n$$

This is Laurent's series within the annulus $2 < |z| < R$, where R is large.

Example 11. *Find the Taylor's and Laurent's series which represent the function* $\dfrac{z^2-1}{(z+2)(z+3)}$ *when* (U.K.T.U. 2011)

(*i*) $|z| < 2$ (*ii*) $2 < |z| < 3$ (*iii*) $|z| > 3$.

Sol. Let $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)} = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}.$

(*i*) **| z | < 2**

$$f(z) = 1 + \frac{3}{2}\left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\sum_{0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

It is a Taylor's series within a circle $|z| = 2$.

(*ii*) **2 < | z | < 3**

$$f(z) = 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

| Arranging suitably to make the binomial expansion valid for $2 < |z| < 3$

$$= 1 + \frac{3}{z}\sum_{0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

It is a Laurent's series within the annulus $2 < |z| < 3$.

(*iii*) **| z | > 3**

$$f(z) = 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1 + \frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{0}^{\infty} (-1)^n \left(\frac{3}{z}\right)^n$$

It is a Laurent's series within the annulus $3 < |z| < R$, where R is large.

Example 12. *Expand* $\dfrac{1}{(z+1)(z+3)}$ *in the regions*

(*i*) $|z| < 1$ (*ii*) $1 < |z| < 3$

(*iii*) $|z| > 3$ (*iv*) $1 < |z + 1| < 2$.

Sol. $f(z) = \dfrac{1}{(z+1)(z+3)} = \dfrac{1}{2}\left[\dfrac{1}{z+1} - \dfrac{1}{z+3}\right]$

(*i*) **| z | < 1**

$$f(z) = \frac{1}{2}\left[(1+z)^{-1} - \frac{1}{3}\left(1 + \frac{z}{3}\right)^{-1}\right] = \frac{1}{2}\left[\sum_{0}^{\infty} (-1)^n z^n - \frac{1}{3}\sum_{0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n\right]$$

It is a Taylor's series within a circle $|z| = 1$.

(*ii*) **1 < | z | < 3**

$$f(z) = \frac{1}{2}\cdot\left[\frac{1}{z}\left(1 + \frac{1}{z}\right)^{-1} - \frac{1}{3}\left(1 + \frac{z}{3}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\sum_{0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^n - \frac{1}{3}\sum_{0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n\right]$$

It is a Laurent's series within the annulus $1 < |z| < 3$.

(*iii*) **| z | > 3**

$$f(z) = \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\sum_{0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{z}\sum_{0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n\right]$$

It is a Laurent's series within the annulus $3 < |z| < R$ where R is large.

(*iv*) $\mathbf{1 < |z+1| < 2}$

$\Rightarrow \quad 1 < |u| < 2 \qquad$ where $z + 1 = u$

$$f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right] = \frac{1}{2}\left[\frac{1}{u} - \frac{1}{u+2}\right]$$

$$= \frac{1}{2}\cdot\frac{2}{u(u+2)} = \frac{1}{u(u+2)} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1} = \frac{1}{2u}\sum_{0}^{\infty}(-1)^n\left(\frac{u}{2}\right)^n$$

$$= \frac{1}{2(z+1)}\sum_{0}^{\infty}(-1)^n\left(\frac{z+1}{2}\right)^n$$

It is Laurent's series in the annulus $1 < |z+1| < 2$.

Example 13. *Find the Laurent's expansion for:*

$$f(z) = \frac{7z-2}{z^3 - z^2 - 2z} \qquad \text{(U.K.T.U. 2010)}$$

in the regions given by:

(*i*) $0 < |z+1| < 1$ (*ii*) $1 < |z+1| < 3$ (*iii*) $|z+1| > 3$.

Sol. We have

$$f(z) = \frac{7z-2}{z^3 - z^2 - 2z} = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2} = \frac{1}{(z+1)-1} - \frac{3}{z+1} + \frac{2}{(z+1)-3}$$

(*i*) $\mathbf{0 < |z+1| < 1}$

$$f(z) = -\{1-(z+1)\}^{-1} - \frac{3}{z+1} - \frac{2}{3}\left\{1-\left(\frac{z+1}{3}\right)\right\}^{-1}$$

$$= -\frac{3}{z+1} - \sum_{n=0}^{\infty}(z+1)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

This is a series in negative and positive powers of $(z + 1)$ hence it is an expansion of $f(z)$ in Laurent's series within the annulus $0 < |z+1| < 1$.

(*ii*) $\mathbf{1 < |z+1| < 3}$

$$f(z) = \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{3}{z+1} - \frac{2}{3}\left\{1-\left(\frac{z+1}{3}\right)\right\}^{-1}$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{z+1} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

This is also a series in negative and positive powers of $(z + 1)$ hence it is an expansion of $f(z)$ in Laurent's series within the annulus $1 < |z + 1| < 3$.

(*iii*) **| z + 1 | > 3**

$$f(z) = \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{3}{z+1} + \frac{2}{z+1}\left(1-\frac{3}{z+1}\right)^{-1}$$

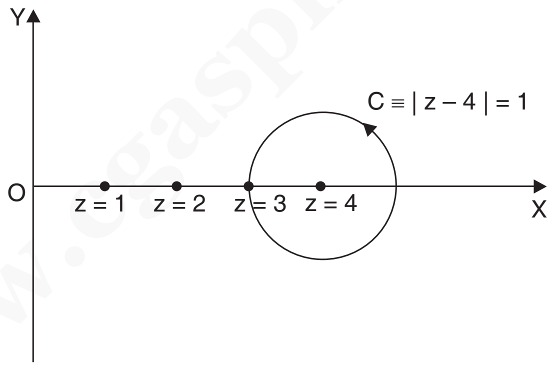
$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^{n} - \frac{3}{z+1} + \frac{2}{z+1}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^{n}$$

This is a series in negative powers of $(z + 1)$ hence it is an expansion of $f(z)$ in Laurent's series within the annulus $3 < |z + 1| < R$ where R is large.

Example 14. (*i*) *Obtain the Taylor's series expansion of* $f(z) = \dfrac{1}{z^2 - 4z + 3}$ *about the point* $z = 4$. *Find its region of convergence.*

(*ii*) *Obtain Taylor's series expansion of* $f(z) = \dfrac{1}{z^2 + 4}$ *about the point* $z = -i$. *Find the region of convergence.* (U.P.T.U. 2006)

Sol. (*i*) If the centre of the circle is at $z = 4$, then the distances of the singularities $z = 1$ and $z = 3$ from centre are 3 and 1. Hence if a circle is drawn with centre at $z = 4$ and radius 1 then within a circle $|z - 4| = 1$, the given function $f(z)$ is analytic hence it can be expanded in Taylor's series within the circle $|z - 4| = 1$ which is therefore the circle of convergence.



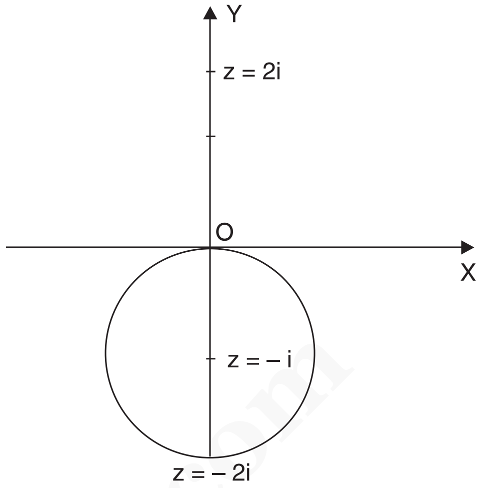
$$f(z) = \frac{1}{(z-1)(z-3)} = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right] = \frac{1}{2}\left[\frac{1}{z-4+1} - \frac{1}{z-4+3}\right]$$

$$= \frac{1}{2}\left[\{1+(z-4)\}^{-1} - \frac{1}{3}\left\{1+\left(\frac{z-4}{3}\right)\right\}^{-1}\right]$$

$$\Rightarrow \qquad f(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n (z-4)^n - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-4}{3}\right)^n\right]$$

(*ii*) If the centre of the circle is at $z = -i$, then the distances of the singularities $z = 2i$ and $z = -2i$ from centre are 3 and 1 respectively.

Hence if a circle is drawn with centre at $z = -i$ and radius 1 then within a circle $|z + i| = 1$, the given function $f(z)$ is analytic hence it can be expanded in Taylor's series within the circle $|z + i| = 1$ which is therefore the circle of convergence.



$$f(z) = \frac{1}{z^2+4} = \frac{1}{(z-2i)(z+2i)}$$

$$= \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right)$$

$$= \frac{1}{4i}\left[\frac{1}{(z+i)-3i} - \frac{1}{(z+i)+i}\right]$$

$$= \frac{1}{4i}\left[\frac{-1}{3i}\left\{1-\left(\frac{z+i}{3i}\right)\right\}^{-1} - \frac{1}{i}\left\{1+\left(\frac{z+i}{i}\right)\right\}^{-1}\right]$$

$$= \frac{1}{4i}\left[\frac{i}{3}\sum_{n=0}^{\infty}\left(\frac{z+i}{3i}\right)^n + i\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+i}{i}\right)^n\right]$$

$$= \frac{1}{4}\left[\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z+i}{3i}\right)^n + \sum_{n=0}^{\infty}(-1)^n\left(\frac{z+i}{i}\right)^n\right]$$

Example 15. (*i*) *If the function f(z) is analytic and one-valued in* $|z-a| < R$, *prove that for* $0 < r < R$,

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta \quad \textit{where } P(\theta) \textit{ is the real part of } (a + re^{i\theta}).$$

(*ii*) *Prove that:* $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n, \; |t| > 0$

where $\quad J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta$ [G.B.T.U. (C.O.) 2008]

Sol. (*i*) $\because$ $f(z)$ is regular in $|z-a| < R$

$\therefore$ $f(z)$ is regular in $|z-a| = r$ $\quad | \because r < R$

$\therefore$ $f(z)$ can be expanded in a Taylor's series within the circle $|z-a| = r$. Thus,

$$f(z) = \sum_{0}^{\infty} a_m (z-a)^m \text{ where } z - a = re^{i\theta}$$

$$= \sum_{0}^{\infty} a_m\, r^m\, e^{mi\theta} \qquad ...(1)$$

$\Rightarrow \qquad \overline{f(z)} = \sum_0^\infty \bar{a}_m r^m e^{-mi\theta}$ ...(2)

Now, $\int_C \overline{f(z)} \frac{dz}{(z-a)^{n+1}} = \int_0^{2\pi} \sum_0^\infty \bar{a}_m r^m e^{-mi\theta} \frac{re^{i\theta} id\theta}{r^{n+1} e^{i(n+1)\theta}}$

$$= \sum_0^\infty \bar{a}_m r^{m-n} i \int_0^{2\pi} e^{-i(m+n)\theta} d\theta$$

$= 0$, for all values of n ...(3)

Particularly, $\int_C \overline{f(z)} \frac{dz}{(z-a)^2} = 0$ ...(4)

We know that $f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-a)^2} = \frac{1}{2\pi i} \int_C \frac{f(z) + \overline{f(\bar{z})}}{(z-a)^2} dz$ | Using (4)

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta}) + \overline{f(a+re^{i\theta})}}{r^2 e^{2i\theta}} re^{i\theta} id\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{2\,\text{Re}(a+re^{i\theta})}{re^{i\theta}} d\theta \qquad | \because \; z + \bar{z} = 2\,\text{Re}\,(z)$$

$$= \frac{1}{\pi r} \int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta \text{ where } P(\theta) = \text{Re}\,(a + re^{i\theta}).$$

(*ii*) The function $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)}$ is analytic everywhere in the t-plane except at $t = 0$ and $t = \infty$ *i.e.,* it is analytic in the ring shaped region $r \le |\, t \,| \le R$ where r is small and R is large. Therefore this function can be expanded in Laurent's series in the form

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=0}^\infty a_n t^n + \sum_{n=1}^\infty b_n t^{-n}$$

where $\quad a_n = \frac{1}{2\pi i} \int_C e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} \frac{dt}{t^{n+1}}$ and $b_n = \frac{1}{2\pi i} \int_C e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} \frac{dt}{t^{-n+1}}$

where C is any circle with centre as origin.

Taking $C \equiv |\, t \,| = 1$ so that $t = e^{i\theta}$ and $dt = ie^{i\theta}\, d\theta$, we get

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} e^{\frac{z}{2}(e^{i\theta} - e^{-i\theta})} \frac{ie^{i\theta}\, d\theta}{e^{(n+1)\, i\theta}}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{iz \sin\theta} e^{-ni\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(n\theta - z\sin\theta)} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - z\sin\theta)\, d\theta \qquad | \text{ Since second part vanishes}$$

$$\Rightarrow \quad a_n = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta \qquad | \text{ Using prop. of definite integrals}$$

Clearly, the function $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)}$ remains unaltered if t is replaced by $-\dfrac{1}{t}$ so that $b_n = (-1)^n a_n$. Therefore,

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=0}^{\infty} a_n t^n + \sum_{n=1}^{\infty} b_n t^{-n}$$

$$= \sum_{n=0}^{\infty} a_n t^n + \sum_{n=1}^{\infty} (-1)^n a_n t^{-n} = \sum_{n=-\infty}^{\infty} a_n t^n$$

Here, a_n is $J_n(z)$ hence, $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n$

where, $$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta.$$

Example 16. *Prove that* $\cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$

where $a_n = \dfrac{1}{2\pi}\int_0^{2\pi} \cosh\theta\ \cosh(2\cos\theta)\, d\theta.$ (M.T.U. 2013)

Sol. The function $f(z) = \cosh\left(z+\frac{1}{z}\right)$ is analytic everywhere in the finite part of the plane except at $z = 0$ *i.e.,* it is analytic in the annulus $r \le |z| \le R$ where r is small and R is large. Hence $f(z)$ can be expanded in Laurent's series in the annulus $r < |z| < R$. Thus,

$$\cosh\left(z+\frac{1}{z}\right) = \sum_{0}^{\infty} a_n z^n + \sum_{1}^{\infty} b_n z^{-n}$$

where $$a_n = \frac{1}{2\pi i}\int_C \frac{\cosh\left(z+\frac{1}{z}\right)}{z^{n+1}}\, dz \quad \text{and} \quad b_n = \frac{1}{2\pi i}\int_C \frac{\cosh\left(z+\frac{1}{z}\right)}{z^{-n+1}}\, dz$$

where C is any circle lying in the annulus with origin as centre.

$$a_n = \frac{1}{2\pi i}\int_C \frac{\cosh\left(z+\frac{1}{z}\right)}{z^{n+1}}\, dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\cosh(2\cos\theta)\, ie^{i\theta}\, d\theta}{e^{i(n+1)\theta}}$$

| Take C as a circle $|z| = 1$ on which $z = e^{i\theta}$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta \qquad \left|\ \because \int_0^{2\pi} \cosh(2\cos\theta)\sin n\theta\, d\theta = 0\right.$$

$$b_n = a_{-n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos(-n\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\,d\theta = a_n$$

Hence, $\cosh\left(z+\frac{1}{z}\right) = \sum_0^\infty a_n z^n + \sum_1^\infty b_n z^{-n} = \sum_0^\infty a_n z^n + \sum_1^\infty a_n z^{-n}$ $\quad | \because \quad a_n = b_n$

$$= a_0 + \sum_1^\infty a_n z^n + \sum_1^\infty a_n z^{-n} = a_0 + \sum_1^\infty a_n (z^n + z^{-n}).$$

Example 17. *If C is a closed contour around origin, prove that* $\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{a^n e^{az}}{n!\, z^{n+1}}\,dz$

Hence deduce $\sum_0^\infty \left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\,d\theta.$

Sol. Let $f(z) = e^{az}$

$\therefore \quad f^n(z) = a^n e^{az}$

$\therefore \quad f^n(0) = a^n$

$$\Rightarrow \quad a^n = \frac{1}{2\pi i}\, n!\int_C \frac{f(z)\,dz}{z^{n+1}}$$

$$\Rightarrow \quad \left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\frac{1}{n!}\int_C \frac{a^n e^{az}}{z^{n+1}}\,dz$$

$$\Rightarrow \quad \sum_0^\infty \left(\frac{a^n}{n!}\right)^2 = \sum_0^\infty \frac{1}{2\pi i\, n!}\int_C \frac{a^n e^{az}}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int_C \sum_0^\infty \frac{a^n e^{az}}{n!\, z^{n+1}}\,dz$$

$$= \frac{1}{2\pi i}\int_C e^{az}\sum_0^\infty \frac{a^n}{n!}\frac{1}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int_C e^{az}\left\{\sum_0^\infty \left(\frac{a}{z}\right)^n \frac{1}{n!}\right\}\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_C e^{az} e^{(a/z)}\frac{dz}{z} = \frac{1}{2\pi i}\int_C e^{a\left(z+\frac{1}{z}\right)}\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} e^{2a\cos\theta}\frac{i e^{i\theta}\,d\theta}{e^{i\theta}}$$

| where the circle C is taken as $|z| = 1$ so that $z = e^{i\theta}$ on C
$\therefore \quad dz = i e^{i\theta}\,d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\,d\theta$$

Example 18. *Prove that for real k, $k^2 < 1$;* $\sum_{n=0}^{\infty} k^n \sin (n+1)\theta = \frac{\sin\theta}{1 - 2k\cos\theta + k^2}$

and
$$\sum_{n=0}^{\infty} k^n \cos (n+1)\theta = \frac{\cos\theta - k}{1 - 2k\cos\theta + k^2}.$$

Sol.
$$\frac{1}{z-k} = \frac{1}{z}\left(1 - \frac{k}{z}\right)^{-1} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{k}{z}\right)^n = \sum_{n=0}^{\infty}\frac{k^n}{z^{n+1}} \; ; \; |z| > k \qquad ...(1)$$

Again, put $z = e^{i\theta}$ in (1),
$$\frac{1}{z-k} = \frac{1}{e^{i\theta} - k} = \sum_{n=0}^{\infty} k^n \, e^{-(n+1)i\theta}$$

$$\Rightarrow \quad \frac{1}{\cos\theta + i\sin\theta - k} = \sum_{n=0}^{\infty} k^n \, [\cos (n+1)\theta - i \sin (n+1)\theta]$$

$$\Rightarrow \quad \frac{(\cos\theta - k) - i\sin\theta}{1 - 2k\cos\theta + k^2} = \sum_{n=0}^{\infty} k^n \, [\cos (n+1)\theta - i \sin (n+1)\theta] \qquad ...(2)$$

Comparing real and imaginary parts of (2), we get the required results.

Example 19. (*i*) *Show that* $\operatorname{cosec} z = \frac{1}{z} + \frac{1}{3!}z + \frac{7}{360}z^3 + \ldots\ldots ; \quad 0 < |z| < \pi.$

(*ii*) *Find the Taylor's series expansion of* $f(z) = \frac{a}{bz + c}$ *about the point* z_0.

Sol. (*i*) $\operatorname{cosec} z = \frac{1}{\sin z}$ has singular points at $z = 0, \pm n\pi$.

We expand the series in $0 < |z| < \pi$.

$$\operatorname{cosec} z = \frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots} = \frac{1}{z\left[1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \ldots\ldots\right)\right]}$$

$$= \frac{1}{z}\left[1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \ldots\ldots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \ldots\ldots\right)^2 + \ldots\ldots\right]$$

$$= \frac{1}{z}\left[1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^4}{(3!)^2} + \ldots\ldots\right] = \frac{1}{z} + \frac{z}{3!} + \left\{\frac{1}{(3!)^2} - \frac{1}{5!}\right\} z^4 + \ldots\ldots$$

(*ii*)
$$f(z) = \frac{a}{bz + c} = \frac{a}{b(z - z_0) + bz_0 + c} = \frac{1}{bz_0 + c}\left[\frac{a}{1 + \frac{b(z - z_0)}{bz_0 + c}}\right]$$

$$= \frac{a}{d}\left[\frac{1}{1 + e\,(z - z_0)}\right] \qquad \Big| \text{ where } bz_0 + c = d, \frac{b}{d} = e$$

$$= \frac{a}{d} \sum_{n=0}^{\infty} (-1)^n e^n (z - z_0)^n \text{ if } | e (z - z_0) | < 1$$

$$= \frac{a}{bz_0 + c} \sum_{n=0}^{\infty} (-1)^n \left(\frac{b}{bz_0 + c}\right)^n (z - z_0)^n \text{ if } | z - z_0 | < \frac{1}{e}.$$

Example 20. *Find Taylor's series expansion of* $\frac{4z-1}{z^4-1}$ *about the point* $z = 0$.

(U.P.T.U. 2007; M.T.U. 2012)

Sol.

$$f(z) = \frac{4z-1}{z^4-1} = \frac{4z-1}{(z-1)(z+1)(z^2+1)}$$

$$= \frac{3}{4}\left(\frac{1}{z-1}\right) + \frac{5}{4(z+1)} + \frac{\left(-2z + \frac{1}{2}\right)}{z^2+1}$$

Expanding about the point $z = 0$, we get

$$f(z) = -\frac{3}{4}(1-z)^{-1} + \frac{5}{4}(1+z)^{-1} + \left(-2z + \frac{1}{2}\right)\left(1+z^2\right)^{-1}$$

$$= -\frac{3}{4} \sum_{n=0}^{\infty} z^n + \frac{5}{4} \sum_{n=0}^{\infty} (-1)^n z^n + \left(-2z + \frac{1}{2}\right) \sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

ASSIGNMENT

Expand the following functions as a Taylor's series (1–3):

1. (*i*) $\log(1 + z)$ about $z = 0$ [*U.P.T.U. (C.O.) 2008*]

(*ii*) $\tan^{-1} z$ in powers of z [*U.P.T.U. (C.O.) 2009*]

(*iii*) $\sin^{-1} z$ in powers of z (*U.P.T.U. 2007*)

2. (*i*) $\sin z$ about $z = \frac{\pi}{4}$ (*ii*) $\tan^{-1}z$ about $z = \frac{\pi}{4}$ (*U.P.T.U. 2015*)

3. $\frac{z}{(z+1)(z+2)}$ about $z = 2$.

Expand the following functions in Laurent's series (4–6):

4. $\frac{1}{z-2}$, for $| z | > 2$ **5.** $\frac{1}{z^2 - 4z + 3}$, for $1 < | z | < 3$ **6.** $\frac{1}{z(z-1)(z-2)}$, for $| z | > 2$

7. (*i*) Find Taylor's expansion of $\frac{2z^3+1}{z(z+1)}$ about the point $z = 1$.

(*ii*) Define the Laurent series expansion of a function. Expand $f(z) = e^{\frac{z}{(z-2)}}$ in a Laurent series about the point $z = 2$. (*U.P.T.U. 2009*)

8. Expand $f(z) = \dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ in the region:

(*a*) $|z| < 1$ (*b*) $1 < |z| < 4$ (*c*) $|z| > 4$.

[*U.P.T.U. (C.O.) 2008*]

9. Expand the function $f(z) = \dfrac{1}{z^2 - z - 6}$ about (*i*) $z = -1$ (*ii*) $z = 1$

10. (*i*) Find the Laurent's series expansion of the function $f(z) = \dfrac{z^2 - 6z - 1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$.

(*ii*) Find the Laurent's series expansion of $f(z) = \dfrac{7z-2}{z(z+1)(z+2)}$ in the region $1 < |z+1| < 3$.

(*G.B.T.U. 2012*)

11. (*i*) Obtain the Taylor series expansion of $f(z) = \dfrac{1}{z^2 + (1+2i)z + 2i}$ about $z = 0$

(*ii*) Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ is Laurent series valid for

(*a*) $|z-1| > 1$ and (*b*) $0 < |z-2| < 1$ (*G.B.T.U. 2011, 2013*)

(*iii*) Expand $f(z) = \dfrac{z}{(z-1)(z-2)}$ in Laurent's series valid for region:

(*a*) $|z-1| > 1$ (*b*) $0 < |z-2| < 1$ (*M.T.U. 2014*)

12. Find Laurent's series of $f(z) = \dfrac{1}{z^2+1}$ about its singular points. Determine the region of convergence.

13. Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \dfrac{1}{(z+1)(z+2)^2}$ about the point $z = 1$. Consider the regions

(*i*) $|z-1| < 2$ (*ii*) $2 < |z-1| < 3$ (*iii*) $|z-1| > 3$

14. The series expansions of the functions $\dfrac{1}{1-z}$ and $\dfrac{1}{z-1}$ are

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots\ldots \text{ and } \frac{1}{z-1} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\ldots\right)$$

Adding, we get $(1 + z + z^2 + \ldots\ldots) + \dfrac{1}{z}\left(1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \ldots\ldots\right) = 0$

Is this result true? If not, give the reason.

15. Expand $f(z) = \dfrac{7z^2 + 9z - 18}{z^3 - 9z}$ in Laurent series valid for the regions:

(*i*) $0 < |z| < 3$ (*ii*) $|z| > 3$ (*G.B.T.U. 2013*)

16. If $f(z) = \dfrac{z+4}{(z+3)(z-1)^2}$, find Laurent's series expansion in (*i*) $0 < |z-1| < 4$ and (*ii*) $|z-1| > 4$.

(*M.T.U. 2013*)

17. Expand $f(z) = \dfrac{z}{(z^2-1)(z^2+4)}$ in Laurent series in $1 < |z| < 2$. (*G.B.T.U. 2011, 2012*)

18. Find all Taylor and Laurent series expansion of the following function about $z = 0$.

$$f(z) = \frac{-2z+3}{z^2 - 3z + 2}.$$

(*U.P.T.U. 2014*)

Answers

1. (*i*) $z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ...$ (*ii*) $z - \frac{z^3}{3} + \frac{z^5}{5} - ......$ (*iii*) $z + \frac{z^3}{6} + \frac{3}{40} z^5 + ...$

2. (*i*) $\frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\left(z - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(z - \frac{\pi}{4}\right)^3 + ...\right]$

(*ii*) $\tan^{-1} z = \tan^{-1}\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right) . \frac{16}{\pi^2 + 16} - 64\pi \frac{(z - \pi/4)^2}{(\pi^2 + 16)^2} + ...$

3. $\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2^3} - \frac{1}{3^2}\right)(z - 2) + \left(\frac{1}{2^5} - \frac{1}{3^3}\right)(z - 2)^2 - ......$

4. $f(z) = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$ **5.** $f(z) = -\frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n - \frac{1}{2z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$

6. $f(z) = \frac{1}{2z} - \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n + \frac{1}{2z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$

7. (*i*) $f(z) = 2z - 2 + \sum_{n=0}^{\infty} (-1)^n (z - 1)^n + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{2}\right)^n$ (*ii*) $f(z) = e \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{2}{z-2}\right)^n$

8. (*a*) $f(z) = 1 - \sum_{n=0}^{\infty} (-1)^n z^n - \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{4}\right)^n$

(*b*) $f(z) = 1 - \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{4}\right)^n$

(*c*) $f(z) = 1 - \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^n - \frac{4}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{4}{z}\right)^n$

9. (*i*) $f(z) = -\frac{1}{20} \sum_{n=0}^{\infty} \left(\frac{z+1}{4}\right)^n - \frac{1}{5} \sum_{n=0}^{\infty} (-1)^n (z + 1)^n$

(*ii*) $f(z) = -\frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n - \frac{1}{15} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{3}\right)^n$

10. (*i*) $f(z) = \frac{1}{z+2} \sum_{n=0}^{\infty} \left(\frac{3}{z+2}\right)^n + \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{z+2}{5}\right)^n + \frac{1}{z+2}.$

(*ii*) $f(z) = \frac{9}{z+1} - \frac{1}{z+1} \sum_{n=0}^{\infty} \frac{1}{(z+1)^n} - \frac{8}{z+1} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(z+1)^n}$

11. (*i*) $f(z) = \frac{1}{1 - 2i}\left[\frac{1}{2i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2i}\right)^n - \sum_{n=0}^{\infty} (-1)^n z^n\right]$

(*ii*) (*a*) $f(z) = \frac{1}{z-1} - \frac{2}{z-1} \sum_{n=0}^{\infty} \frac{1}{(z-1)^n}$ (*b*) $f(z) = \sum_{n=0}^{\infty} (-1)^n (z - 2)^n - \frac{2}{z-2}$

(*iii*) (*a*) $f(z) = \dfrac{-1}{z-1} + \dfrac{2}{z-1} \sum_{n=0}^{\infty} \left(\dfrac{1}{z-1}\right)^n$ (*b*) $f(z) = \dfrac{2}{z-2} - \sum_{n=0}^{\infty} (-1)^n (z-2)^n$

12. (*i*) $f(z) = \dfrac{1}{2i(z-i)} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z-i}{2i}\right)^n$; $|z-i| < 2$

(*ii*) $f(z) = \dfrac{-1}{2i(z+i)} \sum_{n=0}^{\infty} \left(\dfrac{z+i}{2i}\right)^n$; $|z+i| < 2$

13. (*i*) $\dfrac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z-1}{2}\right)^n - \dfrac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z-1}{3}\right)^n - \dfrac{1}{9} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\dfrac{z-1}{3}\right)^n$

(*ii*) $\dfrac{1}{z-1} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{2}{z-1}\right)^n - \dfrac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z-1}{3}\right)^n - \dfrac{1}{9} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\dfrac{z-1}{3}\right)^n$

(*iii*) $\dfrac{1}{z-1} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{2}{z-1}\right)^n - \dfrac{1}{z-1} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{3}{z-1}\right)^n - \dfrac{1}{(z-1)^2} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\dfrac{3}{z-1}\right)^n$

14. *No*. The first series is valid for $|z| < 1$ and the second series is valid for $|z| > 1$. There is no common point where both the series are valid.

15. (*i*) $f(z) = \dfrac{2}{z} + \dfrac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z}{3}\right)^n - \dfrac{4}{3} \sum_{n=0}^{\infty} \left(\dfrac{z}{3}\right)^n$ (*ii*) $f(z) = \dfrac{2}{z} + \dfrac{1}{z} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{3}{z}\right)^n + \dfrac{4}{z} \sum_{n=0}^{\infty} \left(\dfrac{3}{z}\right)^n$

16. (*i*) $f(z) = \dfrac{1}{64} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z-1}{4}\right)^n - \dfrac{1}{16(z-1)} + \dfrac{5}{4(z-1)^2}$

(*ii*) $f(z) = \dfrac{1}{16(z-1)} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{4}{z-1}\right)^n - \dfrac{1}{16(z-1)} + \dfrac{5}{4(z-1)^2}$

17. $f(z) = \dfrac{1}{10}\left[\dfrac{1}{z} \sum_{n=0}^{\infty} \left(\dfrac{1}{z}\right)^n + \dfrac{1}{z} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{1}{z}\right)^n + \dfrac{1}{2i} \sum_{n=0}^{\infty} \left(\dfrac{z}{2i}\right)^n - \dfrac{1}{2i} \sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z}{2i}\right)^n\right]$.

18. (*i*) $f(z) = \sum_{n=0}^{\infty} z^n + \dfrac{1}{2} \sum_{n=0}^{\infty} \left(\dfrac{z}{2}\right)^n$; $|z| < 1$

(*ii*) $f(z) = -\dfrac{1}{z} \sum_{n=0}^{\infty} \left(\dfrac{1}{z}\right)^n + \dfrac{1}{2} \sum_{n=0}^{\infty} \left(\dfrac{z}{2}\right)^n$; $1 < |z| < 2$

(*iii*) $f(z) = -\dfrac{1}{z} \sum_{n=0}^{\infty} \left(\dfrac{1}{z}\right)^n - \dfrac{1}{z} \sum_{n=0}^{\infty} \left(\dfrac{2}{z}\right)^n$; $|z| > 2$

1.31 ZERO OF AN ANALYTIC FUNCTION [*U.P.T.U. (C.O.) 2008*]

A zero of an analytic function $f(z)$ is a value of z such that $f(z) = 0$.

If $f(z)$ is analytic in the neighbourhood of $z = a$, then by Taylor's theorem

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + a_3(z - a)^3 + \ldots\ldots + a_n(z - a)^n + \ldots\ldots\infty$$

If $a_0 = a_1 = a_2 = \ldots\ldots = a_{n-1} = 0$ but $a_n \neq 0$, then $f(z)$ is said to have a zero of order n at $z = a$. The zero is said to be simple if $n = 1$.

$$\because \qquad a_n = \frac{f^n(a)}{n!}$$

$\therefore$ for a zero of order m at $z = a$,

$$f(a) = f'(a) = f''(a) = \ldots\ldots = f^{n-1}(a) = 0 \text{ but } f^n(a) \neq 0.$$

Thus in the neighbourhood of the zero at $z = a$ of order n,

$$f(z) = a_n(z - a)^n + a_{n+1}(z - a)^{n+1} + \ldots\ldots = (z - a)^n \,[a_n + a_{n+1}(z - a) + \ldots\ldots] = (z - a)^n\, \phi(z)$$

where $\phi(z) = a_n + a_{n+1}(z - a) + \ldots\ldots$ is analytic and non-zero at and in the neighbourhood of $z = a$.

1.32 SINGULARITY [*M.T.U. 2013, U.P.T.U. (C.O.) 2008, 2009*]

A singularity of a function $f(z)$ is a point at which the function ceases to be analytic.

1.33 ISOLATED AND NON-ISOLATED SINGULARITY [*M.T.U. 2012*]

If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$, otherwise it is called non-isolated.

Example. Consider the function $f(z) = \dfrac{z+1}{z(z-2)}$.

It is analytic everywhere except at $z = 0$ and $z = 2$. Thus $z = 0$ and $z = 2$ are the only singularities of this function. There are no other singularities of $f(z)$ in the neighbourhood of $z = 0$, $z = 2$. Hence $z = 0$ and $z = 2$ are the isolated singularities of this function.

Again, consider the function

$$f(z) = \frac{1}{\tan\left(\dfrac{\pi}{z}\right)} = \cot\left(\frac{\pi}{z}\right)$$

It is not analytic at the points where $\tan\left(\dfrac{\pi}{z}\right) = 0 = \tan n\pi$ *i.e.,* at the points where $\dfrac{\pi}{z} = n\pi$

$$\Rightarrow \qquad z = \frac{1}{n} \quad (n = 1, 2, 3, \ldots\ldots)$$

Thus $z = 1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\ldots, z = 0$ are the singularities of the function all of which are isolated except $z = 0$ because in the neighbourhood of $z = 0$, there are infinite number of other singularities $z = \dfrac{1}{n}$ where n is large. Therefore $z = 0$ is the non-isolated singularity of the given function.

1.34 TYPES OF SINGULARITY

Let $f(z)$ be analytic within a domain D except at $z = a$ which is an isolated singularity. Draw a circle C with its centre $z = a$ and radius as small as we wish and another large concentric circle C of any radius R lying wholly within the domain D. Now in the annulus between these two circles, $f(z)$ is analytic. If z is any point of the annulus, then by Laurent's theorem,

$$f(z) = \sum_{0}^{\infty} a_n (z-a)^n + \sum_{1}^{\infty} b_n (z-a)^{-n} \qquad \text{where } 0 < |z-a| < \text{R}.$$

The second term $\sum_{1}^{\infty} b_n (z-a)^{-n}$ on the RHS is called the **Principal Part** of $f(z)$ at the isolated singularity $z = a$. Now there arise three possibilities :

(*i*) All b_n's are zero $\Rightarrow$ no term in P.P. **(Removable singularity)** (*M.T.U. 2012*)

(*ii*) Infinite number of terms in P.P. **(Essential singularity)** (*M.T.U. 2012*)

(*iii*) Finite number of terms in P.P. **(Pole)** [*U.P.T.U. (C.O.) 2008*]

(*i*) **Removable Singularity.** Here $f(z) = \sum_{0}^{\infty} a_n (z-a)^n$ which is analytic for $|z-a| < \text{R}$ except at $z = a$. Let $\varphi(z)$ be the sum function of the power series $\sum_{0}^{\infty} a_n (z-a)^n$. Now $\varphi(z)$ differs from $f(z)$ only at $z = a$, where there is singularity. To avoid this singularity, we can suitably define $f(z)$ at $z = a$, so that we have

$$\phi(z) = \begin{cases} f(z) & \text{for } \quad 0 < |z-a\,\} < \text{R} \\ a_0 & \text{for } z = a \end{cases}$$

This type of singularity which can be made to disappear by defining the function suitably is called *removable singularity*.

Example. The function $\dfrac{\sin(z-a)}{z-a}$ has removable singularity at $z = a$ because

$$\frac{\sin(z-a)}{z-a} = \frac{1}{z-a}\left\{(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \ldots\ldots\right\}$$

$$= 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - \ldots\ldots$$

has no terms containing negative powers of $z - a$. However this singularity can be removed and the function made analytic by defining

$$\frac{\sin(z-a)}{z-a} = 1 \text{ at } z = a.$$

(*ii*) **Essential Singularity.** Here the series $\sum_{1}^{\infty} (z-a)^{-n}$ does not terminate.

Example. $f(z) = \sin\left(\dfrac{1}{z-a}\right)$ has essential singularity at $z = a$, because

$$\sin\left(\frac{1}{z-a}\right) = \frac{1}{z-a} - \frac{1}{3!}\frac{1}{(z-a)^3} + \frac{1}{5!}\frac{1}{(z-a)^5} - \ldots\ldots$$

has infinite number of terms in the negative powers of $z - a$.

(*iii*) **Pole.** Here the series $\sum_{1}^{\infty} (z-a)^n$ consists of finite number of terms. Then $z = a$ is said to be a pole of order m of the function $f(z)$. When $m = 1$, the pole is said to be simple.

Example. $f(z) = \dfrac{\sin(z-a)}{(z-a)^4}$ has a pole at $z = a$ because

$$\frac{\sin(z-a)}{(z-a)^4} = \frac{1}{(z-a)^4}\left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \frac{(z-a)^7}{7!} + \ldots\ldots\right]$$

$$= \frac{1}{(z-a)^3} - \frac{1}{3!}\frac{1}{(z-a)} + \frac{1}{5!}(z-a) - \frac{1}{7!}(z-a)^3 + \ldots\ldots$$

has finite number of terms (here first two terms only) in negative powers of $z - a$.

Thus if $z = a$ is a pole of order m of the function $f(z)$, then

$$f(z) = \sum_{0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots\ldots + \frac{b_m}{(z-a)^m}$$

$$= \frac{1}{(z-a)^m}\left[\left(\sum_{0}^{\infty} a_n (z-a)^{n+m}\right) + \{b_m + b_{m-1}(z-a) + \ldots\ldots + b_1 (z-a)^m\}\right]$$

$$= \frac{1}{(z-a)^m}\, \varphi(z)$$

Clearly, $\varphi(z) \to b_m$ as $z \to a$. Hence $\varphi(z)$ is analytic in the neighbourhood of the pole $z = a$.

1.35 THEOREMS

(1) The limit point of the zeros of a function f(z) is an isolated essential singularity.

Proof. Let $z_1, z_2, z_3, \ldots\ldots$ be an infinite set of zeros of $f(z)$. Let z_0 be their limit point.

(*i*) If z_0 is a point of the set, then z_0 will be a zero of $f(z)$ and will have in its neighbourhood, a cluster of zeros. But zeros are isolated, so, z_0 cannot be a zero of $f(z)$ unless $f(z)$ is identically zero in D.

(*ii*) If $f(z)$ is not identically zero in D, then z_0 is not a zero of $f(z)$. But z_0 is surrounded by many zeros. So z_0 is a singularity. Also z_0 is not a pole since $f(z)$ does not tend to infinity in the neighbourhood of z_0. Therefore z_0 is an essential singularity. But the singularity is isolated since in the neighbourhood of z_0, $f(z)$ is analytic. Hence z_0 is an isolated essential singularity.

(2) The limit point of the poles of a function f(z) is a non-isolated essential singularity.

Proof. Let $p_1, p_2, p_3, \ldots\ldots$ be an infinite set of poles of $f(z)$. Let p_0 be their limit point.

(*i*) If p_0 is a point of the set, then p_0 will be a pole of $f(z)$ and will have in its neighbourhood a cluster of poles. But poles are isolated, so, p_0 cannot be a pole of $f(z)$.

(*ii*) p_0 cannot be a zero of $f(z)$ since the function is not analytic (has poles) in the neighbourhood of p_0. So, p_0 is an essential singularity. This singularity is not isolated, since these are poles around p_0. Hence p_0 is a non-isolated essential singularity.

1.36 DETECTION OF SINGULARITY

(1) Removable Singularity: $\lim\limits_{z \to a} f(z)$ exists and is finite.

Example. $$f(z) = \frac{z^2 - a^2}{z - a}$$

$$\lim_{z \to a} f(z) = 2a$$

So, $f(z)$ has a removable singularity at $z = a$.

(2) Pole: $\lim\limits_{z \to a} f(z) = \infty$.

Example. $$f(z) = \frac{z^2 + a^2}{z - a}$$

$$\lim_{z \to a} f(z) = \infty.$$

So, $f(z)$ has a pole at $z = a$.

Moreover, the pole is said to be of order n, if there are n terms in the principal part.

Example. $$\frac{e^{z-a}}{(z-a)^2} = \frac{1}{(z-a)^2}\left[1 + (z-a) + \frac{(z-a)^2}{2!} + \ldots\ldots\right]$$

$$= \frac{1}{(z-a)^2} + \frac{1}{(z-a)} + \frac{1}{2!} + \ldots\ldots$$

Since there are only two terms in the negative powers of $z - a$ *i.e.*, there are only 2 (a finite number) terms in the principal part of the function. Hence the function has a pole of order 2.

(3) Essential singularity: $\lim\limits_{z \to a} f(z)$ does not exist.

Example. $\lim\limits_{z \to a} e^{\frac{1}{z-a}}$ does not exist, so $f(z)$ has an essential singularity at $z = a$.

EXAMPLES

Example 1. *Find out the zero and discuss the nature of the singularity of* $f(z) = \dfrac{z-2}{z^2} \sin \dfrac{1}{z-1}$.

Sol. Zeros of $f(z)$ are given by $f(z) = 0$

$$\Rightarrow \quad z - 2 = 0, \ \sin \frac{1}{z-1} = 0$$

$$\Rightarrow \quad z = 2, \ \frac{1}{z-1} = n\pi \ (n = 0, \pm 1, \pm 2, \ldots\ldots)$$

$$\Rightarrow \quad z = 2, \ 1 + \frac{1}{n\pi} \ (n = 0, \pm 1, \pm 2, \ldots\ldots)$$

Clearly, $z = 1$ is an isolated essential singularity.

Poles of $f(z)$ are given by

$$z^2 = 0$$

$$\Rightarrow \quad z = 0$$

Hence $z = 0$ is a pole of order 2.

Example 2. *Show that the function e^z has an isolated essential singularity at $z = \infty$.*

Sol. Put $z = \dfrac{1}{\rho}$ $\qquad e^{\frac{1}{\rho}} = 1 + \dfrac{1}{\rho} + \dfrac{1}{2!}\dfrac{1}{\rho^2} + \ldots\ldots\infty$

We have an infinite number of terms in the negative powers of ρ. So the function $e^{\frac{1}{\rho}}$ has an isolated essential singularity at $\rho = 0$. This implies that e^z has an isolated essential singularity at $z = \infty$.

Example 3. *Discuss singularity of $\dfrac{1}{1-e^z}$ at $z = 2\pi i$.*

Sol.
$$f(z) = \frac{1}{1-e^z}$$

For poles $\qquad 1 - e^z = 0$

$\Rightarrow \qquad e^z = 1 = e^{2n\pi i}$

$\Rightarrow \qquad z = 2n\pi i \ (n = 0, \pm 1, \pm 2, \ldots\ldots)$

Clearly, $z = 2\pi i$ is a simple pole.

Example 4. *Discuss singularity of $\dfrac{\cot \pi z}{(z-a)^2}$ at $z = a$ and $z = \infty$.* [U.P.T.U. (C.O.) 2008]

Sol.
$$f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{\sin \pi z\,(z-a)^2}$$

For poles $\qquad \sin \pi z(z-a)^2 = 0$

$\Rightarrow \qquad z = a, \ \pi z = n\pi \ (n \in \text{I})$

$\Rightarrow \qquad z = a, n$

Clearly $z = \infty$ is the limit point of these poles. Hence $z = \infty$ is a non-isolated essential singularity. Also $z = a$, being repeated twice, gives a double pole.

Example 5. *Discuss the nature of singularity of $f(z) = \dfrac{z - \sin z}{z^3}$ at $z = 0$.*

Sol.
$$f(z) = \frac{1}{z^3}(z - \sin z) = \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\ldots\right)\right]$$

$$= \frac{1}{z^3}\left(\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \ldots\ldots\right) = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \ldots\ldots$$

Since, there is no term in the principal part of given function hence $z = 0$ is a **removable singularity.**

ASSIGNMENT

1. Discuss singularity of $\dfrac{1}{\sin z - \cos z}$ at $z = \dfrac{\pi}{4}$. [*U.P.T.U. (C.O.) 2008*]
2. Discuss the nature of singularity of the function $f(z) = z \operatorname{cosec} z$ at $z = \infty$.
3. What is the nature of the singularity at $z = \infty$ of the function $f(z) = \cos z - \sin z$?
4. Discuss the singularity of the function $f(z) = \dfrac{1}{\cos \frac{1}{z}}$ at $z = 0$.
5. Discuss the nature of singularity of $f(z) = \sin \dfrac{1}{z}$ at $z = 0$. [*U.P.T.U. (C.O.) 2009*]
6. Find the singularity of the function $g(z) = \dfrac{e^{1/z}}{z^2}$. [*U.P.T.U. (C.O.) 2009*]
7. Prove that the singularity of $\cot z$ at $z = \infty$ is a non-isolated essential singularity.
8. Find the nature of singularities of the following functions:

   (*i*) $\dfrac{1 - e^z}{1 + e^z}$ at $z = \infty$ [*U.P.T.U. (C.O.) 2008*] (*ii*) $\operatorname{cosec} \dfrac{1}{z}$ at $z = 0$.
9. Discuss singularity of $f(z) = \sin \dfrac{1}{1 - z}$ at $z = 1$.

Answers

1. Simple pole **2.** Non-isolated essential **3.** Isolated essential
4. Non-isolated essential **5.** Isolated essential
6. Isolated essential singularity ($z = 0$)
8. (*i*) Non-isolated essential, (*ii*) Non-isolated essential **9.** Isolated essential.

1.37 DEFINITION OF THE RESIDUE AT A POLE

Let $z = a$ be a pole of order m of a one valued function $f(z)$ and γ any circle of radius r with centre at $z = a$ which does not contain any other singularities except at $z = a$, then $f(z)$ is analytic within the annulus $r < | z - a | < R$ hence it can be expanded within the annulus in a Laurent's series in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{n=1}^{\infty} b_n (z - a)^{-n} \qquad ...(1)$$

where
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - a)^{n+1}} \qquad ...(2)$$

and
$$b_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - a)^{-n+1}}\, dz \qquad ...(3)$$

$| z - a | = r$ being the circle γ.

Particularly, $\boxed{b_1 = \dfrac{1}{2\pi i}\displaystyle\int_\gamma f(z)\,dz}$

The coefficient b_1 is called residue of $f(z)$ at the pole $z = a$. It is denoted by symbol Res. $(z = a) = b_1$.

1.38 RESIDUE AT INFINITY

Residue of $f(z)$ at $z = \infty$ is defined as $-\dfrac{1}{2\pi i}\displaystyle\int_C f(z)\,dz$ where the integration is taken round C in anti-clockwise direction.

1.39 CAUCHY'S RESIDUE THEOREM OR THE THEOREM OF RESIDUES

[*M.T.U. 2013, G.B.T.U.* (*C.O.*) *2011*]

Let f(z) be one valued and analytic within and on a closed contour C except at a finite number of poles $z_1, z_2, z_3, ..., z_n$ and let $R_1, R_2, R_3, ..., R_n$ be respectively the residues of f(z) at these poles, then

$$\boxed{\begin{aligned}\oint_C f(z)\,dz &= 2\pi i\,(R_1 + R_2 + R_3 + ... + R_n)\\ &= 2\pi i\,(\textit{Sum of the residues at the poles within C}).\end{aligned}}$$

Proof. Let $\gamma_1, \gamma_2, \gamma_3, ..., \gamma_n$ be the circles with centres at $z_1, z_2, z_3, ..., z_n$ respectively and radii so small that they lie entirely within the closed curve C and do not overlap. Then $f(z)$ is analytic within the region enclosed by the curve C and these circles. Hence by Cauchy's theorem for multi-connected regions, we have

$$\int_C f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + ... + \int_{\gamma_n} f(z)\,dz$$

But by definition of residue,

$$R_1 = \frac{1}{2\pi i}\int_{\gamma_1} f(z)\,dz$$

$$\Rightarrow \qquad \int_{\gamma_1} f(z)\,dz = 2\pi i\, R_1$$

Similarly,

$$\int_{\gamma_2} f(z)\,dz = 2\pi i\, R_2$$

$$\int_{\gamma_3} f(z)\,dz = 2\pi i\, R_3$$

$$\vdots \qquad\qquad \vdots$$

$$\int_{\gamma_n} f(z)\,dz = 2\pi i\, R_n$$

Hence,

$$\begin{aligned}\int_C f(z)\,dz &= 2\pi i\, R_1 + 2\pi i R_2 + 2\pi i\, R_3 + ... + 2\pi i\, R_n\\ &= 2\pi i(R_1 + R_2 + R_3 + ... + R_n).\end{aligned}$$

1.40 METHODS OF FINDING OUT RESIDUES

(1) *If f(z) has a simple pole (i.e., pole of order 1) at z = a, then*

$$\boxed{\textbf{Res }\{\mathbf{f(z)}\} = \underset{\mathbf{z \to a}}{\mathbf{Lt}}\ \mathbf{(z - a)\ f(z).}}$$

Since $z = a$ is a pole of order 1, the Laurent's series becomes

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots\ldots + b_1(z - a)^{-1}.$$

Multiplying both sides by $(z - a)$, we get

$$(z - a)\, f(z) = a_0(z - a) + a_1(z - a)^2 + a_2(z - a)^3 + \ldots\ldots + b_1$$

$$\therefore \quad \underset{z \to a}{\text{Lt}}\ (z - a)\, f(z) = b_1 = \text{Res }\{f(z)\}$$

(2) *If f(z) has a pole of order m at z = a, then*

$$\textbf{Res }\{\mathbf{f(z)}\} = \frac{\mathbf{1}}{\mathbf{(m - 1)\,!}}\ \underset{\mathbf{z \to a}}{\mathbf{Lt}}\ \frac{\mathbf{d^{m-1}}}{\mathbf{dz^{m-1}}}\ \mathbf{[(z - a)^m\ f(z)]}$$

Since $z = a$ is a pole of order m, the Laurent's series becomes

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots\ldots + b_1(z - a)^{-1} + b_2(z - a)^{-2} + \ldots.. + b_m(z - a)^{-m}$$

Multiplying both sides by $(z - a)^m$, we get

$$(z - a)^m\, f(z) = a_0(z - a)^m + a_1(z - a)^{m+1} + a_2(z - a)^{m+2} + \ldots\ldots$$
$$+ b_1(z - a)^{m-1} + b_2(z - a)^{m-2} + \ldots\ldots + b_m$$

Differentiating both sides $(m - 1)$ times w.r.t. z and taking the limit as $z \to a$, we get

$$\underset{z \to a}{\text{Lt}}\ \frac{d^{m-1}}{dz^{m-1}}\ [(z - a)^m\, f(z)] = b_1(m - 1)\,!$$

or
$$\frac{1}{(m - 1)\,!}\ \underset{z \to a}{\text{Lt}}\ \frac{d^{m-1}}{dz^{m-1}}\ [(z - a)^m\, f(z)] = b_1 = \text{Res }\{f(z)\}.$$

Or

$$\boxed{\text{Res. }\{f(z)\} = \frac{1}{(m - 1)\,!}\left[\frac{d^{m-1}}{dz^{m-1}}\{(z - a)^m\ f(z)\}\right]_{z=a}}$$

(3) If $f(z)$ is of the form given by

$$f(z) = \frac{\phi(z)}{\psi(z)}\ ;\ \psi(a) = 0,\ \phi(a) \neq 0$$

where $z = a$ is the simple pole of $f(z)$, then Residue of $f(z)$ at $z = a$ is $= \dfrac{\phi(a)}{\psi'(a)}$.

(4) Residue of $f(z)$ at $z = a$ pole {simple or of order m}

= Coefficient of $\dfrac{1}{t}$ in $f(a + t)$ expanded in powers of t, where t, is sufficiently small.

(5) Residue of $f(z)$ at $z = \infty$

$$= \lim_{z \to \infty} \{- z f(z)\}$$

Or

$$= -\left[\text{coefficient of } \frac{1}{z} \text{ in the expansion of } f(z) \text{ for values of } z \text{ in the neighbourhood of } z = \infty\right].$$

EXAMPLES

Example 1. *Determine the poles of the following functions and residue at each pole:*

(*i*) $\dfrac{z^2}{(z-1)(z-2)^2}$ (*ii*) $\dfrac{1}{z^4+1}$ (*iii*) $\dfrac{1-e^{2z}}{z^4}$

Sol. (*i*) $$f(z) = \frac{z^2}{(z-1)(z-2)^2}.$$

Poles are given by

$(z-1)(z-2)^2 = 0 \quad \Rightarrow \quad z = 1, 2.$

$z = 1$ is a simple pole while $z = 2$ is a double pole.

Residue of $f(z)$ at simple pole $(z = 1)$ is

$$R_1 = \lim_{z \to 1} (z-1) \cdot \frac{z^2}{(z-1)(z-2)^2} = \lim_{z \to 1} \frac{z^2}{(z-2)^2} = \frac{(1)^2}{(1-2)^2} = 1.$$

Residue of $f(z)$ at double pole $(z = 2)$ is

$$R_2 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-2)^2 \cdot \frac{z^2}{(z-1)(z-2)^2}\right\}\right]_{z=2}$$

$$= \left[\frac{d}{dz}\left(\frac{z^2}{z-1}\right)\right]_{z=2} = \left[\frac{(z-1) \cdot 2z - z^2}{(z-1)^2}\right]_{z=2} = \left[\frac{z^2 - 2z}{(z-1)^2}\right]_{z=2} = 0$$

(*ii*) $$f(z) = \frac{1}{z^4+1}$$

Poles of $f(z)$ are given by

$z^4 + 1 = 0$

$\Rightarrow \quad z = (-1)^{1/4} = \{e^{(2n+1)\pi i}\}^{1/4}$

$\therefore$ Poles are, $z = e^{(2n+1)\pi i/4}$ where, $n = 0, 1, 2, 3, \ldots\ldots$

These are all of order 1 since the four factors occur linearly in $z^4 + 1$.

Since the roots repeat themselves, we can write them more conveniently as $e^{(2n+1)\pi i/4}$ where, $n = -2, -1, 0, 1$

i.e., $e^{m\pi i/4}$ where, $m = \pm 1, \pm 3$. Let denote it by z_m.

Residue at $(z = z_m)$ is

$$R = \lim_{z \to z_m} (z - z_m) \cdot \frac{1}{z^4+1} = \lim_{z \to z_m} \frac{1}{4z^3} \quad | \text{ By L' Hospital's Rule}$$

$$= \frac{1}{4z_m^3} = \frac{z_m}{4z_m^4} = -\frac{z_m}{4} = -\frac{1}{4} e^{m\pi i/4} h \text{ where, } m = \pm 1, \pm 3.$$

(*iii*) Pole of $\dfrac{1-e^{2z}}{z^4}$ is evidently $z = 0$. But this is not of the fourth order since

$$\frac{1-e^{2z}}{z^4} = \frac{1}{z^4}\left[1-\left\{1+2z+\frac{4z^2}{2}+\frac{8z^3}{6}+\frac{16z^4}{24}+\ldots\ldots\right\}\right]$$

$$= -\frac{\left(2+2z+\frac{4}{3}z^2+\frac{2}{3}z^3+\ldots\ldots\right)}{z^3} \qquad \ldots(1)$$

Therefore the pole is of order 3.

Residue at this pole is

$$\text{R} = \lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{(1-e^{2z})z^3}{z^4}\right\}$$

$$= \lim_{z\to 0}\frac{1}{2!}\left[\frac{d^2}{dz^2}\left\{\frac{1}{z}\left(1-1-2z-\frac{4z^2}{2!}-\frac{8z^3}{3!}-\ldots\ldots\right)\right\}\right]$$

$$= \lim_{z\to 0}\frac{1}{2}\left[\frac{d^2}{dz^2}\left(-2-2z-\frac{4}{3}z^2-\ldots\ldots\right)\right]$$

$$= \lim_{z\to 0}\frac{1}{2}\left[-\frac{8}{3}-\frac{2}{3}.6z-\ldots\ldots\right] = -\frac{4}{3}$$

Example 2. *Find the residue at $z = 0$ of the following functions:*

(*i*) $\dfrac{1+e^z}{\sin z+z\cos z}$ (*ii*) $z\cos\dfrac{1}{z}$.

Sol. (*i*) $z = 0$ is a pole of order 1.

$$\text{Residue} = \lim_{z\to 0}\frac{z(1+e^z)}{\sin z+z\cos z} = \lim_{z\to 0}\frac{1+e^z}{\left(\frac{\sin z}{z}\right)+\cos z} = \frac{1+1}{1+1} = 1.$$

(*ii*) Expanding the function in powers of z, we have

$$z\cos\frac{1}{z} = z\left[1-\frac{1}{2z^2}+\frac{1}{4!\,z^4}-\ldots\ldots\right] = z-\frac{1}{2z}+\frac{1}{24z^3}-\ldots\ldots$$

This is the Laurent 's expansion about $z = 0$.

The coefficient of $\dfrac{1}{z}$ in it is $-\dfrac{1}{2}$. So the residue of $z\cos\dfrac{1}{z}$ at $z = 0$ is $-\dfrac{1}{2}$.

Example 3. (*a*) *Give an example of a function having residue at infinity yet analytic there.*

(*b*) *Find the residue of $f(z) = \dfrac{z^3}{z^2-1}$ at $z = \infty$.*

Sol. (*a*) $$f(z) = \frac{z^2}{(z-\alpha)(z-\beta)(z-\gamma)}$$

Residue of $f(z)$ at $z = \infty$ is

$$= \lim_{z \to \infty} \left\{ -z \cdot \frac{z^2}{(z-\alpha)(z-\beta)(z-\gamma)} \right\}$$

$$= \lim_{z \to \infty} \frac{-1}{\left(1-\frac{\alpha}{z}\right)\left(1-\frac{\beta}{z}\right)\left(1-\frac{\gamma}{z}\right)} = -1$$

Now, $$f\left(\frac{1}{\lambda}\right) = \frac{\frac{1}{\lambda^2}}{\frac{(1-\alpha\lambda)}{\lambda} \cdot \frac{(1-\beta\alpha)}{\lambda} \cdot \frac{(1-\gamma\lambda)}{\lambda}} = \frac{\lambda}{(1-\alpha\lambda)(1-\beta\lambda)(1-\gamma\lambda)}$$

At $\lambda = 0$, $$f\left(\frac{1}{\lambda}\right) = 0 \ (\neq \infty)$$

$\therefore$ $f\left(\frac{1}{\lambda}\right)$ is analytic at $\lambda = 0$

$\Rightarrow$ $f(z)$ is analytic at $z = \infty$.

(*b*) Required residue $= \underset{z \to \infty}{\text{Lt}} \left(-z \cdot \frac{z^3}{z^2-1} \right)$ which does not exist

Hence, $$f(z) = \frac{z^3}{z^2\left(1-\frac{1}{z^2}\right)} = z\left(1-\frac{1}{z^2}\right)^{-1} = z\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\ldots\ldots\right) = z + \frac{1}{z} + \frac{1}{z^3} + \ldots\ldots$$

Required residue $= -\left(\text{coefficient of } \frac{1}{z}\right) = -1.$

Example 4. *Evaluate* $\oint_C \frac{e^z}{(z+1)^2}\, dz$, *where C is the circle* $|z-1| = 3$.

Sol. Here $f(z) = \frac{e^z}{(z+1)^2}$ has only one singular point $z = -1$ which is a pole of order 2 and it lies inside the circle $|z-1| = 3$.

Residue of $f(z)$ at $z = -1$ is $\underset{z \to -1}{\text{Lt}} \frac{d}{dz} [(z+1)^2 f(z)] = \underset{z \to -1}{\text{Lt}} \frac{d}{dz}(e^z) = \underset{z \to -1}{\text{Lt}} e^z = e^{-1}$

$\therefore$ By Residue theorem, we have $\oint_C \frac{e^z}{(z+1)^2}\, dz = 2\pi i(e^{-1}) = \frac{2\pi i}{e}$.

Example 5. *Determine the poles of the function* $f(z) = \dfrac{z^2}{(z-1)^2 (z+2)}$ *and the residue at each pole. Hence evaluate* $\int_C \dfrac{z^2}{(z-1)^2 (z+2)}\, dz$ *where* $C \equiv |z| = 3$. (U.P.T.U. 2015)

Sol. The function $f(z)$ has a pole of order 2 at $z = 1$ and a simple pole at $z = -2$.

Residue of $f(z)$ at $z = 1$ is

$$R_1 = \underset{z \to 1}{\text{Lt}} \frac{d}{dz} [(z-1)^2 f(z)] = \underset{z \to 1}{\text{Lt}} \frac{d}{dz}\left(\frac{z^2}{z+2}\right)$$

or
$$R_1 = \underset{z \to 1}{\text{Lt}} \frac{(z+2)\,.\,2z - z^2\,.\,1}{(z+2)^2} = \underset{z \to 1}{\text{Lt}} \frac{z^2+4z}{(z+2)^2} = \frac{5}{9}$$

Residue of $f(z)$ at $z = -2$ is

$$R_2 = \underset{z \to -2}{\text{Lt}} [(z+2) f(z)] = \underset{z \to -2}{\text{Lt}} \frac{z^2}{(z-1)^2} = \frac{4}{9}.$$

Since both the poles lie inside the given curve $C \equiv |\, z \,| = 3$,

$$\therefore \quad \int_C \frac{z^2}{(z-1)^2 (z+2)}\, dz = 2\pi i\, (R_1 + R_2) = 2\pi i \left[\frac{5}{9} + \frac{4}{9}\right] = 2\pi i.$$

| By Cauchy's Residue theorem

Example 6. *Determine the poles of the following function and residues at each pole:*

$f(z) = \dfrac{z-1}{(z+1)^2 (z-2)}$ *and hence evaluate* $\oint_C f(z)\, dz$, *where C is the circle* $|\, z - i \,| = 2$.

(U.K.T.U. 2011)

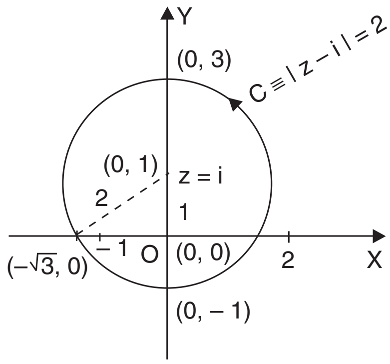
Sol. Poles of $f(z)$ are given by

$(z+1)^2 (z-2) = 0 \;\;\Rightarrow\;\; z = -1$ (double pole), 2(simple pole)

Residue of $f(z)$ at $z = -1$ is

$$R_1 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z+1)^2\,.\,\frac{z-1}{(z+1)^2 (z-2)}\right\}\right]_{z=-1}$$

$$= \left[\frac{d}{dz}\left(\frac{z-1}{z-2}\right)\right]_{z=-1} = \left[\frac{-1}{(z-2)^2}\right]_{z=-1} = \frac{-1}{9}$$



Residue of $f(z)$ at $z = 2$ is

$$R_2 = \underset{z \to 2}{\text{Lt}} (z-2) \frac{z-1}{(z+1)^2 (z-2)}$$

$$= \underset{z \to 2}{\text{Lt}} \frac{z-1}{(z+1)^2} = \frac{1}{9}$$

The given curve $C \equiv |\, z - i \,| = 2$ is a circle whose centre is at $z = i$ [*i.e.,* at (0, 1)] and radius is 2. Clearly, only the pole $z = -1$ lies inside the curve C.

Hence, by Cauchy's residue theorem

$$\oint_C f(z)\, dz = 2\pi i\,(R_1) = 2\pi i\left(\frac{-1}{9}\right) = -\frac{2\pi i}{9}.$$

Example 7. *Evaluate* $\int_C \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}\, dz$, *where C is the circle* $|\,z\,| = 10$.

(U.P.T.U. 2009)

Sol. Singularities are given by

$$(z + 1)^2 (z^2 + 4) = 0 \quad \Rightarrow \quad z = -1 \text{ (double pole)}, \pm 2i \text{ (simple poles)}$$

All the poles lie inside the given circle $c \equiv |\,z\,| = 10$.

$\therefore$ Residue (at $z = -1$) is

$$R_1 = \frac{1}{2-1!}\left[\frac{d}{dz}\left\{(z+1)^2 \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}\right\}\right]_{z=-1}$$

$$= \left[\frac{d}{dz}\left(\frac{z^2 - 2z}{z^2+4}\right)\right]_{z=-1} = \left[\frac{2z^2 + 8z - 8}{(z^2+4)^2}\right]_{z=-1} = -\frac{14}{25}$$

Residue (at $z = 2i$) is

$$R_2 = \lim_{z \to 2i} (z - 2i)\frac{z^2 - 2z}{(z+1)^2 (z-2i)(z+2i)}$$

$$= \frac{-4-4i}{(2i+1)^2 (4i)} = \frac{1+i}{3i+4} = \frac{7+i}{25}$$

Similarly, Residue (at $z = -2i$) is

$$R_3 = \frac{7-i}{25}$$

By Cauchy's Residue theorem,

$$\int_C \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}\, dz = 2\pi i\,(R_1 + R_2 + R_3) = 2\pi i\left[-\frac{14}{25} + \frac{7+i}{25} + \frac{7-i}{25}\right] = 0$$

Example 8. *Evaluate* $\int_C \frac{12z-7}{(z-1)^2 (2z+3)}\, dz$, *where C is the circle*

(*i*) $|\,z\,| = 2$ (G.B.T.U. 2011) (*ii*) $|\,z + i\,| = \sqrt{3}$.

Sol. $$f(z) = \frac{12z-7}{(z-1)^2 (2z+3)}.$$

Poles are given by

$$z = 1 \text{ (double pole)} \quad \text{and} \quad z = -\frac{3}{2} \text{ (simple pole)}$$

Residue at $(z = 1)$ is

$$R_1 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-1)^2 \cdot \frac{12z-7}{(z-1)^2 (2z+3)}\right\}\right]_{z=1}$$

$$= \left[\frac{d}{dz}\left(\frac{12z-7}{2z+3}\right)\right]_{z=1} = \left[\frac{(2z+3)\,.\,12-(12z-7)\,.\,2}{(2z+3)^2}\right]_{z=1}$$

$$= \frac{60-10}{25} = 2.$$

Residue at simple pole $\left(z = -\frac{3}{2}\right)$ is

$$R_2 = \lim_{z \to -3/2} \left(z + \frac{3}{2}\right) . \frac{12z-7}{(z-1)^2(2z+3)}$$

$$= \lim_{z \to -3/2} \frac{1}{2} . \frac{(12z-7)}{(z-1)^2} = -2.$$

(i) The contour $|z| = 2$ encloses both the poles 1 and $-\frac{3}{2}$.

$\therefore$ The given integral $= 2\pi i\,(R_1 + R_2) = 2\pi i\,(2-2) = 0.$

(ii) The contour $|z+i| = \sqrt{3}$ is a circle of radius $\sqrt{3}$ and centre at $z = -i$. The distances of the centre from $z = 1$ and $-\frac{3}{2}$ are respectively $\sqrt{2}$ and $\sqrt{\frac{13}{4}}$. The first of these is $< \sqrt{3}$ and the second is $> \sqrt{3}$.

$\therefore$ The second contour includes only the first singularity $z = 1$.

Hence, the given integral $= 2\pi i\,(R_1) = 2\pi i\,(2) = 4\pi i$.

Example 9. *Evaluate* $\oint_C \frac{z-3}{z^2+2z+5}\,dz$, *where C is the circle*

(i) $|z| = 1$ (ii) $|z+1-i| = 2$ (iii) $|z+1+i| = 2$.

(G.B.T.U. 2013)

Sol. The poles of $f(z) = \frac{z-3}{z^2+2z+5}$ are given by

$$z^2 + 2z + 5 = 0 \quad \Rightarrow \quad z = -1 \pm 2i$$

(i) Both the poles lie outside the circle $|z| = 1$.

$\therefore$ By Cauchy's integral theorem, we have $\oint_C \frac{z-3}{z^2+2z+5}\,dz = 0$

(ii) Only the pole $z = -1+2i$ lies inside the circle $|z+1-i| = 2$

Residue of $f(z)$ at $z = -1+2i$ is

$$\operatorname*{Lt}_{z \to -1+2i} (z+1-2i)\,f(z) = \operatorname*{Lt}_{z \to \alpha} \frac{(z-\alpha)(z-3)}{z^2+2z+5}, \text{ where } \alpha = -1-2i \qquad \Big|\ \text{Form } \frac{0}{0}$$

$$= \operatorname*{Lt}_{z \to \alpha} \frac{(z-\alpha)+(z-3)}{2z+2} \qquad |\ \text{By L' Hospital's Rule}$$

$$= \frac{\alpha-3}{2\alpha+2} = \frac{-1+2i-3}{-2+4i+2} = \frac{i-2}{2i}$$

$\therefore$ By Cauchy's residue theorem, $\oint_C \frac{z-3}{z^2+2z+5}\,dz = 2\pi i\left(\frac{i-2}{2i}\right) = \pi(i-2).$

(*iii*) Only the pole $z = -1 - 2i$ lies inside the circle $|z + 1 + i| = 2$.

Residue of $f(z)$ at $z = -1 - 2i$ is

$$\underset{z \to -1-2i}{\text{Lt}} (z + 1 + 2i) f(z) = \underset{z \to \beta}{\text{Lt}} \frac{(z-\beta)(z-3)}{z^2 + 2z + 5}, \text{ where } \beta = -1 - 2i \qquad \Big| \text{ Form } \frac{0}{0}$$

$$= \underset{z \to \beta}{\text{Lt}} \frac{(z-\beta)+(z-3)}{2z+2} \qquad | \text{ By L' Hospital's Rule}$$

$$= \frac{\beta - 3}{2\beta + 2} = \frac{-4-2i}{-4i} = \frac{i+2}{2i}$$

$\therefore$ By Cauchy's residue theorem, $\oint_C \frac{z-3}{z^2+2z+5}\, dz = 2\pi i \left(\frac{i+2}{2i}\right) = \pi(i + 2)$

Example 10. *Find the residue of* $f(z) = \dfrac{z^3}{(z-1)^4 (z-2)(z-3)}$ *at its pole and hence evaluate* $\int_C f(z)\, dz$*, where C is the circle* $|z| = 5/2$.

Sol. Poles of $f(z)$ are given by $(z-1)^4 (z-2)(z-3) = 0 \quad \Rightarrow \quad z = 1, 2, 3$

$z = 1$ is a pole of order 4 while $z = 2$ and $z = 3$ are simple poles.

Residue of $f(z)$ at $z = 2$ is

$$R_1 = \underset{z \to 2}{\text{Lt}} (z-2) \frac{z^3}{(z-1)^4 (z-2)(z-3)} = \underset{z \to 2}{\text{Lt}} \frac{z^3}{(z-1)^4 (z-3)} = \frac{8}{(-1)} = -8$$

Residue of $f(z)$ at $z = 3$ is

$$R_2 = \underset{z \to 3}{\text{Lt}} (z-3) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} = \underset{z \to 3}{\text{Lt}} \frac{z^3}{(z-1)^4 (z-2)} = \frac{27}{16}$$

Residue of $f(z)$ at $z = 1$ is

$$R_3 = \frac{1}{(4-1)!}\left[\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)}\right\}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}\right]_{z=1} = \frac{1}{6}\left[\frac{d^3}{dz^3}\left\{z + 5 + \frac{19z - 30}{z^2 - 5z + 6}\right\}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d^3}{dz^3}\left\{z + 5 + \frac{27}{z-3} - \frac{8}{z-2}\right\}\right]_{z=1} = \frac{1}{6}\left[\frac{d^2}{dz^2}\left\{1 - \frac{27}{(z-3)^2} + \frac{8}{(z-2)^2}\right\}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d}{dz}\left\{\frac{54}{(z-3)^3} - \frac{16}{(z-2)^3}\right\}\right]_{z=1} = \frac{1}{6}\left[\frac{-162}{(z-3)^4} + \frac{48}{(z-2)^4}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{-162}{16} + 48\right] = 8 - \frac{27}{16} = \frac{101}{16}$$

The given curve $C \equiv |z| = 5/2$ is a circle with centre at (0, 0) and radius 5/2.

Clearly, only the poles $z = 1$ and $z = 2$ lie inside this circle.

Hence, By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(R_3 + R_1) = 2\pi i\left(\frac{101}{16} - 8\right) = 2\pi i\left(\frac{-27}{16}\right) = -\frac{27\pi i}{8}.$$

Example 11. (*i*) *Find the value of* $\oint_C ze^{1/z}\,dz$ *around the unit circle.*

(*ii*) *Using Residue theorem, evaluate* $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{e^{zt}}{z^2\,(z^2+2z+2)}\,dz$*, where C is the circle* $|z| = 3$. (U.P.T.U. 2009)

Sol. (*i*) The only singularity of $ze^{1/z}$ is at the origin. Expanding $e^{1/z}$, we have

$$ze^{1/z} = z\left[1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \ldots\ldots\right] = z + 1 + \frac{1}{2z} + \frac{1}{6z^2} + \ldots\ldots$$

Residue at origin = coefficient of $\dfrac{1}{z} = \dfrac{1}{2}$.

Hence, the required integral $= 2\pi i\left(\dfrac{1}{2}\right) = \pi i$.

(*ii*) Singularities are given by

$$z^2\,(z^2 + 2z + 2) = 0 \quad\Rightarrow\quad z = 0, -1 \pm i$$

$z = 0$ is a pole of order 2. $z = -1 \pm i$ are simple poles. All these poles lie inside the circle $|z| = 3$.

Residue (at $z = 0$) is

$$R_1 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{z^2 \cdot \frac{e^{zt}}{2\pi i\, z^2(z^2+2z+2)}\right\}\right]_{z=0} = \left[\frac{d}{dz}\left\{\frac{e^{zt}}{2\pi i\,(z^2+2z+2)}\right\}\right]_{z=0}$$

$$= \frac{1}{2\pi i}\left[\frac{(z^2+2z+2)\,t\,e^{zt} - e^{zt}\,(2z+2)}{(z^2+2z+2)^2}\right]_{z=0} = \frac{1}{2\pi i}\left(\frac{t-1}{2}\right)$$

Let $-1 + i = \alpha$ and $-1 - i = \beta$ then

Residue at ($z = \alpha = -1 + i$) is

$$R_2 = \underset{z\to\alpha}{\text{Lt}}\,(z-\alpha)\cdot\frac{1}{2\pi i}\,\frac{e^{zt}}{z^2\,(z-\alpha)\,(z-\beta)} = \frac{1}{2\pi i}\,\frac{e^{\alpha t}}{\alpha^2(\alpha-\beta)} = \frac{1}{2\pi i}\left[\frac{1}{4}\,e^{(-1+i)t}\right]$$

Residue at ($z = \beta = -1 - i$) is

$$R_3 = \frac{1}{2\pi i}\left[\frac{1}{4}\,e^{(-1-i)t}\right]$$

By Residue theorem,

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = 2\pi i\left[\frac{1}{2\pi i}\left(\frac{t-1}{2}\right)+\frac{1}{2\pi i}\left\{\frac{e^{(-1+i)t}+(e^{(-1-i)t}}{4}\right\}\right]$$

$$= \frac{t-1}{2}+\frac{e^{-t}}{2}\cos t.$$

Example 12. *Obtain Laurent's expansion for the function* $f(z) = \dfrac{1}{z^2 \sinh z}$ *at the isolated singularity and hence evaluate* $\oint_C \dfrac{1}{z^2 \sinh z}\,dz$*, where C is the circle* $|z-1| = 2$.

Sol. Here, $f(z) = \dfrac{1}{z^2 \sinh z} = \dfrac{2}{z^2(e^z - e^{-z})}$

$$= \frac{2}{z^2\left[\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots\ldots\right)-\left(1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\ldots\ldots\right)\right]}$$

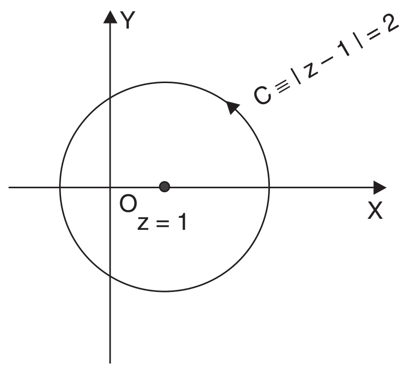
$$= \frac{2}{z^2\left(2z+\frac{2z^3}{3!}+\frac{2z^5}{5!}+\ldots\ldots\right)}$$

$$= \frac{1}{z^3\left(1+\frac{z^2}{3!}+\frac{z^4}{5!}+\ldots\ldots\right)}$$

$$= z^{-3}\left[1+\left(\frac{z^2}{6}+\frac{z^4}{120}\right)+\ldots\ldots\right]^{-1}$$

$$= z^{-3}\left(1-\frac{z^2}{6}-\frac{z^4}{120}+\frac{z^4}{36}+\ldots\ldots\right)$$

$$= \frac{1}{z^3}-\frac{1}{6z}+\frac{7}{360}z+\ldots\ldots$$



Only pole $z = 0$ of order two lies inside the circle $C \equiv |z-1| = 2$.

Residue of $f(z)$ at $(z = 0)$ is = coeff. of $\dfrac{1}{z}$ in the Laurent's expansion of $f(z) = -\dfrac{1}{6}$.

By Cauchy's Residue theorem,

$$\oint_C \frac{dz}{z^2\sinh z} = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}.$$

ASSIGNMENT

Determine the poles of the following functions and the residue at each pole:

1. $\dfrac{2z+1}{z^2-z-2}$ **2.** $\dfrac{z+1}{z^2(z-2)}$ **3.** $\dfrac{e^z}{z^2+\pi^2}$.

Evaluate the following integrals using Cauchy's residue theorem:

4. $\oint_C \left[\dfrac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)}\right] dz$; $C \equiv |z| = 3$

5. $\oint_C \left[\dfrac{3z^2+z+1}{(z^2-1)(z+3)}\right] dz$, where C is the circle $|z| = 2$.

6. $\oint_C \dfrac{z^2+2z-2}{z-4}\, dz$, where C is a closed curve containing the point $z = 4$ in its interior.

7. $\oint_C \dfrac{1-2z}{z(z-1)(z-2)}\, dz$, where C is the circle $|z| = 1.5$.

8. $\oint_C \dfrac{z}{(z-1)(z-2)^2}\, dz$, where C is the circle $|z-2| = \frac{1}{2}$.

9. $\oint_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}\, dz$, where C is the circle $|z| = 3$.

10. $\oint_C \dfrac{5z-2}{z(z-1)}\, dz$; $C \equiv |z| = 2$

11. $\oint_C \dfrac{e^z-1}{z(z-i)^2(z-1)}\, dz$; $C \equiv |z| = 1/2$

12. $\oint_C \dfrac{z \cos z}{\left(z-\dfrac{\pi}{2}\right)^3}\, dz$, where $C \equiv |z-1| = 1$

13. $\oint_C \dfrac{dz}{(z^2+4)^2}$, where $C \equiv |z-i| = 2$

14. $\oint_C \dfrac{3z^2+2}{(z-1)(z^2+9)}\, dz$, where $C \equiv |z-2| = 2$

15. $\oint_C \dfrac{2z+1}{(2z-1)^2}\, dz$, where $C \equiv |z| = 1$

16. $\oint_C \dfrac{dz}{(z^2+1)(z^2-4)}\, dz$, where $C \equiv |z| = 1.5$

17. (*i*) $\oint_C \dfrac{4z^2-4z+1}{(z-2)(4+z^2)}\, dz$, where $C \equiv |z| = 1$.

(*ii*) $\oint_C \dfrac{24z-7}{(z-1)^2(2z+3)}\, dz$, where C is the circle of radius 2 with centre at the origin.

(*M.T.U. 2012*)

18. $\oint_C \dfrac{z^2+4}{z(z^2+2z+2)}\, dz$, where C is

(*i*) $|z| = 1$ (*ii*) $|z+1-i| = 1$

(*iii*) $|z+1+i| = 1$ (*iv*) $|z-1| = 5$

19. $\oint_C \dfrac{e^{-z}}{z^2}\, dz$; $C \equiv |z| = 1$

20. $\oint_C z^2 e^{1/z}\, dz$; $C \equiv |z| = 1$

21. $\oint_C \frac{1}{z^2 \sin z}\, dz$ where C is the triangle with vertices (0, 1), (2 – 2) and (7, 1). (*G.B.T.U. 2012*)

22. Determine the poles and residues at each pole of the function $f(z) = \frac{z}{z^2 - 3z + 2}$ and hence evaluate $\oint_C f(z)\, dz$ where C is the circle $| z - 2 | = \frac{1}{2}$. (*G.B.T.U. 2011*)

Answers

1. $z = -1, 2 ; \frac{1}{3}, \frac{5}{3}$	**2.** $z = 0, 2 ; -\frac{3}{4}, \frac{3}{4}$	**3.** $z = \pm \pi i ; \pm \frac{i}{2\pi}$
4. $-4\pi i$	**5.** $-\frac{\pi i}{4}$	**6.** $44\pi i$
7. $3\pi i$	**8.** $-2\pi i$	**9.** $4\pi i(\pi + 1)$
10. $10\pi i$	**11.** 0	**12.** $-2\pi i$
13. $\pi/16$	**14.** πi	
15. πi	**16.** 0	**17.** (*i*) 0 (*ii*) 0
18. (*i*) $4\pi i$	(*ii*) $-\pi(3 + i)$	(*iii*) $\pi(3 - i)$ (*iv*) $2\pi i$
19. $-2\pi i$	**20.** $\frac{\pi i}{3}$.	**21.** $\frac{2i(-1)^n}{n^2 \pi}$
22. $z = 1, 2 ; -1, 2 ; 4\pi i$.		

1.41 CONTOUR INTEGRATION

We take a closed curve C, find the poles of $f(z)$ within C and calculate residue at these poles. Then by Cauchy's residue theorem

$$\int_C f(z)\, dz = 2\pi i \text{ [sum of the residues of } f(z) \text{ at the poles within C]}$$

The curve is called a contour.

The process of integration along a contour is called *contour integration.*

1.42 APPLICATION OF RESIDUE THEOREM TO EVALUATE REAL INTEGRALS

The residue theorem provides a simple and elegant method for evaluating many important definite integrals of real variables. Some of these are illustrated below.

1.42.1. Integrals of the Type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$, where F(cos θ, sin θ) is a Rational Function of cos θ and sin θ.

Such integrals can be reduced to complex line integrals by the substitution $z = e^{i\theta}$, so that

$$dz = ie^{i\theta}\, d\theta, \text{ i.e., } d\theta = \frac{dz}{iz}.$$

Also,
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

As θ varies from 0 to 2π, z moves once round the unit circle in the anti-clockwise direction.

$$\therefore \quad \boxed{\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}}$$

where C is the unit circle $|z| = 1$.

The integral on the right can be evaluated by using the residue theorem.

EXAMPLES

Example 1. *Using contour integration, evaluate* $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$ *where* $a > |b|$

Hence or otherwise evaluate (U.K.T.U. 2010)

(i) $\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta}$ $\qquad$ (ii) $\int_0^{\pi} \frac{d\theta}{a + b\cos\theta}$; $a > |b|$

Sol. Consider the integration round a unit circle $C \equiv |z| = 1$ so that $z = e^{i\theta}$

$$\therefore \quad dz = ie^{i\theta}\, d\theta = iz\, d\theta \quad \Rightarrow \quad d\theta = \frac{dz}{iz}$$

Also, $$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

Then the given integral reduces to

$$I = \oint_C \frac{1}{\left[a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right]}\left(\frac{dz}{iz}\right) = \oint_C \frac{2z}{bz^2 + 2az + b}\left(\frac{dz}{iz}\right)$$

$$= \frac{2}{ib}\oint_C \frac{dz}{z^2 + \frac{2a}{b}z + 1} = \frac{2}{ib}\oint_C \frac{dz}{(z-\alpha)(z-\beta)}$$

where, $$\alpha = -\frac{a}{b} + \frac{\sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \beta = -\frac{a}{b} - \frac{\sqrt{a^2 - b^2}}{b}$$

Poles are given by $(z - \alpha)(z - \beta) = 0 \Rightarrow z = \alpha, \beta$

Both are simple poles.

Since $a > |b| \quad \therefore \quad |\beta| > 1$

Since $\alpha\beta = 1$

$\therefore \quad |\alpha\beta| = 1$

$|\alpha|\,|\beta| = 1$

$\Rightarrow \quad |\alpha| < 1 \qquad\qquad | \because \ |\beta| > 1$

Hence $z = \alpha$ is the only pole which lies inside the circle $C \equiv |z| = 1$.

Residue of $f(z)$ at $(z = \alpha)$ is

$$R = \lim_{z\to\alpha} (z-\alpha) f(z) = \lim_{z\to\alpha} (z-\alpha) \cdot \frac{2}{ib(z-\alpha)(z-\beta)}$$

$$= \frac{2}{ib(\alpha - \beta)} = \frac{2(b)}{ib(2\sqrt{a^2 - b^2})} = \frac{1}{i\sqrt{a^2 - b^2}}$$

By Cauchy's Residue theorem,

$$I = 2\pi i(R) = 2\pi i\left(\frac{1}{i\sqrt{a^2 - b^2}}\right)$$

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}} \qquad \ldots(1)$$

(i) Putting $a = \sqrt{2}$ and $b = -1$ in (1), we get

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = \frac{2\pi}{\sqrt{2 - 1}} = 2\pi$$

(ii) From (1),

$$2\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}} \qquad | \text{ Using prop. of definite integrals}$$

$$\Rightarrow \qquad \int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}.$$

Example 2. *Evaluate by contour integration:*

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}, \quad \text{where } a > |b|$$

[U.P.T.U. (C.O.) 2010; G.B.T.U. 2012]

Hence or otherwise evaluate $\int_0^{2\pi} \frac{d\theta}{1 - 2a\sin\theta + a^2}$, $0 < a < 1$.

Sol. Consider the integration round a unit circle $C \equiv |z| = 1$

so that $\quad z = e^{i\theta} \quad \therefore \quad d\theta = \frac{dz}{iz}$.

Also, $$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Then the given integral reduces to

$$I = \oint_C \frac{1}{\left[a + \frac{b}{2i}\left(z - \frac{1}{z}\right)\right]}\left(\frac{dz}{iz}\right) = \oint_C \frac{2iz}{bz^2 + 2iaz - b}\left(\frac{dz}{iz}\right)$$

$$= \frac{2}{b}\oint_C \frac{dz}{z^2 + \frac{2ia}{b}z - 1}$$

Poles are given by

$$z^2 + \frac{2ia}{b}z - 1 = 0$$

$$\Rightarrow \qquad z = \frac{\dfrac{-2ia}{b} \pm \sqrt{\dfrac{-4a^2}{b^2}+4}}{2} = \frac{-ia}{b} \pm \frac{\sqrt{b^2-a^2}}{b}$$

$$= \frac{-ia}{b} \pm \frac{i\sqrt{a^2-b^2}}{b} = \alpha, \beta \text{ (simple poles)}$$

where, $\alpha = \dfrac{-ia}{b} + \dfrac{i\sqrt{a^2-b^2}}{b}$ and $\beta = \dfrac{-ia}{b} - \dfrac{i\sqrt{a^2-b^2}}{b}$

Clearly, $|\beta| > 1$

But $\alpha\beta = -1$

$\therefore$ $|\alpha\beta| = 1 \Rightarrow |\alpha|\,|\beta| = 1 \Rightarrow |\alpha| < 1$

Hence $z = \alpha$ is the only pole which lies inside circle $C \equiv |z| = 1$.

Residue of $f(z)$ at $(z = \alpha)$ is

$$R = \operatorname*{Lt}_{z\to\alpha} (z-\alpha)\cdot\frac{2}{b(z-\alpha)(z-\beta)} = \frac{2}{b(\alpha-\beta)}$$

$$= \frac{2}{b\left(\dfrac{2i\sqrt{a^2-b^2}}{b}\right)} = \frac{1}{i\sqrt{a^2-b^2}}$$

$\therefore$ By Cauchy's Residue theorem,

$$I = 2\pi i\,(R) = 2\pi i\left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \qquad \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} \qquad \ldots(1)$$

If we replace a by $1 + a^2$ and b by $-2a$, then

$$\int_0^{2\pi} \frac{d\theta}{(1+a^2)-2a\sin\theta} = \frac{2\pi}{\sqrt{(1+a^2)^2-4a^2}} = \frac{2\pi}{\sqrt{1+a^4-2a^2}} = \frac{2\pi}{1-a^2}.$$

Example 3. *Use contour integration method to evaluate the following integral:*

$\displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$, *(a > 0).*

Sol.

$$I = \int_0^{\pi} \frac{a\,d\theta}{a^2 + \dfrac{(1-\cos 2\theta)}{2}}$$

$$= 2a\int_0^{\pi} \frac{d\theta}{(2a^2+1)-\cos 2\theta} \qquad \text{Put } 2\theta = \phi,\ d\theta = \frac{d\phi}{2}$$

$$= a\int_0^{2\pi} \frac{d\phi}{(2a^2+1)-\cos\phi}$$

$$I = 2a\int_0^{2\pi} \frac{d\phi}{(4a^2+2)-(e^{i\phi}+e^{-i\phi})} \qquad \ldots(1)$$

But $z = e^{i\phi}$ so that $d\phi = \dfrac{dz}{iz}$ then (1) reduces to

$$I = 2a \int_C \frac{1}{(4a^2+2) - \left(z + \dfrac{1}{z}\right)} \cdot \frac{dz}{iz} = \frac{2a}{i} \int_C \frac{dz}{4a^2 z + 2z - z^2 - 1}$$

$$= 2ai \int_C \frac{dz}{z^2 - 2z(1+2a^2) + 1} = 2ai \int_C \frac{dz}{(z-\alpha)(z-\beta)}$$

where, $\alpha = (1 + 2a^2) + 2a\sqrt{1+a^2}$ and $\beta = (1 + 2a^2) - 2a\sqrt{1+a^2}$

Clearly, $|\alpha| > 1$

$\because$ $|\alpha\beta| = 1$ $\therefore$ $|\beta| < 1$

$\therefore$ Only β lies inside C.

Residue (at $z = \beta$) is $= \underset{z \to \beta}{\text{Lt}} (z-\beta) \cdot \dfrac{2ai}{(z-\alpha)(z-\beta)} = \dfrac{2ai}{\beta - \alpha} = \dfrac{2ai}{-4a\sqrt{1+a^2}} = \dfrac{-i}{2\sqrt{1+a^2}}$

By Cauchy Residue theorem,

$$I = 2\pi i \left(\frac{-i}{2\sqrt{1+a^2}} \right) = \frac{\pi}{\sqrt{1+a^2}}.$$

Example 4. *Apply Calculus of residues to prove that*

$$\int_0^{2\pi} \frac{d\phi}{(a + b\cos\phi)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}} \quad \textit{where } a > 0,\ b > 0,\ a > b.$$

Sol. Let, $I = \displaystyle\int_0^{2\pi} \frac{d\phi}{(a + b\cos\phi)^2} = \int_0^{2\pi} \frac{d\phi}{\left\{a + \dfrac{b}{2}(e^{i\phi} + e^{-i\phi})\right\}^2}$ ...(1)

Put $e^{i\phi} = z$ so that $d\phi = \dfrac{dz}{iz}$ then,

From (1), $I = \displaystyle\oint_C \frac{1}{\left\{a + \dfrac{b}{2}\left(z + \dfrac{1}{z}\right)\right\}^2} \frac{dz}{iz} = \oint_C \frac{-4iz\,dz}{(bz^2 + 2az + b)^2}$

$$= -\frac{4i}{b^2} \oint_C \frac{z\,dz}{\left(z^2 + \dfrac{2az}{b} + 1\right)^2}$$

Poles are given by,

$$\left(z^2 + \frac{2az}{b} + 1\right)^2 = 0 \quad \Rightarrow \quad (z-\alpha)^2 (z-\beta)^2 = 0 \text{ where, } \alpha + \beta = -\frac{2a}{b} \text{ and } \alpha\beta = 1.$$

Also, $$\alpha = \frac{-\frac{2a}{b} + \sqrt{\frac{4a^2}{b^2} - 4}}{2} = \frac{-a + \sqrt{a^2 - b^2}}{b}$$

$$\beta = \frac{-\frac{2a}{b} - \sqrt{\frac{4a^2}{b^2} - 4}}{2} = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

There are two poles, at $z = \alpha$ and at $z = \beta$ each of order 2.

Since, $$|\alpha\beta| = 1$$

or $$|\alpha|\,|\beta| = 1$$

But $|\beta| > 1 \quad \therefore \quad |\alpha| < 1$ $\qquad | \because \quad a > b$

$\therefore$ Only $z = \alpha$ lies inside the unit circle $|z| = 1$.

Residue of $f(z)$ at the double pole $z = \alpha$ is

$$= \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-\alpha)^2 \cdot \frac{-4iz}{b^2(z-\alpha)^2(z-\beta)^2}\right\}\right]_{z=\alpha}$$

$$= \left[\frac{d}{dz}\left\{\frac{-4iz}{b^2(z-\beta)^2}\right\}\right]_{z=\alpha} = -\frac{4i}{b^2}\cdot\left[\frac{(z-\beta)^2 \,.\, 1 - z \,.\, 2(z-\beta)}{(z-\beta)^4}\right]_{z=\alpha}$$

$$= -\frac{4i}{b^2}\left[\frac{(-\beta - z)}{(z-\beta)^3}\right]_{z=\alpha} = -\frac{4i}{b^2}\frac{(-\alpha-\beta)}{(\alpha-\beta)^3} = \frac{4i}{b^2}\frac{(\alpha+\beta)}{(\alpha-\beta)^3}$$

$$= \frac{4i}{b^2}\cdot\frac{\left(-\frac{2a}{b}\right)}{\left(\frac{2}{b}\sqrt{a^2-b^2}\right)^3} = -\frac{ia}{(a^2-b^2)^{3/2}}$$

$\therefore$ $$I = 2\pi i\left[\frac{-ia}{(a^2-b^2)^{3/2}}\right] = \frac{2\pi a}{(a^2-b^2)^{3/2}}.$$

Example 5. *Apply Calculus of residues to prove that*

$$\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1 - 2p\cos\theta + p^2} = \frac{\pi p^2}{1-p^2} \quad (0 < p < 1).$$

Sol. $$I = \int_0^{\pi} \frac{\cos 2\theta\, d\theta}{1 - 2p\cos\theta + p^2} = \frac{1}{2}\int_0^{2\pi}\frac{\cos 2\theta\, d\theta}{1 - p(e^{i\theta} + e^{-i\theta}) + p^2}$$

$$= \frac{1}{2} \text{ real part of } \int_0^{2\pi} \frac{e^{2i\theta}}{(1-pe^{i\theta})(1-pe^{-i\theta})}\, d\theta$$

$$= \frac{1}{2} \text{ real part of } \oint_C \frac{z^2}{(1-pz)\left(1-\frac{p}{z}\right)}\frac{dz}{iz} \qquad \left| \text{ writing } e^{i\theta} = z, d\theta = \frac{dz}{iz}\right.$$

$$= \frac{1}{2} \text{ real part of } \oint_C \frac{-iz^2}{(1-pz)(z-p)}\, dz$$

$$= \frac{1}{2} \text{ real part of } \oint_C f(z)\,dz \qquad \text{where, } f(z) = \frac{-iz^2}{(1-pz)(z-p)}$$

Poles of $f(z)$ are given by $(1 - pz)(z - p) = 0$.

Thus $z = \frac{1}{p}$ and $z = p$ are the simple poles. Only $z = p$ lies within the unit circle C as $p < 1$.

The residue of $f(z)$ at $z = p$ is

$$= \lim_{z \to p} (z-p) f(z) = \lim_{z \to p} (z-p) \frac{-iz^2}{(1-pz)(z-p)} = -\frac{ip^2}{1-p^2}$$

Hence by Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = 2\pi i \times [\text{Sum of residues within the contour}]$$

$$= 2\pi i \left(-\frac{ip^2}{1-p^2}\right) = \frac{2\pi p^2}{1-p^2} \text{ which is purely real.}$$

Hence, $\quad I = \frac{1}{2} \text{ real part of } \oint_C f(z)\,dz = \frac{\pi p^2}{1-p^2}$.

Example 6. *Use Complex integration method to prove that*

$$\int_0^{2\pi} \frac{\sin^2 \theta}{a + b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right), \text{ where } 0 < b < a.$$

Sol. Let $\quad I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\,d\theta = \int_0^{2\pi} \frac{1 - \cos 2\theta}{2(a + b\cos\theta)}\,d\theta$

$$= \text{Real part of } \int_0^{2\pi} \frac{1 - e^{2i\theta}}{2a + 2b\cos\theta}\,d\theta$$

Put $z = e^{i\theta}$ so that $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$

Then $\int_0^{2\pi} \frac{1 - e^{2i\theta}}{2a + 2b\cos\theta}\,dz = \oint_C \frac{1 - z^2}{2a + b\left(z + \frac{1}{z}\right)}\left(\frac{dz}{iz}\right) = \oint_C \frac{1 - z^2}{i(bz^2 + 2az + b)}\,dz$

where C is the circle $|z| = 1$.

The poles of the integrand are the roots of $bz^2 + 2az + b = 0$, *viz.*

$$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

Let $\quad \alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}$ and $\beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$

Clearly, $|\beta| > 1$ so that $z = \alpha$ is the only simple pole inside C.

Also, $bz^2 + 2az + b = b(z - \alpha)(z - \beta)$

Residue at $z = \alpha$ is

$$\underset{z \to a}{\text{Lt}} (z - \alpha) \cdot \frac{1 - z^2}{ib(z - \alpha)(z - \beta)} = \underset{z \to \alpha}{\text{Lt}} \frac{1 - z^2}{ib(z - \beta)} = \frac{1 - \alpha^2}{ib(\alpha - \beta)}$$

$$= \frac{\alpha\left(\frac{1}{\alpha} - \alpha\right)}{ib(\alpha - \beta)} = \frac{\alpha(\beta - \alpha)}{ib(\alpha - \beta)} \qquad | \because \quad \alpha\beta = 1$$

$$= -\frac{\alpha}{ib} = \frac{a - \sqrt{a^2 - b^2}}{ib^2}$$

$\therefore$ By Residue theorem,

$$\oint_C \frac{1 - z^2}{i(bz^2 + 2az + b)} dz = 2\pi i \cdot \frac{a - \sqrt{a^2 - b^2}}{ib^2} = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$$

Hence I = Real part of $\oint_C \frac{1 - z^2}{i(bz^2 + 2az + b)} dz = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$.

Example 7. *Using complex integration method, evaluate* $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$.

(M.T.U. 2012, G.B.T.U. 2010)

Sol. Let I = Real part of $\int_0^{2\pi} \frac{e^{2i\theta}}{5 + 2(e^{i\theta} + e^{-i\theta})} d\theta$

$$= \text{Real part of} \oint_C \frac{z^2}{5 + 2\left(z + \frac{1}{z}\right)}\left(\frac{dz}{iz}\right) \qquad \left| \begin{array}{l} \text{writing } e^{i\theta} = z \\ \therefore \quad d\theta = \frac{dz}{iz} \end{array} \right.$$

$$= \text{Real part of } \frac{1}{i}\oint_C \frac{z^2}{2z^2 + 5z + 2} dz$$

Singularities are given by

$$2z^2 + 5z + 2 = 0 \qquad \Rightarrow \qquad z = -\frac{1}{2}, -2$$

$z = -\frac{1}{2}$ is the only pole which lies inside the unit circle $C \equiv |z| = 1$.

Residue of $f(z)$ at $\left(z = -\frac{1}{2}\right)$ is

$$R = \underset{z \to -\frac{1}{2}}{\text{Lt}} \left(z + \frac{1}{2}\right) \cdot \frac{z^2}{i(2z + 1)(z + 2)} = \underset{z \to -\frac{1}{2}}{\text{Lt}} \frac{z^2}{2i(z + 2)} = \frac{1}{2i}\left(\frac{1}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{12i}$$

Hence by Cauchy's Residue theorem,

$$I = \oint_C f(z)\, dz = 2\pi i\left(\frac{1}{12i}\right) = \frac{\pi}{6}.$$

Example 8. *Evaluate:* $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$

Sol. Let $\quad I = \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = \text{Real part of } \int_0^{2\pi} \frac{1+2e^{i\theta}}{5+4\cos\theta}\, d\theta$

$$= \text{Real part of } \oint_C \frac{1+2z}{5+2\left(z+\frac{1}{z}\right)}\left(\frac{dz}{iz}\right) \qquad \left|\begin{array}{l}\text{Putting } e^{i\theta} = z \\ \therefore \quad d\theta = \dfrac{dz}{iz}\end{array}\right.$$

$$= \text{Real part of } \frac{1}{i}\oint_C \frac{1+2z}{2z^2+5z+2}\, dz$$

Poles are given by

$$(2z+1)(z+2) = 0 \quad \Rightarrow \quad z = -\frac{1}{2}, -2 \quad \text{(simple poles)}$$

$z = -\frac{1}{2}$ lies inside unit circle $C \equiv |z| = 1$

$$\text{Residue}\left(\text{at } z = -\frac{1}{2}\right) = \underset{z\to -\frac{1}{2}}{\text{Lt}} \left(z+\frac{1}{2}\right).\frac{1}{i}\,\frac{1+2z}{(2z+1)(z+2)} = \left(\frac{1}{2i}\right)\underset{z\to -\frac{1}{2}}{\text{Lt}} \frac{1+2z}{z+2} = 0$$

Hence by Cauchy's Residue theorem,

$$I = 2\pi i\,(0) = 0$$

$$\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0$$

$$\Rightarrow \quad \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0 \qquad | \text{ Using property of definite integrals}$$

Example 9. *Evaluate by Contour integration:* $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta$.

Sol. Let $\quad I = \int_0^{2\pi} e^{\cos\theta}\,[\cos(\sin\theta - n\theta) + i\sin(\sin\theta - n\theta)]\, d\theta$

$$= \int_0^{2\pi} e^{\cos\theta}\,.\,e^{i(\sin\theta - n\theta)}\, d\theta = \int_0^{2\pi} e^{e^{i\theta}}\,.\,e^{-in\theta}\, d\theta \qquad \text{...(1)}$$

Put $e^{i\theta} = z$ so that $d\theta = \frac{dz}{iz}$ then,

$$I = \int_C e^z\,.\,\frac{1}{z^n}\,.\,\frac{dz}{iz} = -i\int_C \frac{e^z}{z^{n+1}}\, dz$$

Poles are given by

$$z = 0 \qquad \text{[of order } (n+1)]$$

It lies inside the unit circle.

Residue of $f(z)$ at $z = 0$ is

$$R = \frac{1}{(n+1-1)!}\left[\frac{d^n}{dz^n}\left\{z^{n+1}\,.\,\frac{-ie^z}{z^{n+1}}\right\}\right]_{z=0} = \frac{-i}{n!}\left[\frac{d^n}{dz^n}(e^z)\right]_{z=0} = \frac{-i}{n!}$$

$\therefore$ By Cauchy's Residue theorem,

$$I = 2\pi i\left(\frac{-i}{n!}\right) = \frac{2\pi}{n!}$$

Comparing real parts, we have

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}.$$

Example 10. *Evaluate the integral:* $\int_0^{\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta$ (U.P.T.U. 2015)

Sol. Let,
$$I = \frac{1}{2}\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta} d\theta = \frac{1}{4}\int_0^{2\pi} \frac{1+\cos 6\theta}{5 - 4\cos 2\theta} d\theta \quad ...(1)$$

Consider the integration round a unit circle $c \equiv |z| = 1$ so that $z = e^{i\theta}$

$$\therefore \quad dz = ie^{i\theta}\, d\theta = iz\, d\theta \quad \Rightarrow \quad d\theta = \frac{dz}{iz}$$

Also,
$$\cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$$

and
$$\cos 6\theta = \frac{1}{2}\left(z^6 + \frac{1}{z^6}\right)$$

Then the given integral (1) reduces to

$$I = \frac{1}{4}\oint_c \frac{\left(1 + \frac{z^{12}+1}{2z^6}\right)}{5 - 2\left(\frac{z^4+1}{z^2}\right)} \cdot \frac{dz}{iz} = -\frac{1}{16i}\oint_c \frac{z^{12} + 2z^6 + 1}{z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)}\, dz$$

Singularities are, $z = 0$ (order 5), $z = \pm\sqrt{2},\ \pm\frac{1}{\sqrt{2}}$ (order 1)

Clearly, $z = 0$ and $z = \pm\frac{1}{\sqrt{2}}$ lie inside C.

Now we will find residues at $z = 0$ and $z = \pm\frac{1}{\sqrt{2}}$.

Let
$$f(z) = \frac{z^{12} + 2z^6 + 1}{z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} = \frac{z^{12} + 2z^6 + 1}{z^5}\left[1 - \left(\frac{5}{2}z^2 - z^4\right)\right]^{-1}$$

$$= \frac{(z^6+1)^2}{z^5}\left[1 + \frac{5}{2}z^2 - z^4 + \frac{25}{4}z^4 + z^8 - 5z^6 + ...\right]$$

Residue of $f(z)$ at $z = 0$ is the coefficient of $\frac{1}{z}$ in this laurent series expansion. Hence

$$R_1 = \text{Residue of } f(z) \text{ at } z = 0 = -1 + \frac{25}{4} = \frac{21}{4}$$

$$R_2 = \text{Residue of } f(z) \text{ at } z = \frac{1}{\sqrt{2}}$$

$$= \underset{z \to \frac{1}{\sqrt{2}}}{\text{Lt.}} \left(z - \frac{1}{\sqrt{2}}\right) \cdot \frac{(z^6+1)^2}{z^5 (z^2-2)\left(z - \frac{1}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}}\right)}$$

$$= \underset{z \to \frac{1}{\sqrt{2}}}{\text{Lt.}} \frac{(z^6+1)^2}{z^5 (z^2-2)\left(z + \frac{1}{\sqrt{2}}\right)} = -\frac{27}{8}$$

$$R_3 = \text{Residue of } f(z) \text{ at } z = -\frac{1}{\sqrt{2}}$$

$$= \underset{z \to -\frac{1}{\sqrt{2}}}{\text{Lt}} \left(z + \frac{1}{\sqrt{2}}\right) \cdot \frac{(z^6+1)^2}{z^5 (z^2-2)\left(z - \frac{1}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}}\right)}$$

$$= \underset{z \to -\frac{1}{\sqrt{2}}}{\text{Lt}} \frac{(z^6+1)^2}{z^5 (z^2-2)\left(z - \frac{1}{\sqrt{2}}\right)} = -\frac{27}{8}$$

Now, by Cauchy-Residue theorem,

$$\text{I} = -\frac{1}{16i}[2\pi i\,(\text{R}_1 + \text{R}_2 + \text{R}_3)] = -\frac{\pi}{8}\left(\frac{21}{4} - \frac{27}{4}\right) = \frac{3\pi}{16}.$$

ASSIGNMENT

Evaluate the following integrals by using contour integration:

1. (*i*) $\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$ (*U.K.T.U. 2011*) (*ii*) $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$ (*U.P.T.U. 2015*)

2. (*i*) $\int_0^{\pi} \frac{d\theta}{5 + 4\cos\theta}$ (*ii*) $\int_0^{\pi} \frac{d\theta}{17 - 8\cos\theta}$

(*iii*) $\int_0^{\pi} \frac{a\,d\theta}{1 + 2a^2 - \cos 2\theta}$ [*G.B.T.U. (C.O.) 2011*]

3. (*i*) $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$ (*G.B.T.U. 2011*) (*ii*) $\int_0^{\pi} \frac{d\theta}{3 + \sin^2\theta}$ (*G.B.T.U. 2013*)

4. (*i*) $\int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2}$ (*ii*) $\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2}$

5. (*i*) $\int_0^{\pi} \frac{a\,d\phi}{a^2 + \cos^2\phi}$ $(a > 0)$ (*ii*) $\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2p\cos\theta + p^2}\,d\theta$, $0 < p < 1$

6. (*i*) $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta$ (*U.P.T.U. 2009*) (*ii*) $\int_0^{\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta$

(*iii*) $\int_0^{2\pi} \frac{\cos 3\theta}{5 + 4\cos\theta}\, d\theta$ (*U.P.T.U. 2007*) (*iv*) $\int_0^{\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta$ (*U.P.T.U. 2014*)

7. $\int_{-\pi}^{\pi} \frac{a\cos\theta}{a + \cos\theta}\, d\theta \;;\, a > 1$ **8.** $\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta$

9. $\int_0^{2\pi} e^{-\cos\theta} \cos(n\theta + \sin\theta)\, d\theta \;;\, n \in \mathrm{I}$

10. (*i*) $\int_0^{2\pi} \cos^{2n}\theta\, d\theta \;;\, n \in \mathrm{I}$ (*ii*) $\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$

[*G.B.T.U. 2013; U.P.T.U. 2014*]

Answers

1. (*i*) $\frac{\pi}{2}$ (*ii*) $\frac{2\pi}{\sqrt{3}}$

2. (*i*) $\frac{\pi}{3}$ (*ii*) $\frac{\pi}{15}$ (*iii*) $\frac{\pi}{2\sqrt{1+a^2}}$

3. (*i*) $\frac{2\pi}{3}$ (*ii*) $\frac{\pi}{2\sqrt{3}}$

4. (*i*) $\frac{5\pi}{32}$ (*ii*) $\frac{\pi a}{(a^2 - 1)^{3/2}}$ **5.** (*i*) $\frac{\pi}{\sqrt{1+a^2}}$ (*ii*) $\frac{2\pi p^2}{1 - p^2}$

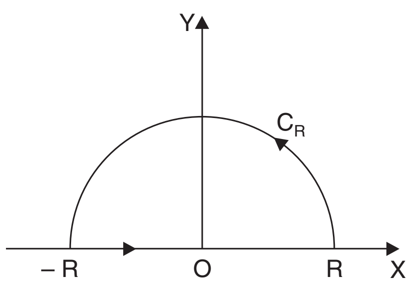
6. (*i*) $\frac{\pi}{12}$ (*ii*) $\frac{\pi}{24}$ (*iii*) $-\frac{\pi}{12}$ (*iv*) $\frac{\pi}{12}$

7. $2\pi a\left(1 - \frac{a}{\sqrt{a^2 - 1}}\right)$ **8.** $\frac{\pi}{4}$ **9.** $\frac{2\pi}{n!}(-1)^n$

10. (*i*) $\frac{\pi\,(2n)!}{(2)^{2n-1}\,(n!)^2}$ (*ii*) π.

1.42.2. Integrals of the Type $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx$, where f(x) and F(x) are Polynomials in x such that $\frac{x\,f(x)}{F(x)} \to 0$ as $x \to \infty$ and F(x) has no Zeros on the Real axis.

Consider the integral $\oint_C \frac{f(z)}{F(z)}\, dz$ over the closed contour C consisting of the real axis from – R to R and the semi-circle C_R of radius R in the upper half plane.



We take R large enough so that all the poles of $\dfrac{f(z)}{F(z)}$ in the upper half plane lie within C.

By residue theorem, we have

$$\oint_C \frac{f(z)}{F(z)}\,dz = 2\pi i\left[\text{sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane}\right]$$

$$\oint_{C_R} \frac{f(z)}{F(z)}\,dz + \int_{-R}^{R} \frac{f(x)}{F(x)}\,dx = 2\pi i\left[\text{sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane}\right]$$

...(1) $(\because$ on the real axis, $z = x)$

If we put $z = Re^{i\theta}$ in the first integral on the left side, then R is constant on C_R and as z moves along C_1, θ varies from 0 to π.

$$\therefore \qquad \int_{C_R} \frac{f(z)}{F(z)}\,dz = \int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})}\,Re^{i\theta}\; i\,d\theta$$

For large R, $\left|\displaystyle\int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})}\,Re^{i\theta}\,id\theta\right| \to 0$ $\qquad\Big|$ is of the order $\dfrac{Rf(R)}{F(R)}$

$$\therefore \qquad \int_0^{\pi} \frac{f(Re^{i\theta})}{F(Re^{i\theta})}\,Re^{i\theta}\,id\theta \to 0 \text{ when } R \to \infty$$

Hence from (1), we have

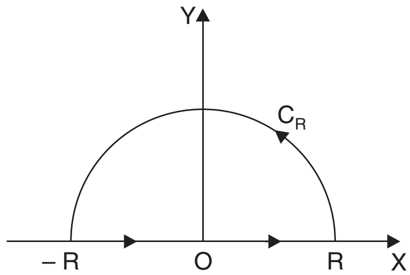
$$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx = 2\pi i\left[\text{sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane}\right]$$

EXAMPLES

Example 1. *Using contour integration, prove that* $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$.

Hence or otherwise evaluate $\displaystyle\int_0^{\infty} \frac{dx}{(1+x^2)^2}$.

Sol. Consider the integral $\displaystyle\int_C f(z)\,dz$ where $f(z) = \dfrac{1}{(1+z^2)^2}$ taken round the closed contour C consisting of the semi-circle C_R which is upper half of a large circle $|z| = R$ and the part of real axis from – R to R.



For poles, $\qquad (1 + z^2)^2 = 0$

$\Rightarrow \quad z^2 = -1$

$\Rightarrow \quad z = \pm i$ (Poles of order 2)

$z = -i$ is outside C.

So $z = i$ is the only pole inside C and is of order 2.

Residue of $f(z)$ at $z = i$ is

$$= \left[\frac{d}{dz}\left\{(z-i)^2 \cdot \frac{1}{(z-i)^2(z+i)^2}\right\}\right]_{z=i}$$

$$= \left[\frac{-2}{(z+i)^3}\right]_{z=i} = -\frac{i}{4}$$

By Cauchy's residue theorem,

$$\int_{-R}^{R} \frac{dx}{(1+x^2)^2} + \int_{C_R} \frac{dz}{(1+z^2)^2} = 2\pi i \left(\frac{-i}{4}\right) = \frac{\pi}{2}$$

Taking limit as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} + \lim_{R \to \infty} \int_{C_R} \frac{dz}{(1+z^2)^2} = \frac{\pi}{2} \qquad \ldots(1)$$

Now, $$\left| \int_{C_R} \frac{dz}{(1+z^2)^2} \right| \le \int_{C_R} \frac{|dz|}{|1+z^2|^2}$$

$$\le \int_{C_R} \frac{|dz|}{\{|z|^2 - 1\}^2}$$

$$= \int_0^{\pi} \frac{R\, d\theta}{(R^2-1)^2} \qquad \left| \begin{array}{l} \because\ |z| = R \text{ on } C_R \\ \text{and } |dz| = R\, d\theta \\ \text{also, } 0 < \theta < \pi \end{array} \right.$$

$$= \frac{\pi R}{(R^2-1)^2} \to 0 \text{ as } R \to \infty$$

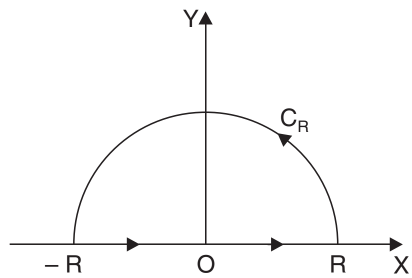
Hence, $$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$$

Now, $$2\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2} \quad \Rightarrow \quad \int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}.$$

Example 2. *Apply calculus of residues to prove that*

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}\ ;\ a > 0. \qquad \text{(M.T.U. 2013)}$$

Sol. Consider the integral $\int_C f(z)\, dz$ where $f(z) = \dfrac{1}{(a^2+z^2)^2}$ taken round the closed contour C consisting of the semi-circle c_R which is upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.



Poles of $f(z)$ are given by

$$(a^2 + z^2)^2 = 0$$

i.e., $$a^2 + z^2 = 0$$

or $z = \pm ai$ each repeated twice.

The only pole within the contour is $z = ai$, and is of the order 2.

Here, $$f(z) = \frac{1}{(z-ai)^2(z+ai)^2}.$$

Residue (at $z = ai$) is $$= \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-ai)^2 \cdot \frac{1}{(z-ai)^2(z+ai)^2}\right\}\right]_{z=ai}$$

$$= \left[\frac{d}{dz}\left\{\frac{1}{(z+ai)^2}\right\}\right]_{z=ai} = \left[\frac{-2}{(z+ai)^3}\right]_{z=ai}$$

$$= \frac{-2}{(2ai)^3} = \frac{-1}{4a^3 i^3} = \frac{1}{4a^3 i}.$$

Hence by Cauchy's residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i \text{ (Sum of residues within C)}$$

i.e.,
$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\left(\frac{1}{4a^3 i}\right)$$

or
$$\int_{-R}^{R} \frac{1}{(a^2+x^2)^2}\,dx + \int_{C_R} \frac{1}{(a^2+z^2)^2}\,dz = \frac{\pi}{2a^3} \qquad ...(1)$$

Now,
$$\left|\int_{C_R} \frac{1}{(a^2+z^2)^2}\,dz\right| \le \int_{C_R} \frac{|dz|}{|a^2+z^2|^2}$$

$$\le \int_{C_R} \frac{|dz|}{(|z|^2-a^2)^2}$$

$$= \int_0^{\pi} \frac{R\,d\theta}{(R^2-a^2)^2}, \text{ since } z = Re^{i\theta}$$

$$= \frac{\pi R}{(R^2-a^2)^2} \to 0 \text{ as } R \to \infty$$

Hence taking $R \to \infty$, relation (1) becomes,

$$\int_{-\infty}^{\infty} \frac{1}{(a^2+x^2)^2}\,dx = \frac{\pi}{2a^3}$$

or
$$\int_0^{\infty} \frac{1}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a^3}.$$

Example 3. *Apply Calculus of residues to prove that*

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a+b} \quad (a > 0,\ b > 0).$$

Sol. Consider the integral $\int_C f(z)\,dz$ where $f(z) = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$ taken round the closed contour C consisting of the semi-circle C_R which is upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

The poles of $f(z) = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$ are $z = \pm ia, \pm ib$. Of these, $z = ia$ and $z = ib$ lie in the upper half of the z-plane.

Residue of $f(z)$ at $z = ia$ is

$$= \lim_{z \to ia} (z-ia)\frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$= \lim_{z \to ia} \frac{z^2}{(z+ia)(z^2+b^2)} = -\frac{a^2}{2ia(-a^2+b^2)} = \frac{a}{2i(a^2-b^2)}.$$

Residue of $f(z)$ at $z = ib$ is

$$= \lim_{z \to ib} (z-ib)\frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$= \lim_{z \to ib} \frac{z^2}{(z^2+a^2)(z+ib)} = \frac{-b^2}{(-b^2+a^2)(2ib)} = \frac{-b}{2i(a^2-b^2)}.$$

By Cauchy's residue theorem,

$$\int_{-R}^{R} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx + \int_{C_R} \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz = 2\pi i\left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right]$$

Taking limit as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx + \underset{R\to\infty}{\text{Lt}} \int_{C_R} \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz = \pi\left[\frac{a-b}{a^2-b^2}\right]$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx + \underset{R\to\infty}{\text{Lt}} \int_{C_R} \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz = \frac{\pi}{a+b} \qquad ...(1)$$

Now, $$\left|\int_{C_R} \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz\right| \le \int_{C_R} \frac{|z|^2}{|z^2+a^2|\,|z^2+b^2|}\,|dz|$$

$$\le \int_{C_R} \frac{|z|^2}{(|z|^2-a^2)(|z|^2-b^2)}\,|dz|$$

$$= \frac{R^2}{(R^2-a^2)(R^2-b^2)} \int_0^{\pi} R\,d\theta \qquad |\because |z| = R \text{ on } C_R$$

$$= \frac{\pi R^3}{(R^2-a^2)(R^2-b^2)} \to 0 \text{ as } R \to \infty$$

$\therefore$ From (1), $$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a+b}.$$

Example 4. (*i*) *Apply Calculus of residues to prove that*

$$\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3},\ (a > 0).$$

(*ii*) *Using contour integration, evaluate* $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4}$. (U.P.T.U. 2007)

Sol. (*i*) Consider the integral $\int_C f(z)\,dz$ where $f(z) = \dfrac{1}{z^4+a^4}$.

The poles of $f(z)$ are given by

$$z^4 + a^4 = 0 \Rightarrow z^4 = -a^4 = a^4 e^{\pi i} = a^4 e^{2n\pi i + \pi i}$$

or $$z = ae^{(2n+1)\pi i/4};\ n = 0, 1, 2, 3.$$

Since there is no pole on the real axis, therefore, we may take the closed contour C consisting of the upper half C_R of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

$\therefore$ By Cauchy's residue theorem, we have

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = \int_C f(z)\,dz$$

or $$\int_{-R}^{R} \frac{1}{x^4+a^4}\,dx + \int_{C_R} \frac{1}{z^4+a^4}\,dz = 2\pi i \sum R^+ \qquad ...(1)$$

| where $\sum R^+$ = sum of residues of $f(z)$ at poles within C.

The poles $z = ae^{\frac{\pi i}{4}}$ and $z = ae^{3\pi i/4}$ are the only two poles which lie within the contour C. Let α denote any one of these poles, then

$$\alpha^4 + a^4 = 0 \quad \Rightarrow \quad \alpha^4 = -a^4.$$

Residue of $f(z)$ (at $z = \alpha$) is $= \left[\dfrac{1}{\dfrac{d}{dz}(z^4 + a^4)}\right]_{z=\alpha} = \dfrac{1}{4\alpha^3} = \dfrac{\alpha}{-4a^4}$

$\therefore$ Residue at $z = ae^{\pi i/4}$ is $= -\dfrac{1}{4a^3}\, e^{\pi i/4}$

and residue at $z = ae^{3\pi i/4}$ is $= -\dfrac{1}{4a^3}\, e^{3\pi i/4} = \dfrac{e^{-\pi i/4}}{4a^3}$

$\therefore$ Sum of residues $= -\dfrac{1}{2a^3}\left[\dfrac{e^{i\pi/4} - e^{-i\pi/4}}{2}\right] = -\dfrac{1}{2a^3}\, i \sin\dfrac{\pi}{4} = -\dfrac{i}{2\sqrt{2}\, a^3} = \sum R^+$.

$\therefore$ From (1),

$$\int_{-R}^{R} \frac{dx}{x^4 + a^4} + \int_{C_R} \frac{dz}{z^4 + a^4} = 2\pi i\left(\frac{-i}{2\sqrt{2}\, a^3}\right) = \frac{\pi\sqrt{2}}{2a^3} \qquad ...(2)$$

Now, $$\left| \int_{C_R} \frac{1}{z^4 + a^4}\, dz \right| \le \int_{C_R} \frac{|dz|}{|z^4 + a^4|} \le \int_{C_R} \frac{|dz|}{|z|^4 - |a^4|}$$

$$= \int_0^{\pi} \frac{R\,d\theta}{R^4 - a^4} \qquad | \because |z| = R \text{ on } C_R$$

$$= \frac{\pi R}{R^4 - a^4} \to 0 \text{ as } R \to \infty.$$

Hence when $R \to \infty$, relation (2) becomes

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{2a^3} \quad \text{or} \quad \int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3}.$$

(*ii*) Consider the integral $\int_C f(z)\, dz$ where $f(z) = \dfrac{1}{1 + z^4}$ taken round a closed contour C, consisting of the semi-circle C_R which is upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

The poles of $f(z) = \dfrac{1}{z^4 + 1}$ are obtained by solving $z^4 + 1 = 0$.

Now $\quad z^4 + 1 = 0$

$\Rightarrow \quad z = (-1)^{1/4} = (\cos \pi + i \sin \pi)^{1/4} = [\cos(2n\pi + \pi) + i \sin(2n\pi + \pi)]^{1/4}$

$$= \cos\frac{(2n+1)\pi}{4} + i \sin\frac{(2n+1)\pi}{4} \quad \text{where } n = 0, 1, 2, 3.$$

| By De Moivre's theorem

When $n = 0$, $\quad z = \cos\dfrac{\pi}{4} + i \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}$

When $n = 1$, $z = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

When $n = 2$, $z = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

When $n = 3$, $z = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

Of these, only the poles corresponding to $n = 0, 1$, *viz*, $z = e^{i\frac{\pi}{4}}$ and $z = e^{3i\frac{\pi}{4}}$ lie in the upper half of z-plane.

Residue of $f(z)$ at $z = e^{i\frac{\pi}{4}}$ is $\underset{z \to e^{i\frac{\pi}{4}}}{\text{Lt}} \frac{z - e^{i\frac{\pi}{4}}}{z^4 + 1}$ | Form $\frac{0}{0}$

$$= \underset{z \to e^{i\frac{\pi}{4}}}{\text{Lt}} \frac{1}{4z^3} \quad \text{| By L' Hospital's rule}$$

$$= \frac{1}{4e^{3i\frac{\pi}{4}}} = \frac{1}{4}e^{-3i\frac{\pi}{4}}$$

Similarly, residue of $f(z)$ at $z = e^{3i\frac{\pi}{4}}$ is $\frac{1}{4}e^{-9i\frac{\pi}{4}}$

Sum of residues $= \frac{1}{4}\left(e^{-3i\pi/4} + e^{-9i\pi/4}\right)$

$$= \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right]$$

$$= \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \frac{-i\sqrt{2}}{4}$$

By Cauchy Residue theorem

$$\int_{-R}^{R} \frac{dx}{1+x^4} + \int_{C_R} \frac{dz}{1+z^4} = 2\pi i\left(\frac{-i\sqrt{2}}{4}\right) = \frac{\pi\sqrt{2}}{2}$$

Taking Limit $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} + \underset{R\to\infty}{\text{Lt}} \int_{C_R} \frac{dz}{1+z^4} = \frac{\pi\sqrt{2}}{2} \qquad \text{...(1)}$$

Now, $\left|\int_{C_R} \frac{dz}{1+z^4}\right| \le \int_{C_R} \frac{|dz|}{|z^4+1|} \le \int_{C_R} \frac{|dz|}{|z|^4 - 1}$

$$= \frac{R}{R^4 - 1}\int_0^{\pi} d\theta \qquad \text{| } \because \ |z| = R \text{ on } C_R$$

$$= \frac{\pi R}{R^4 - 1} \to 0 \ \text{ as } \ R \to \infty$$

$\therefore$ From (1), $\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{2}$ or $\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{4}$.

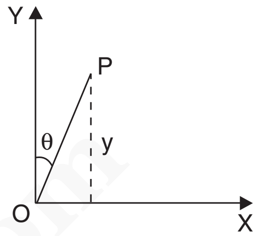
Note. The above method can also be applied to some cases where $f(x)$ contains trigonometric functions also.

1.42.2. (a) Jordan's Inequality

Consider the relation $y = \cos\theta$. As θ increases, $\cos\theta$ decreases and therefore y decreases.

The mean ordinate between 0 and $\theta = \frac{1}{\theta}\int_0^{\theta} \cos\theta \, d\theta = \frac{\sin\theta}{\theta}$

when $\theta = 0$, ordinate is cos 0 *i.e.* 1



when $\theta = \frac{\pi}{2}$, mean ordinate is $\frac{\sin \pi/2}{\pi/2}$ *i.e.* $\frac{2}{\pi}$

Hence, when $0 < \theta < \pi/2$,

Mean ordinate lies between 1 and $\frac{2}{\pi}$

i.e.,
$$\frac{2}{\pi} < \frac{\sin\theta}{\theta} < 1$$

This is known as **Jordan's Inequality.**

1.42.2. (b) Jordan's Lemma

If $f(z) \to 0$ uniformly as $|z| \to \infty$, then $\underset{R\to\infty}{\text{Lt}} \int_{C_R} e^{imz} f(z)\, dz = 0$, $(m > 0)$

where C_R denotes the semi-circle $|z| = R$, $I(z) > 0$.

Example 5. *Apply calculus of residues to evaluate*

$$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx, \; a > 0. \qquad \text{(G.B.T.U. 2010)}$$

Sol. Consider the integral $\int_C f(z)\, dz$ where $f(z) = \frac{z e^{iz}}{z^2 + a^2}$ taken round a closed contour C, consisting of a semi-circle C_R which is upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

For poles, $z^2 + a^2 = 0$

$\Rightarrow$ $z = \pm ai$

$z = ai$ is the only pole which lie inside C.

$\therefore$ Residue of $f(z)$ at $(z = ai) = \lim_{z \to ai} (z - ai) . \frac{z e^{iz}}{(z - ai)(z + ai)} = \frac{ai e^{-a}}{2ai} = \frac{e^{-a}}{2}$

$\therefore$ By Cauchy Residue theorem,

$$\int_{-R}^{R} \frac{x e^{ix}}{x^2 + a^2}\, dx + \int_{C_R} \frac{z e^{iz}}{z^2 + a^2}\, dz = 2\pi i \left(\frac{e^{-a}}{2}\right) = \pi i e^{-a}$$

Taking limit as $R \to \infty$,

$$\underset{R\to\infty}{\text{Lt}} \int_{-R}^{R} \frac{x e^{ix}}{x^2 + a^2}\, dx + \underset{R\to\infty}{\text{Lt}} \int_{C_R} \frac{z e^{iz}}{z^2 + a^2}\, dz = \pi i e^{-a}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2}\, dx + \underset{R\to\infty}{\text{Lt}} \int_{C_R} \frac{z e^{iz}}{z^2 + a^2}\, dz = \pi i e^{-a} \qquad ...(1)$$

Since $\frac{z}{z^2 + a^2} \to 0$ as $|z| \to \infty$, therefore by Jordan's Lemma,

$$\operatorname*{Lt}_{R \to \infty} \int_{C_R} \frac{ze^{iz}}{z^2 + a^2}\, dz = 0$$

$\therefore$ From (1), $\displaystyle\int_{-\infty}^{\infty} \frac{x\,e^{ix}}{x^2 + a^2}\, dx = \pi i\, e^{-a}$

Comparing imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx = \pi e^{-a}$$

or
$$\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx = \frac{\pi}{2}\, e^{-a}.$$

Example 6. *Evaluate by using Contour integration* $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2 + 1}\, dx\ ;\ a \geq 0.$

(U.P.T.U. 2006, G.B.T.U. 2011)

Sol. Consider the integral $\displaystyle\int_C f(z)\, dz$ where $f(z) = \dfrac{e^{iaz}}{z^2 + 1}$ taken round a closed contour C, consisting of a semi-circle C_R which is upper half of a large circle $|z| = R$ and the part of real axis from – R to R.

For poles, $z^2 + 1 = 0$

$\Rightarrow$ $z = \pm i$

$z = i$ is the only pole which lies inside C.

$\therefore$ Res. $(z = i) = \displaystyle\operatorname*{Lt}_{z \to i}\, (z - i) \cdot \frac{e^{iaz}}{(z - i)(z + i)} = \frac{e^{-a}}{2i}$

$\therefore$ By Cauchy Residue theorem,

$$\int_{-R}^{R} \frac{e^{iax}}{x^2 + 1}\, dx + \int_{C_R} \frac{e^{iaz}}{z^2 + 1}\, dz = 2\pi i \left(\frac{e^{-a}}{2i}\right) = \pi e^{-a}$$

Taking Limit as $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + 1}\, dx + \operatorname*{Lt}_{R \to \infty} \int_{C_R} \frac{e^{iaz}}{z^2 + 1}\, dz = \pi\, e^{-a} \qquad \text{...(1)}$$

Since $\dfrac{1}{z^2 + 1} \to 0$ as $|z| \to \infty$, therefore by Jordan's Lemma,

$$\operatorname*{Lt}_{R \to \infty} \int_{C_R} \frac{e^{iaz}}{z^2 + 1}\, dz = 0 \qquad (a > 0)$$

$\therefore$ From (1), $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + 1}\, dx = \pi\, e^{-a}$

Equating real parts, we get

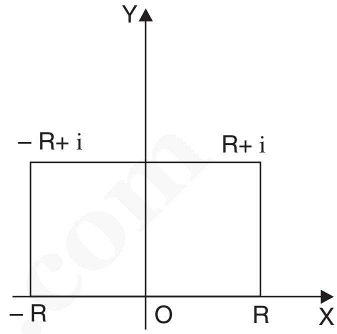
$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + 1}\, dx = \pi e^{-a}$$

or
$$\int_{0}^{\infty} \frac{\cos ax}{x^2 + 1}\, dx = \frac{\pi\, e^{-a}}{2}.$$

Example 7. *Apply calculus of residues to prove that*

$$\int_0^\infty \frac{\cosh ax}{\cosh \pi x}\, dx = \frac{1}{2} \sec \frac{a}{2}, \ -\pi < a < \pi. \qquad \textit{(M.T.U. 2014)}$$

Sol. Consider $\int_c f(z)\, dz$ where $f(z) = \dfrac{e^{az}}{\cosh \pi z}$, c is the rectangle with vertices at $-$ R, R, R + i and $-$ R + i.



$f(z)$ has simple poles given by

$$\cosh \pi z = 0$$

or $$e^{\pi z} + e^{-\pi z} = 0$$

or $$e^{\pi z} = -e^{-\pi z} = e^{(2n+1)\pi i - \pi z}$$

whence, $z = \dfrac{(2n+1)i}{2}$; $n = 0, \pm 1, \pm 2$. Of these poles, only $z = \dfrac{i}{2}$ lies inside c.

$$\text{Residue}\left(\text{at } z = \frac{i}{2}\right) = \left[\frac{e^{az}}{\dfrac{d}{dz}(\cosh \pi z)}\right]_{z=\frac{i}{2}}$$

$$= \frac{e^{ia/2}}{\pi \sinh \dfrac{i\pi}{2}} = \frac{e^{ia/2}}{\pi i \sin \dfrac{\pi}{2}} = \frac{1}{\pi i} e^{ia/2}$$

By residue theorem, we get

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_0^1 f(R+iy)\,.\, idy + \int_R^{-R} f(x+i)\, dx + \int_1^0 f(-R+iy)\,.\, idy$$

$$= 2\pi i \,.\, \frac{1}{\pi i} e^{ia/2} = 2e^{ia/2} \qquad \text{...(1)}$$

or $$I_1 + I_2 + I_3 + I_4 = 2e^{ia/2}.$$

Now, $$|\, I_2 \,| = \left| \int_0^1 \frac{e^{a(R+iy)}}{\cosh \pi (R+iy)}\, idy \right|$$

$$\le \int_0^1 \frac{2e^{aR} |e^{aiy}|\,|i|\, dy}{|e^{\pi(R+iy)} + e^{-\pi(R+iy)}|}$$

$$= \int_0^1 \frac{2e^{aR}\, dy}{e^{\pi R} - e^{-\pi R}} = \frac{2e^{aR}}{e^{\pi R} - e^{-\pi R}} \to 0 \text{ as } R \to \infty \quad |\text{ since } -\pi < a < \pi.$$

In the same way $I_4 \to 0$. Hence when R $\to \infty$, we get from (1),

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx + \int_{\infty}^{-\infty} \frac{e^{a(x+i)}}{\cosh \pi (x+i)}\, dx = 2e^{ia/2}$$

or $$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx - \int_{-\infty}^{\infty} \frac{e^{ax}\,.\, e^{ai}}{-\cosh \pi x}\, dx = 2e^{ia/2} \qquad [\because \ \cosh \pi(x+i) = -\cosh \pi x]$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} \frac{(1+e^{ia})\,.\, e^{ax}}{\cosh \pi x}\, dx = 2e^{ia/2}$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} \frac{(e^{ia/2} + e^{-ia/2})\, e^{ax}}{\cosh \pi x}\, dx = 2$$

or $$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx = \frac{1}{\cos \frac{a}{2}}$$

or $$\int_{-\infty}^{0} \frac{e^{ax}}{\cosh \pi x}\, dx + \int_{0}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx = \frac{1}{\cos \frac{a}{2}} \qquad ...(2)$$

Putting $x = -t$ in the first integral, we get

$$\int_{-\infty}^{0} \frac{e^{ax}}{\cosh \pi x}\, dx = -\int_{\infty}^{0} \frac{e^{-at}}{\cosh \pi t}\, dt = \int_{0}^{\infty} \frac{e^{-ax}}{\cosh \pi x}\, dx$$

$\therefore$ From (2),

$$\int_{0}^{\infty} \frac{e^{ax} + e^{-ax}}{\cosh \pi x}\, dx = \frac{1}{\cos \frac{a}{2}}$$

or $$\int_{0}^{\infty} \frac{\cosh ax}{\cosh \pi x}\, dx = \frac{1}{2\cos \frac{a}{2}} = \frac{1}{2} \sec \frac{a}{2}.$$

Example 8. *Using contour integration, prove that:* $\displaystyle\int_0^{\infty} \frac{\log (1+x^2)}{1+x^2}\, dx = \pi \log_e 2.$

Sol. Consider the integral $\displaystyle\int_C f(z)\, dz$ where $f(z) = \dfrac{\log (z+i)}{z^2+1}$ taken round a closed contour C which consists of semi-circle C_R, the upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

For poles, $z^2 + 1 = 0 \Rightarrow z = \pm i$

Only the pole $z = i$ lies inside C.

$$\text{Res. } (z = i) = \operatorname*{Lt}_{z \to i} (z-i) \cdot \frac{\log (z+i)}{(z-i)(z+i)} = \frac{\log (2i)}{2i} = \frac{\log 2 + i\frac{\pi}{2}}{2i}$$

$\therefore$ By Cauchy Residue theorem,

$$\operatorname*{Lt}_{R\to\infty} \int_{-R}^{R} \frac{\log (x+i)}{1+x^2}\, dx + \operatorname*{Lt}_{R\to\infty} \int_{C_R} \frac{\log (z+i)}{1+z^2}\, dz = \frac{2\pi i}{2i}\left(\log 2 + i\frac{\pi}{2}\right) = \pi\left(\log 2 + \frac{\pi i}{2}\right) \qquad ...(1)$$

Now, $$\operatorname*{Lt}_{z\to\infty} \frac{z \log (z+i)}{z^2+1} = \operatorname*{Lt}_{z\to\infty} \left[\frac{z}{z-i} \cdot \frac{\log (z+i)}{z+i}\right]$$

$$= \operatorname*{Lt}_{z\to\infty} \frac{z}{z-i} \operatorname*{Lt}_{z\to\infty} \frac{\log (z+i)}{z+i} = 1.0 = 0$$

Hence, $$\operatorname*{Lt}_{z\to\infty} \int_{C_R} \frac{z \log (z+i)}{1+z^2}\, dz = 0$$

$\Rightarrow$ $$\operatorname*{Lt}_{z\to\infty} \int_{C_R} \frac{\log (z+i)}{1+z^2}\, dz = 0$$

From (1), $$\int_{-\infty}^{\infty} \frac{\log (x+i)}{1+x^2}\, dx = \pi\left(\log 2 + \frac{i\pi}{2}\right)$$

Equating real parts, we get

$$\int_{-\infty}^{\infty} \frac{1}{2} \frac{\log (x^2+1)}{1+x^2}\, dx = \pi \log 2$$

$$\Rightarrow \qquad \int_{0}^{\infty} \frac{\log (1+x^2)}{1+x^2}\, dx = \pi \log 2.$$

ASSIGNMENT

Evaluate the following integrals using Contour integration:

1. (*i*) $\displaystyle\int_0^{\infty} \frac{dx}{1+x^2}$ (*ii*) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$

(*iii*) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\, dx$ (*G.B.T.U. 2012*) (*iv*) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\, dx\,;\ a > 0$

2. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}\,;\ a > b > 0$

3. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx$ (*U.P.T.U. 2008*)

4. $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2}\, dx$

5. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\, dx$

6. (*i*) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\, dx$ (*ii*) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+2x+2)}\, dx$

[*U.P.T.U. (C.O.) 2009*]

7. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2}\, dx$

8. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^6+1}$

9. (*i*) $\displaystyle\int_0^{\infty} \frac{\cos mx}{x^2+a^2}\, dx\,;\ (m \ge 0,\ a > 0)$ [*U.P.T.U. (C.O.) 2008*] (*ii*) $\displaystyle\int_0^{\infty} \frac{\cos mx}{(a^2+x^2)^2}\, dx\,;\ m \ge 0,\ a > 0$

10. (*i*) $\displaystyle\int_0^{\infty} \frac{x \sin ax}{x^2+k^2}\, dx\,;\ a > 0$ (*ii*) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5}\, dx$ [*U.P.T.U. (C.O.) 2009*]

11. (*i*) $\displaystyle\int_0^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\, dx\ (a > b > 0)$ (*ii*) $\displaystyle\int_0^{\infty} \frac{\sin x}{(x^2+a^2)(x^2+b^2)}\, dx\ (a > b > 0)$

[*U.P.T.U. (C.O.) 2008*]

12. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\, dx$

13. $\displaystyle\int_0^{\infty} \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)}\, dx\,;\ (a > 0,\ b > 0)$

14. If $a > 0$, prove that

(*i*) $\displaystyle\int_{-\infty}^{\infty} \frac{a \cos x + x \sin x}{x^2+a^2}\, dx = 2\pi e^{-a}$ (*ii*) $\displaystyle\int_{-\infty}^{\infty} \frac{x \cos x - a \sin x}{x^2+a^2}\, dx = 0$

[**Hint:** Consider $f(z) = \dfrac{e^{iz}}{z-ia}$. At last multiply both N^r and D^r by $x + ia$ and separate real and imaginary parts]

15. Prove that $\int_0^\infty \frac{\cos^2 x}{(1+x^2)^2}\, dx = \frac{\pi}{2}\left(1+\frac{3}{e^2}\right)$ $\left[\textbf{Hint: } \text{Take } f(z) = \frac{1+e^{2iz}}{(1+z^2)^2} \text{ so that } f(x) = \frac{1+\cos 2x}{(1+x^2)^2}\right]$

16. By integrating e^{-z^2} round the rectangle whose vertices are 0, R, $R + ia$, ia, show that

(*i*) $\int_0^\infty e^{-x^2} \cos 2ax\, dx = \frac{e^{-a^2}}{2}\sqrt{\pi}$ and (*ii*) $\int_0^\infty e^{-x^2} \sin 2ax\, dx = e^{-a^2}\int_0^a e^{y^2}\, dy$.

17. Apply calculus of residues to prove that $\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Answers

1. (*i*) $\pi/2$ (*ii*) $3\pi/8$ (*iii*) $\frac{\pi}{2}$ (*iv*) $\frac{\pi}{8a^3}$

2. $\frac{\pi}{ab(a+b)}$ **3.** $\pi/3$ **4.** $\frac{\pi}{200}$ **5.** $\frac{5\pi}{12}$

6. (*i*) $-\frac{\pi}{5}$ (*ii*) $\frac{7\pi}{50}$ **7.** $-\frac{\pi}{27}$ **8.** $\pi/3$

9. (*i*) $\frac{\pi}{2a} e^{-ma}$ (*ii*) $\frac{\pi e^{-am}(am+1)}{4a^3}$ **10.** (*i*) $\frac{\pi}{2} e^{-ak}$ (*ii*) $-\pi e^{-2\pi}$

11. (*i*) $\frac{\pi}{2(a^2-b^2)}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$ (*ii*) 0 **12.** $-\frac{\pi}{e}\sin 2$

13. $\frac{\pi}{2}\frac{(a^2e^{-a}-b^2e^{-b})}{(a^2-b^2)}$

1.42.3. Integrals of the Type $\int_{-\infty}^{\infty} \frac{\mathbf{f(x)}}{\mathbf{F(x)}}\, \mathbf{dx}$, when F(x) has zeros on the real axis.

When the poles of $f(z)$ lie on the real axis and also within the semi-circular region, then those which lie on the real axis can be avoided by drawing small semi-circles C_r, C_r' etc. about those poles as centres and small radii r and r' in the upper half of the plane.

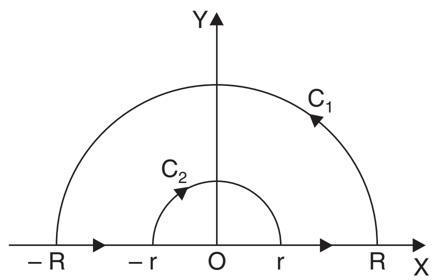
This method is said to be '**indenting the semi-circular contour**'.

When the semi-circle of radius R has been indented then $f(z)$ is analytic along this modified contour C and the integral $\int_C f(z)\, dz$ can be evaluated by Cauchy-Residue theorem.

Example. *Evaluate* $\int_0^\infty \frac{\sin mx}{x}\, dx,\ m > 0.$ [G.B.T.U. (C.O.) 2008, G.B.T.U. 2007]

Sol. Since sin mx is the imaginary part of e^{imx}, we consider the function

$$\phi(z) = \frac{e^{imz}}{z}.$$



This has a simple pole at $z = 0$, which lies on the real axis. Enclose this singularity by a small semi-circle $C_2 : |z| = r$. Evaluate the function $\phi(z)$ over the contour C shown in the figure consisting of parts of the real axis from $-R$ to $-r$ and r to R, the small semi-circle C_2 and the large semi-circle C_1. Since the function has no singularity within this contour, by Cauchy's theorem, we have

$$\oint_C \phi(z)\, dz = 0$$

$$\Rightarrow \quad \int_{-R}^{-r} \phi(x)\, dx + \int_{C_2} \phi(z)\, dz + \int_r^R \phi(x)\, dx + \int_{C_1} \phi(z)\, dz = 0$$

$$\Rightarrow \quad \int_{-R}^{-r} \frac{e^{imx}}{x}\, dx + \int_{C_2} \frac{e^{imz}}{z}\, dz + \int_r^R \frac{e^{imx}}{x}\, dx + \int_{C_1} \frac{e^{imz}}{z}\, dz = 0 \qquad ...(1)$$

Substituting $-x$ for x in the first integral and combining it with the third integral, we get

$$\int_r^R \frac{e^{imx} - e^{-imx}}{x}\, dx + \int_{C_2} \frac{e^{imz}}{z}\, dz + \int_{C_1} \frac{e^{imz}}{z}\, dz = 0$$

or

$$2i \int_r^R \frac{\sin mx}{x}\, dx + \int_{C_2} \frac{e^{imz}}{z}\, dz + \int_{C_1} \frac{e^{imz}}{z}\, dz = 0 \qquad ...(2)$$

Now $$\int_{C_2} \frac{e^{imz}}{z}\, dz = \int_{C_2} \frac{1}{z}\, dz + \int_{C_2} \frac{e^{imz} - 1}{z}\, dz \qquad ...(3)$$

On C_2, $z = re^{i\theta}$

$$\therefore \quad \int_{C_2} \frac{1}{z}\, dz = \int_\pi^0 \frac{re^{i\theta}\, i\, d\theta}{re^{i\theta}} = -\int_0^\pi i\, d\theta = -i\pi$$

Also, $$\left| \int_{C_2} \frac{e^{imz} - 1}{z}\, dz \right| \le M \int_{C_2} \frac{|dz|}{|z|} = \pi M$$

where M is the maximum value on C_2 of $| e^{imz} - 1 | = | e^{imr(\cos\theta + i\sin\theta)} - 1 |$

Clearly, $M \to 0$ as $r \to 0$

$\therefore$ From (3), $\displaystyle\int_{C_2} \frac{e^{imz}}{z}\, dz = -i\pi$

Putting $z = Re^{i\theta}$ in the integral over C_1, we get ...(4)

$$\int_{C_1} \frac{e^{imz}}{z}\, dz = \int_0^\pi \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} Re^{i\theta}\, i\, d\theta = i \int_0^\pi e^{imR\cos\theta} \,.\, e^{-mR\sin\theta}\, d\theta$$

Since, $| e^{imR\cos\theta} | \le 1$

$$\therefore \quad \left| \int_{C_1} \frac{e^{imz}}{z}\, dz \right| \le \int_0^\pi e^{-mR\sin\theta}\, d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta$$

Also, $\dfrac{\sin\theta}{\theta}$ continually decreases from 1 to $\dfrac{2}{\pi}$ as θ increases from 0 to $\dfrac{\pi}{2}$.

$\therefore$ For $0 \le \theta \le \dfrac{\pi}{2}, \dfrac{\sin\theta}{\theta} \ge \dfrac{2}{\pi}$ or $\sin\theta \ge \dfrac{2\theta}{\pi}$

$$\therefore \quad \left| \int \frac{e^{imz}}{z}\, dz \right| \le 2\int_0^{\pi/2} e^{-2mR\,\theta/\pi}\, d\theta = \left[-\frac{\pi}{mR} e^{-2mR\theta/\pi} \right]_0^{\pi/2} = \frac{\pi}{mR}(1 - e^{-mR})$$

As $R \to \infty$, $\dfrac{\pi}{mR}(1 - e^{-mR}) \to 0$

$$\therefore \quad \int_{C_1} \frac{e^{imz}}{z}\, dz = 0$$

Hence from (2), on taking the limit as $r \to 0$ and $R \to \infty$, we get

$$2i \int_0^{\infty} \frac{\sin mx}{x}\, dx - i\pi = 0$$

or

$$\int_0^{\infty} \frac{\sin mx}{x}\, dx = \frac{\pi}{2}.$$

ASSIGNMENT

Apply calculus of residues to prove that:

1. (*i*) $\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin \pi p}$, $0 < p < 1$ (*M.T.U. 2013*) (*ii*) $\int_0^{\infty} \frac{x^{a-1}}{1-x}\, dx = \pi \cot a\pi$, $0 < a < 1$

(*iii*) $\int_0^{\infty} \frac{x^{a-1}}{1+x^2} dx = \frac{\pi}{2} \operatorname{cosec}\left(\frac{\pi a}{2}\right)$, $0 < a < 2$

2. (*i*) $\int_0^{\infty} \frac{\cos x}{x}\, dx = 0$ (*ii*) $\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\, dx = \frac{\pi}{a} \sin a$, $(a > 0)$

3. (*i*) $\int_0^{\infty} \frac{\sin mx}{x(x^2 + a^2)}\, dx = \frac{\pi}{2a^2}(1 - e^{-ma})$; $a > 0$ (*ii*) $\int_0^{\infty} \frac{\sin \pi x}{x(1 - x^2)}\, dx = \pi$

4. (*i*) $\int_0^{\infty} \frac{\log x}{(1+x^2)^2}\, dx = -\frac{\pi}{4}$ (*ii*) $\int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx = \frac{\pi^2}{8}$

5. (*i*) $\int_0^{\infty} \frac{x^a}{(1+x^2)^2}\, dx = \frac{\pi}{4}(1-a)\sec\left(\frac{\pi a}{2}\right)$; $-1 < a < 3$

(*ii*) $\int_0^{\infty} \frac{x^a}{x^2 - x + 1}\, dx = \frac{2\pi}{\sqrt{3}} \sin\left(\frac{2a\pi}{3}\right) \operatorname{cosec}(a\pi)$; $-1 < a < 1$

6. $\int_0^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2}\, dx = \pi(b - a)$ if $a \geq b \geq 0$.

TEST YOUR KNOWLEDGE

1. Define analytic function and state the necessary and sufficient condition for function to be analytic. (*M.T.U. 2012*)

2. If $f(z) = u + iv$ is analytic, then show that the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are mutually orthogonal. (*M.T.U. 2012*)

3. Using the Cauchy-Riemann equations, show that $f(z) = |z|^2$ is not analytic at any point. (*M.T.U. 2013*)

4. Find the constants a, b and c such that the function $f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + y^2)$ is analytic. (*M.T.U. 2013*)

5. Evaluate $\int_0^{1+i} z^2\, dz$. (*U.P.T.U. 2008*)

6. Evaluate the integral $\int_C \frac{e^{iz}}{z^3}\, dz$ where $C : |z| = 1$. (*M.T.U. 2013*)

7. Define isolated and non-isolated singular points. (*M.T.U. 2012*)

8. Define removable and essential singular points with example. (*M.T.U. 2012*)

9. Define singular point of an analytic function. Find nature and location of the singularity of

$$f(z) = \frac{z - \sin z}{z^2}$$ (*M.T.U. 2013*)

10. Find the values of a and b for which the function $f(z) = \cos x\,(\cosh y + a \sinh y) + i \sin x\,(\cosh y + b \sinh y)$ is analytic.

11. If $f(z) = u + iv$ is an analytic function and $u = x^2 - y^2 - y$ then find its conjugate harmonic function $v(x, y)$.

12. If $f(z) = u + iv$ is an analytic function and $v = y^2 - x^2$ then find its conjugate harmonic function $u(x, y)$.

13. If $u = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$ is the real part of analytic function $f(z) = u + iv$ then find $f(z)$ in terms of z.

14. Evaluate $\oint_C \dfrac{dz}{z-2}$ around the circle $|\, z - 2 \,| = 4$.

15. Evaluate $\oint_C (5z^4 - z^3 + 2)\, dz$ around the unit circle $|\, z \,| = 1$.

16. If $F(\alpha) = \oint_C \dfrac{5z^2 - 4z + 3}{z - \alpha}\, dz$ which C is the ellipse $16x^2 + 9y^2 = 144$, then find F(2).

17. Evaluate $\oint_C \dfrac{dz}{z^2 + 9}$ where C is $|\, z - 3i \,| = 4$.

18. Find residue of $f(z) = \left(\dfrac{z+1}{z-1}\right)^3$ at $z = 1$.

19. Find residue of $f(z) = \dfrac{2z + 1}{z^2 - z - 2}$ at the pole $z = -1$. (*M.T.U. 2014*)

20. (*i*) Find residue of $f(z) = \dfrac{z^2}{(z^2 + 3z + 2)^2}$ at the pole $z = -1$.

(*ii*) Find residue of $f(z) = \dfrac{z^2}{z^2 + 3z + 2}$ at the pole -1. (*U.P.T.U. 2014*)

21. Evaluate $\oint_C \dfrac{4 - 3z}{z^2 - z}\, dz$ where C is any simple closed path such that $1 \in C$, $0 \notin C$.

22. Find the nature of singularity of $f(z) = \dfrac{z - \sin z}{z^3}$ at $z = 0$.

23. Evaluate $\oint_C \dfrac{z - 3}{z^2 + 2z + 5}\, dz$ when $C \equiv |\, z \,| = 1$.

24. Let $u(x, y) = 2x(1 - y)$ for all real x and y. Find a function $v(x, y)$ so that $f(z) = u + iv$ is analytic.

25. Let $I = \int_C \dfrac{f(z)}{(z-1)(z-2)}\, dz$ where $f(z) = \sin \dfrac{\pi z}{2} + \cos \dfrac{\pi z}{2}$ and C is the curve $|\, z \,| = 3$ oriented anti-clockwise. Find the value of I.

26. Let $\sum_{n=-\infty}^{\infty} b_n z^n$ be the Laurent's series expansion of the function $\dfrac{1}{z \sinh z}$, $0 < |\, z \,| < \pi$, then find b_{-2}, b_0 and b_2.

27. Let $f(z) = \sum_{n=0}^{15} z^n$ for $z \in \mathbb{C}$. If $\mathbb{C} : |z - i| = 2$, then evaluate $\oint_C \frac{f(z)}{(z-i)^{15}}\, dz$.

28. Let $u(x, y)$ be the real part of an entire function $f(z) = u(x, y) + i\, v(x, y)$ for $z = x + iy \in$ C. If C is the positively oriented boundary of a rectangular region R in R^2 then evaluate $\oint_C \left(\frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy \right)$.

29. Consider the function $f(z) = \frac{e^{iz}}{z(z^2+1)}$. Find the residue of f at the isolated singular point in the upper half plane $\{z = x + iy \in \mathbb{C} : y > 0\}$.

30. Let S be the positively oriented circle given by $|z - 3i| = 2$. Then evaluate $\int_S \frac{dz}{z^2+4}$.

31. Let $f(z)$ be an analytic function. Then evaluate $\int_0^{2\pi} f(e^{it}) \cos(t)\, dt$.

32. Let $f(z) = \frac{1}{z^2 - 3z + 2}$ then find the coefficient of $\frac{1}{z^3}$ in the Laurent's series expansion of $f(z)$ for $|z| > 2$.

33. If $u(x, y) = x^3y - xy^3$ is the real part of analytic function $f(z) = u(x, y) + i\, v(x, y)$, then find its conjugate harmonic function $v(x, y)$. (*M.T.U. 2014*)

34. Define Harmonic function.

Answers

4. $a = -\frac{1}{2}, b = -2, c = \frac{1}{2}$		**5.** $-\frac{2}{3} + \frac{2}{3}i$	**6.** $-\pi i$
9. removable singularity at $z = 0$		**10.** $a = -1, b = -1$	**11.** $2xy + x + c$
12. $2xy + c$	**13.** $\frac{1}{z^2} + c$	**14.** $2\pi i$	**15.** 0
16. $30\pi i$	**17.** $\pi/3$	**18.** 6	**19.** 1/3
20. (*i*) – 4 (*ii*) 1	**21.** $2\pi i$	**22.** removable singularity	
23. 0	**24.** $x^2 - (y-1)^2$	**25.** $-4\pi i$	
26. $b_{-2} = 1, b_0 = -1/6, b_2 = 7/360$		**27.** $2\pi i\,(1 + 15i)$	**28.** 0
29. $-\frac{1}{2e}$	**30.** $\frac{\pi}{2}$	**31.** $\pi f'(0)$	**32.** 3
33. $x^4 + y^4 - 6x^2y^2 + c$.			

UNIT 2

Integral Transforms

2.1 INTRODUCTION

The theory of integral transforms affords mathematical devices through which solutions of numerous boundary value problems of engineering can be obtained *e.g.* conduction of heat, transverse vibrations of a string, transverse oscillations of an elastic beam, transmission lines etc.

The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ease with which the transform can be inverted. An integral transform when applied to a partial differential equation reduces the number of its independent variables by one.

2.2 DEFINITION

The integral transform $f(p)$ of a function $F(x)$ is defined as

$$I\{F(x)\} = f(p) = \int_a^b F(x)\, K(p, x)\, dx,$$

where $K(p, x)$ is a known function of p and x, called the **kernel** of the transform : p is called the **parameter** of the transform and $F(x)$ is called the **inverse transform** of $f(p)$.

Some of the well-known transforms are given below:

(*i*) **Laplace Transform.** $K(p, x) = e^{-px}$

$$L\{F(x)\} = f(p) = \int_0^\infty F(x)\, e^{-px}\, dx$$

(*ii*) **Complex Fourier Transform.** $K(p, x) = e^{ipx}$

$$F\{F(x)\} = f(p) = \int_{-\infty}^\infty F(x)\, e^{ipx}\, dx$$

(*iii*) **Hankel Transform.** $K(p, x) = x\, J_n(px)$

$$H_n\{F(x)\} = f(p) = \int_0^\infty F(x)\, x\, J_n(px)\, dx$$

where $J_n(px)$ is the Bessel function of the first kind and of order n.

(*iv*) **Mellin Transform.** $K(p, x) = x^{p-1}$

$$M\{F(x)\} = f(p) = \int_0^\infty f(x)\, x^{p-1}\, dx.$$

Other special transforms arise when the kernel is a sine or a cosine function. These lead to Fourier sine or cosine transforms respectively.

2.3 FOURIER INTEGRAL THEOREM

Statement. *If*

(*i*) *F*(*x*) *satisfies Dirichlet's conditions in every interval* (– *c*, *c*), *however large.*

(*ii*) $\int_{-\infty}^{\infty} |F(x)|\, dx$ *converges;*

then $$F(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} F(t)\cos\lambda(t-x)\,dt\,d\lambda$$

The integral on the right hand side is called ***Fourier integral*** *of F*(*x*).

Proof. We know that Fourier series of a function F(x) in ($-c$, c) is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right) \qquad ...(1)$$

where, $$a_0 = \frac{1}{c}\int_{-c}^{c} F(t)\,dt$$

$$a_n = \frac{1}{c}\int_{-c}^{c} F(t)\cos\frac{n\pi t}{c}\,dt$$

and $$b_n = \frac{1}{c}\int_{-c}^{c} F(t)\sin\frac{n\pi t}{c}\,dt$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$F(x) = \frac{1}{2c}\int_{-c}^{c} F(t)\,dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c}\left[\cos\frac{n\pi x}{c}\cos\frac{n\pi t}{c} + \sin\frac{n\pi x}{c}\sin\frac{n\pi t}{c}\right]F(t)\,dt$$

$$= \frac{1}{2c}\int_{-c}^{c} F(t)\,dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c}\cos\frac{n\pi(t-x)}{c}\,F(t)\,dt \qquad ...(2)$$

If we assume that $\int_{-\infty}^{\infty} |F(x)|\,dx$ converges *i.e.*, F(x) is absolutely integrable on the x-axis, the first term on right side of (2) approaches 0 as $c \to \infty$ since

$$\left|\frac{1}{2c}\int_{-c}^{c} F(t)\,dt\right| \le \frac{1}{2c}\int_{-\infty}^{\infty} |F(t)|\,dt$$

From (2), $$F(x) = \lim_{c\to\infty}\frac{1}{c}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} F(t)\cos\frac{n\pi(t-x)}{c}\,dt$$

$$= \lim_{\Delta\lambda\to 0}\frac{1}{\pi}\sum_{n=1}^{\infty}(\Delta\lambda)\int_{-\infty}^{\infty} F(t)\cos\{n(\Delta\lambda)(t-x)\}\,dt \qquad ...(3) \qquad \text{where } \frac{\pi}{c} = \Delta\lambda$$

This resembles a Riemann sum of a definite integral and is of the form

$$\lim_{\Delta\lambda\to 0}\sum_{n=1}^{\infty} f(n\,\Delta\lambda) \quad i.e., \quad \int_0^{\infty} f(\lambda)\,d\lambda$$

Hence as $c \to \infty$, (3) reduces to

$$\boxed{F(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} F(t)\cos\lambda(t-x)\,dt\,d\lambda} \qquad ...(4)$$

which is known as **Fourier integral** of F(x). Eqn. (4) is true at a point of continuity.

At a point of discontinuity, the value of the integral on the right is

$$\frac{1}{2}\,[F(x+0)+F(x-0)].$$

2.4 FOURIER SINE AND COSINE INTEGRALS

We know that

$$\cos\lambda(t-x)=\cos\lambda t\cos\lambda x+\sin\lambda t\sin\lambda x$$

∴ Fourier integral of $F(x)$ can be written as

$$F(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty F(t)\{\cos\lambda t\cos\lambda x+\sin\lambda t\sin\lambda x\}\,dt\,d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\cos\lambda x\int_{-\infty}^\infty F(t)\cos\lambda t\,dt\,d\lambda+\frac{1}{\pi}\int_0^\infty\sin\lambda x\int_{-\infty}^\infty F(t)\sin\lambda t\,dt\,d\lambda \quad \text{...(1)}$$

Case I. When F(x) is an odd function

$F(t)\cos\lambda t$ is odd while $F(t)\sin\lambda t$ is even. Thus the first integral in (1) vanishes and we get

$$\boxed{F(x)=\frac{2}{\pi}\int_0^\infty\sin\lambda x\int_0^\infty F(t)\sin\lambda t\,dt\,d\lambda}$$

This is called **Fourier sine integral.**

Case II. When F(x) is an even function

$F(t)\cos\lambda t$ is even while $F(t)\sin\lambda t$ is odd. Thus, the second integral in (1) vanishes and we get

$$\boxed{F(x)=\frac{2}{\pi}\int_0^\infty\cos\lambda x\int_0^\infty F(t)\cos\lambda t\,dt\,d\lambda}$$

This is called **Fourier cosine integral.**

2.5 COMPLEX FORM OF FOURIER INTEGRAL

Fourier integral of $F(x)$ is

$$F(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty F(t)\cos\lambda(t-x)\,dt\,d\lambda \quad \text{...(1)}$$

Since $\cos\lambda(t-x)$ is an even function of λ, we have from (1),

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty F(t)\cos\lambda(t-x)\,dt\,d\lambda \quad \text{...(2)}$$

Also, since $\sin\lambda(t-x)$ is an odd function of λ, we have

$$0=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty F(t)\sin\lambda(t-x)\,dt\,d\lambda \quad \text{...(3)}$$

Multiplying (3) by i and adding to (2), we get

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty F(t)\,[\cos\lambda(t-x)+i\sin\lambda(t-x)]\,dt\,d\lambda$$

$$=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty F(t)\,.\,e^{i\lambda(t-x)}\,dt\,d\lambda$$

$$\boxed{F(x)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\lambda x}\int_{-\infty}^\infty F(t)\,.\,e^{i\lambda t}\,dt\,d\lambda}$$

which is known as the **complex form of Fourier integral.**

EXAMPLES

Example 1. *Express the function* $F(x) = \left.\begin{matrix} 1, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{matrix}\right\}$ *as a Fourier integral. Hence evaluate* $\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda$.

Sol. Fourier integral for F(x) is

$$
\begin{aligned}
F(x) &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty F(t)\cos \lambda(t-x)\, dt\, d\lambda \\
&= \frac{1}{\pi}\int_0^\infty \int_{-1}^1 \cos \lambda(t-x)\, dt\, d\lambda \qquad \Bigg|\ \because\ F(t) = \begin{cases} 1, & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin \lambda(t-x)}{\lambda}\right]_{-1}^1 d\lambda \\
&= \frac{1}{\pi}\int_0^\infty \frac{[\sin \lambda(1-x) - \sin \lambda(-1-x)]}{\lambda}\, d\lambda \\
&= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin \lambda(1+x) + \sin \lambda(1-x)}{\lambda}\right] d\lambda \\
&= \frac{2}{\pi}\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda
\end{aligned}
$$

$$
\therefore \qquad \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda = \frac{\pi}{2} F(x) = \left.\begin{matrix} \pi/2, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{matrix}\right\}
$$

At $|x| = 1$ *i.e.*, $x = \pm 1$, F(x) is discontinuous and integral has the value $\frac{1}{2}\left(\frac{\pi}{2} + 0\right) = \frac{\pi}{4}$.

Note. Putting $x = 0$, we get $\int_0^\infty \frac{\sin \lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$ or, $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

Example 2. *Using Fourier sine integral, show that*

$$
\int_0^\infty \left(\frac{1-\cos \pi\lambda}{\lambda}\right) \sin (x\lambda)\, d\lambda = \begin{cases} \pi/2, & \text{when } 0 < x < \pi \\ 0, & \text{when } x > \pi \end{cases}.
$$

Sol. Let, $\qquad F(x) = \left.\begin{matrix} \pi/2, & 0 < x < \pi \\ 0, & x > \pi \end{matrix}\right\}$

Using Fourier sine integral, we have

$$
\begin{aligned}
F(x) &= \frac{2}{\pi}\int_0^\infty \sin \lambda x \int_0^\infty F(t)\sin \lambda t\, dt\, d\lambda \\
&= \frac{2}{\pi}\int_0^\infty \sin \lambda x \left[\int_0^\pi \frac{\pi}{2} \sin \lambda t\, dt\, d\lambda\right] \\
&= \int_0^\infty \sin \lambda x \left(\frac{-\cos \lambda t}{\lambda}\right)_0^\pi d\lambda = \int_0^\pi \left(\frac{1-\cos \pi\lambda}{\lambda}\right) \sin (x\lambda)\, d\lambda
\end{aligned}
$$

$$
\therefore \qquad \int_0^\pi \left(\frac{1-\cos \pi\lambda}{\lambda}\right) \sin (x\lambda)\, d\lambda = F(x) = \begin{cases} \pi/2, & 0 < x < \pi \\ 0, & x > \pi \end{cases}.
$$

Example 3. *Using Fourier integral representation, show that*

(i) $\displaystyle\int_0^\infty \frac{\omega \sin x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x},\ x > 0$ $\qquad$ (ii) $\displaystyle\int_0^\infty \frac{\cos x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x},\ x \ge 0.$

Sol. (i) Fourier sine integral is

$$F(x) = \frac{2}{\pi}\int_0^\infty \sin \lambda x \int_0^\infty F(t) \sin \lambda t\, dt\, d\lambda$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \sin \lambda x \int_0^\infty e^{-t} \sin \lambda t\, dt\, d\lambda \qquad | \text{ Let } F(x) = e^{-x}$$

$$= \frac{2}{\pi}\int_0^\infty \sin \lambda x \left[\frac{e^{-t}}{1+\lambda^2}(-\sin \lambda t - \lambda \cos \lambda t)\right]_0^\infty d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin \lambda x \left(\frac{\lambda}{1+\lambda^2}\right) d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2}\, d\lambda$$

$$\Rightarrow \qquad \int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2}\, d\lambda = \frac{\pi}{2} e^{-x},\ x > 0$$

or,
$$\int_0^\infty \frac{\omega \sin \omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x},\ x > 0$$

(ii) Fourier cosine integral is

$$F(x) = \frac{2}{\pi}\int_0^\infty \cos \lambda x \int_0^\infty F(t) \cos \lambda t\, dt\, d\lambda$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \cos \lambda x \int_0^\infty e^{-t} \cos \lambda t\, dt\, d\lambda \qquad | \text{ Let } F(x) = e^{-x}$$

$$= \frac{2}{\pi}\int_0^\infty \cos \lambda x \left[\frac{e^{-t}}{1+\lambda^2}(-\cos \lambda t + \lambda \sin \lambda t)\right]_0^\infty d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos \lambda x \,.\left(\frac{1}{1+\lambda^2}\right) d\lambda$$

$$\Rightarrow \qquad \int_0^\infty \frac{\cos \lambda x}{1+\lambda^2}\, d\lambda = \frac{\pi}{2} e^{-x},\ x \ge 0 \qquad \text{or,} \qquad \int_0^\infty \frac{\cos \omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x},\ x \ge 0.$$

Example 4. *Using Fourier integral representation, show that*

$$\int_0^\infty \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2}\, d\omega = \begin{Bmatrix} 0, & \text{if } x < 0 \\ \pi/2, & \text{if } x = 0 \\ \pi e^{-x}, & \text{if } x > 0 \end{Bmatrix}.$$

Sol. From example 3, we have

$$\int_0^\infty \frac{\cos x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x},\ x > 0 \quad \text{and} \quad \int_0^\infty \frac{\omega \sin x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x},\ x > 0$$

Adding, we get

$$\int_0^\infty \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x} + \frac{\pi}{2} e^{-x},\ x > 0$$
$$= \pi e^{-x}, \qquad x > 0$$

when $x = 0$, $\displaystyle\int_0^{\infty} \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2}\, d\omega = \int_0^{\infty} \frac{d\omega}{1+\omega^2} = \left(\tan^{-1}\omega\right)_0^{\infty} = \frac{\pi}{2}$

when $x < 0$,

$$\int_0^{\infty} \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2}\, d\omega = \int_0^{\infty} \frac{\cos x\omega - \omega \sin x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x} - \frac{\pi}{2} e^{-x} = 0$$

Hence,

$$\int_0^{\infty} \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2}\, d\omega = \begin{Bmatrix} 0, & \text{if } x < 0 \\ \pi/2, & \text{if } x = 0 \\ \pi e^{-x}, & \text{if } x > 0 \end{Bmatrix}.$$

Example 5. *Find the complex form of the Fourier integral representation of*

$$f(x) = \begin{cases} e^{-kx}, & x > 0 \text{ and } k > 0 \\ 0, & \text{otherwise} \end{cases}$$

Sol. We know that the complex form of Fourier integral representation of $f(x)$ is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \int_{-\infty}^{\infty} f(t)\, e^{i\lambda t}\, dt\, d\lambda \quad ...(1)$$

Here, $$f(t) = \begin{cases} e^{-kt}, & x > 0 \text{ and } k > 0 \\ 0, & \text{otherwise} \end{cases}$$

$\therefore$ From (1),

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \left[\int_0^{\infty} e^{-kt}\, e^{i\lambda t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \left[\int_0^{\infty} e^{-(k-i\lambda)t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \left[\frac{e^{-(k-i\lambda)t}}{-(k-i\lambda)}\right]_0^{\infty} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{k - i\lambda}\, d\lambda.$$

ASSIGNMENT

1. Using Fourier integral representation, show that

$$e^{-ax} = \frac{2a}{\pi} \int_0^{\infty} \frac{\cos \lambda x}{\lambda^2 + a^2}\, d\lambda;\ a > 0,\ x \geq 0.$$

2. Find Fourier sine integral for $F(x) = e^{-\alpha x}$.

3. Using Fourier integral representation, show that

$$\int_0^{\infty} \frac{\lambda \sin \lambda x}{(\lambda^2 + \alpha^2)(\lambda^2 + \beta^2)}\, d\lambda = \frac{\pi}{2} \frac{(e^{-\alpha x} - e^{-\beta x})}{(\beta^2 - \alpha^2)}.$$

Hence find the Fourier sine integral representation of $e^{-x} - e^{-2x}$.

4. If $F(x) = \begin{Bmatrix} 0, & x < 0 \\ x, & 0 \leq x \leq \pi \\ 0, & x > \pi \end{Bmatrix}$, then show that

$$F(x) = \frac{1}{\pi\lambda^2} \int_0^{\infty} [\lambda\pi \sin \lambda(\pi - x) + \cos \lambda(\pi - x) - \cos \lambda x]\, d\lambda$$

5. Using Fourier integral formula, prove that

(i) $\displaystyle\int_0^\infty \left(\frac{\lambda^2+2}{\lambda^4+4}\right)\cos\lambda x\,d\lambda = \frac{\pi}{2}e^{-x}\cos x$, if $x > 0$

(ii) $\displaystyle\int_0^\infty \left(\frac{\lambda^3}{\lambda^4+4}\right)\sin\lambda x\,d\lambda = \frac{\pi}{2}e^{-x}\cos x$, if $x > 0$.

6. Using Fourier integral method, prove that

(i) $\displaystyle\int_0^\infty \left(\frac{\sin\pi\lambda}{1-\lambda^2}\right)\sin\lambda x\,d\lambda = \begin{cases}\frac{\pi}{2}\sin x, & \text{if } 0 \le x \le \pi \\ 0, & \text{if } x > \pi\end{cases}$

(ii) $\displaystyle\int_0^\infty \frac{\cos\left(\frac{\pi\lambda}{2}\right)\cos\lambda x}{1-\lambda^2}\,d\lambda = \begin{cases}\frac{\pi}{2}\cos x, & \text{if } |x| < \frac{\pi}{2} \\ 0, & \text{if } |x| > \frac{\pi}{2}\end{cases}$.

7. Find Fourier sine integral representation of

(i) $F(x) = \begin{cases}0, & 0 < x < 1 \\ k, & 1 < x < 2 \\ 0, & x > 2\end{cases}$, where k is a constant. (ii) $F(x) = x^2$, $0 \le x \le 1$.

8. Find Fourier cosine integral representation of

(i) $F(x) = \begin{cases}\sin x, & 0 \le x \le \pi \\ 0, & x > \pi\end{cases}$ (ii) $F(x) = \begin{cases}x, & 0 \le x \le 2 \\ 0, & \text{otherwise}\end{cases}$

9. Find the complex form of the Fourier integral representation of

$$F(x) = \begin{cases}0, & -\infty < x < -1 \\ x, & -1 < x < 0 \\ 0, & x > 0\end{cases}$$

10. If Fourier cosine integral of a function $F(x)$ can be represented as

$$F(x) = \int_0^\infty A(\lambda)\cos x\lambda\,d\lambda \quad \text{where} \quad A(\lambda) = \frac{2}{\pi}\int_0^\infty F(t)\cos\lambda t\,dt$$

Prove that $\displaystyle F(ax) = \frac{1}{a}\int_0^\infty A\left(\frac{\lambda}{a}\right)\cos x\lambda\,d\lambda.$ $(a > 0)$

Answers

2. $\displaystyle\frac{2}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{\alpha^2+\lambda^2}\,d\lambda$

3. $\displaystyle\frac{6}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+1)(\lambda^2+4)}\,d\lambda$

7. (i) $\displaystyle F(x) = \frac{2k}{\pi}\int_0^\infty \left(\frac{\cos\lambda - \cos 2\lambda}{\lambda}\right)\sin\lambda x\,d\lambda$

(ii) $\displaystyle F(x) = \frac{2}{\pi}\int_0^\infty \left[\left(\frac{-1}{\lambda}+\frac{2}{\lambda^3}\right)\cos\lambda + \frac{2\sin\lambda}{\lambda^2} - \frac{2}{\lambda^3}\right]\sin\lambda x\,d\lambda$

8. (i) $\displaystyle F(x) = -\frac{2}{\pi}\int_0^\infty \left(\frac{1+\cos\lambda\pi}{\lambda^2-1}\right)\cos\lambda x\,d\lambda$

(ii) $\displaystyle F(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{2\sin 2\lambda}{\lambda} + \frac{\cos 2\lambda - 1}{\lambda^2}\right)\cos\lambda x\,d\lambda$

9. $\displaystyle F(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\lambda x}\left\{\frac{1}{\lambda^2} + \left(\frac{1}{i\lambda} - \frac{1}{\lambda^2}\right)e^{-i\lambda}\right\}d\lambda.$

2.6 COMPLEX FOURIER TRANSFORM

If a function F(x) defined on the interval ($-\infty$, ∞) is piecewise continuous in each finite partial interval and absolutely integrable in ($-\infty$, ∞), then

$$f(p) = \mathrm{F}\{\mathrm{F}(x)\} = \int_{-\infty}^{\infty} \mathrm{F}(x) \,.\, e^{ipx} \, dx$$

is defined as the Fourier transform of F(x) and is denoted by $f(p)$. The function F(x) is called the inverse Fourier transform of $f(p)$.

The inverse formula for Fourier transform is given by

$$\mathrm{F}^{-1} \{f(p)\} = \mathrm{F}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(p) \,.\, e^{-ipx} \, dp.$$

Note. Some authors write the formulae as

$$f(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \,.\, \mathrm{F}(x) \, dx \quad \text{and} \quad \mathrm{F}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \,.\, f(p) \, dp.$$

2.7 FOURIER SINE TRANSFORM

The infinite Fourier sine transform of the function F(x), $0 < x < \infty$ is denoted by F_s {F(x)} or $f_s(p)$ and defined by

$$f_s(p) = \int_0^{\infty} \mathrm{F}(x) \,.\, \sin px \, dx$$

and the function F(x) is called the inverse Fourier sine transform of F_s {F(x)}.

The inverse formula is given by

$$\mathrm{F}(x) = \mathrm{F}_s^{-1} \{f_s (p)\} = \frac{2}{\pi} \int_0^{\infty} f_s (p) \,.\, \sin px \, dp.$$

2.8 FOURIER COSINE TRANSFORM

The infinite Fourier cosine transform of the function F(x). $0 < x < \infty$ is denoted by F_c{F(x)} or $f_c(p)$ and defined by

$$f_c(p) = \mathrm{F}_c\{\mathrm{F}(x)\} = \int_0^{\infty} \mathrm{F}(x) \,.\, \cos px \, dx$$

and the function F(x) is called inverse Fourier cosine transform of $f_c(p)$.

The inverse formula for infinite Fourier cosine transform is given by

$$\mathrm{F}(x) = \mathrm{F}_c^{-1} [f_c(p)] = \frac{2}{\pi} \int_0^{\infty} f_c (p) \,.\, \cos px \, dp.$$

2.9 SOME IMPORTANT RESULTS

1. $\displaystyle\int_0^{\infty} \frac{\sin mx}{x} \, dx = \frac{\pi}{2}, \ (m > 0)$

2. $\displaystyle\int_0^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$

3. $\displaystyle\int_0^{\infty} \frac{e^{ax} + e^{-ax}}{e^{\pi x} + e^{-\pi x}} \, dx = \frac{1}{2} \sec \frac{a}{2}, \ (-\pi < a < \pi)$

4. $\displaystyle\int_0^{\infty} \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} \, dx = \frac{1}{2} \tan \frac{a}{2}, \ (-\pi < a < \pi)$

5. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin rx}{(x-b)^2 + a^2} \, dx = \frac{\pi}{a} e^{-ar} \sin br, \ (r > 0).$

2.10 PROPERTIES OF FOURIER TRANSFORMS

2.10.1. Linear Property

If $f_1(p)$ and $f_2(p)$ are the Fourier transforms of $F_1(x)$ and $F_2(x)$ respectively then

$$\boxed{F\,[c_1F_1(x) + c_2F_2(x)] = c_1f_1(p) + c_2f_2(p)}$$

where c_1 and c_2 are constants.

Proof. We have,

$$F\,[c_1F_1(x) + c_2F_2(x)] = \int_{-\infty}^{\infty} [c_1F_1(x) + c_2F_2(x)]\,.\,e^{ipx}\,dx$$

$$= c_1\int_{-\infty}^{\infty} F_1(x)\,.\,e^{ipx}\,dx + c_2\int_{-\infty}^{\infty} F_2(x)\,e^{ipx}\,dx$$

$$= c_1\,F\,[F_1(x)] + c_2F[F_2(x)] = c_1f_1(p) + c_2f_2(p).$$

2.10.2. Change of Scale Property (Similarity Theorem)

If $f(p)$ is the complex Fourier transform of $F(x)$ then

$$\boxed{F\,[F(ax)] = \frac{1}{a}f\left(\frac{p}{a}\right)}\,, \qquad a \neq 0.$$

Proof.

$$F[F(ax)] = \int_{-\infty}^{\infty} e^{ipx}\,.\,F(ax)\,dx \qquad \left|\begin{array}{l}\text{Put } ax = t\\ \Rightarrow\quad dx = \dfrac{dt}{a}\end{array}\right.$$

$$= \int_{-\infty}^{\infty} e^{ip\frac{t}{a}}\,F(t)\,\frac{dt}{a} = \frac{1}{a}\int_{-\infty}^{\infty} e^{i\left(\frac{p}{a}\right)t}\,F(t)\,dt = \frac{1}{a}f\left(\frac{p}{a}\right).$$

From this property, it is evident that if the width of a function is decreased while its height is kept constant then its Fourier transform becomes wider and shorter. If its width is increased, its transform becomes narrower and taller.

Remark. If $f_s(p)$ and $f_c(p)$ are Fourier sine and cosine transforms of $F(x)$ respectively then

$$F_s\,[F(ax)] = \frac{1}{a}f_s\left(\frac{p}{a}\right) \quad \text{and} \quad F_c\,[F(ax)] = \frac{1}{a}f_c\left(\frac{p}{a}\right).$$

2.10.3. Shifting Property

If $f(p)$ is the complex Fourier transform of $F(x)$ then

$$\boxed{F\,[F(x-a)] = e^{iap}\,f(p)}$$

Proof.

$$F\,[F(x-a)] = \int_{-\infty}^{\infty} F(x-a)\,.\,e^{ipx}\,dx \qquad \left|\begin{array}{ll}\text{Put} & x - a = u\\ \therefore & dx = du\end{array}\right.$$

$$= \int_{-\infty}^{\infty} F(u)\,e^{ip(u+a)}\,du = e^{iap}\int_{-\infty}^{\infty} F(u)\,e^{ipu}\,du = e^{iap}\,f(p).$$

2.10.4. Modulation Theorem

If $f(p)$ is the complex Fourier transform of $F(x)$ then

$$\boxed{F\,[F(x)\cos ax] = \frac{1}{2}\,[f(p+a) + f(p-a)]}$$

Proof.
$$\text{F}\,[\text{F}(x)\cos ax] = \int_{-\infty}^{\infty} e^{ipx}\,.\,\text{F}(x)\cos ax\,dx$$
$$= \int_{-\infty}^{\infty} e^{ipx}\,.\,\text{F}(x)\,.\left(\frac{e^{iax}+e^{-iax}}{2}\right)dx$$
$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(p+a)x}\,.\,\text{F}(x)\,dx + \int_{-\infty}^{\infty} e^{i(p-a)x}\,.\,\text{F}(x)\,dx\right]$$
$$= \frac{1}{2}\,[f(p+a)+f(p-a)].$$

This result has application in radio and television where the harmonic carrier wave is modulated by an envelope.

Note. If $f_s(p)$ and $f_c(p)$ are Fourier sine and cosine transforms of F(x) respectively then

(i) $\text{F}_s\,[\text{F}(x)\cos ax] = \frac{1}{2}\,[f_s(p+a)+f_s(p-a)]$

(ii) $\text{F}_s\,[\text{F}(x)\sin ax] = \frac{1}{2}\,[f_c(p-a)-f_c(p+a)]$

(iii) $\text{F}_c\,[\text{F}(x)\sin ax] = \frac{1}{2}\,[f_s(p+a)-f_s(p-a)]$.

2.10.5. Convolution Theorem [*M.T.U. 2014, U.P.T.U. 2010*]

The convolution of two functions F(x) and G(x) over the interval $(-\infty, \infty)$ is defined as

$$\text{F} * \text{G} = \int_{-\infty}^{\infty} \text{F}(u)\,.\,\text{G}(x-u)\,du = \text{H}(x).$$

Statement. *The Fourier transform of the convolution of F(x) and G(x) is the product of their Fourier transform i.e.,*

$$\boxed{F\{F(x) * G(x)\} = F\{F(x)\}\,.\,F\{G(x)\}}$$

Proof. We have,

$$\text{F}\{\text{F}(x) * \text{G}(x)\} = \text{F}\left\{\int_{-\infty}^{\infty} \text{F}(u)\,.\,\text{G}(x-u)\,du\right\}$$
$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} \text{F}(u)\,.\,\text{G}(x-u)\,du\right\} e^{ipx}\,dx$$
$$= \int_{-\infty}^{\infty} \text{F}(u)\left\{\int_{-\infty}^{\infty} \text{G}(x-u)\,.\,e^{ipx}\,dx\right\} du \qquad \Big|\ \text{on changing the order of integration}$$
$$= \int_{-\infty}^{\infty} \text{F}(u)\,.\left\{\int_{-\infty}^{\infty} e^{ip(x-u)}\,.\,\text{G}(x-u)\,d(x-u)\right\} e^{ipu}\,du$$
$$= \int_{-\infty}^{\infty} e^{ipu}\,\text{F}(u)\left\{\int_{-\infty}^{\infty} e^{ipt}\,\text{G}(t)\,dt\right\} du \qquad \Big|\ \text{where } x-u=t \text{ (say)}$$
$$= \int_{-\infty}^{\infty} e^{ipu}\,\text{F}(u)\,du\,.\,\text{F}\{\text{G}(t)\}$$
$$= \int_{-\infty}^{\infty} e^{ipx}\,.\,\text{F}(x)\,dx\,.\,\text{F}\{\text{G}(x)\} = \text{F}\{\text{F}(x)\}\,.\,\text{F}\{\text{G}(x)\}.$$

2.10.6. If $F_s[F(x)] = f_s(p)$ and $F_c[F(x)] = f_c(p)$, then

(*i*) $F_s\,[x\,F(x)] = -\dfrac{d}{dp}\,[f_c(p)]$ (*ii*) $F_c\,[x\,F(x)] = \dfrac{d}{dp}\,[f_s(p)]$.

Proof. (*i*)
$$f_c(p) = \int_0^\infty F(x)\cos px\,dx$$
$$\frac{d}{dp}\{f_c(p)\} = -\int_0^\infty F(x)\,.\,x\sin px\,dx = -F_s\,[x\,F(x)]$$
$$\Rightarrow \qquad \boxed{F_s\,[x\,F(x)] = -\frac{d}{dp}\,[f_c(p)]}$$

(*ii*)
$$f_s(p) = \int_0^\infty F(x)\,.\sin px\,dx$$
$$\frac{d}{dp}\{f_s(p)\} = \int_0^\infty F(x)\,.\,x\cos px\,dx = F_c\,[x\,F(x)]$$
$$\Rightarrow \qquad \boxed{F_c\,[x\,F(x)] = \frac{d}{dp}\,[f_s(p)]}$$

EXAMPLES

Example 1. *Find the Fourier transform of following functions:*

(*i*) $F(x) = \left.\begin{array}{ll} e^{i\omega x}, & \text{for } a < x < b \\ 0, & \text{otherwise} \end{array}\right\}$ (*ii*) $F(x) = \left.\begin{array}{ll} \dfrac{1}{2\varepsilon}, & |x| \le \varepsilon \\ 0\,, & x > \varepsilon \end{array}\right\}$

(*iii*) $F(x) = e^{-|x|}$ (*iv*) $F(x) = e^{-a|x|}$, $a > 0$

(*v*) $F(t) = \left.\begin{array}{ll} t, & \text{for } |t| < a \\ 0, & \text{for } |t| > a \end{array}\right\}$. (U.P.T.U. 2008, 2009)

Sol. (*i*)
$$f(p) = \int_{-\infty}^{\infty} F(x)\,.\,e^{ipx}\,dx = \int_a^b e^{i\omega x}\,.\,e^{ipx}\,dx$$
$$= \left[\frac{e^{i(\omega+p)x}}{i(\omega+p)}\right]_a^b = \frac{e^{i(\omega+p)b} - e^{i(\omega+p)a}}{i(\omega+p)}.$$

(*ii*)
$$f(p) = \int_{-\infty}^{\infty} F(x)\,.\,e^{ipx}\,dx$$
$$= \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\,e^{ipx}\,dx = \frac{1}{2\in}\int_{-\varepsilon}^{\varepsilon} (\cos px + i\sin px)\,dx$$
$$= \frac{1}{2\varepsilon}\,.\,2\int_0^\varepsilon \cos px\,dx = \frac{1}{\varepsilon}\left(\frac{\sin px}{p}\right)_0^\varepsilon = \frac{\sin p\varepsilon}{p\varepsilon}.$$

(*iii*)
$$f(p) = \int_{-\infty}^{\infty} F(x)\,e^{ipx}\,dx = \int_{-\infty}^{\infty} e^{-|x|}\,e^{ipx}\,dx$$
$$= \int_{-\infty}^{0} e^{x}\,e^{ipx}\,dx + \int_0^\infty e^{-x}\,e^{ipx}\,dx$$

$$= \left[\frac{e^{(1+ip)x}}{1+ip}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-ip)x}}{-(1-ip)}\right]_{0}^{\infty} = \frac{1}{1+ip} + \frac{1}{1-ip} = \frac{2}{1+p^2}.$$

(*iv*) We have just proved that

$$F\{e^{-|x|}\} = \frac{2}{1+p^2}$$

using change of scale property, we get

$$F\{e^{-a|x|}\} = \frac{1}{a}\left[\frac{2}{1+\left(\frac{p}{a}\right)^2}\right] = \frac{2a}{p^2+a^2}.$$

(*v*)
$$f(p) = \int_{-\infty}^{\infty} F(t) \,.\, e^{ipt}\, dt = \int_{-a}^{a} t\, e^{ipt}\, dt$$

$$= \int_{-a}^{a} t\,(\cos pt + i \sin pt)\, dt = 2i \int_{0}^{a} t \sin pt\, dt$$

$$= 2i\left[\left\{t\,.\left(-\frac{\cos pt}{p}\right)\right\}_0^a - \int_0^a 1.\left(-\frac{\cos pt}{p}\right) dt\right]$$

$$= 2i\left[-\frac{a}{p}\cos ap + \frac{1}{p}\left(\frac{\sin pt}{p}\right)_0^a\right]$$

$$= 2i\left[-\frac{a}{p}\cos ap + \frac{1}{p^2}\sin ap\right] = \frac{2i}{p^2}(\sin ap - ap \cos ap).$$

Example 2. *Find the Fourier transform of the function*

(*i*) $F(x) = xe^{-a|x|},\ a > 0$ (*ii*) $F(x) = \dfrac{\sin ax}{x},\ a > 0$

(*iii*) $F(x) = \begin{cases} 1+\dfrac{x}{a}, & \text{for } -a < x < 0 \\ 1-\dfrac{x}{a}, & \text{for } 0 < x < a \\ 0, & \text{otherwise} \end{cases}.$

Sol. (*i*)
$$f(p) = \int_{-\infty}^{\infty} x\, e^{-a|x|}\, e^{ipx}\, dx$$

$$= \int_{-\infty}^{0} x\, e^{ax}\, e^{ipx}\, dx + \int_{0}^{\infty} x\, e^{-ax}\, e^{ipx}\, dx = \int_{-\infty}^{0} x\, e^{(a+ip)x}\, dx + \int_{0}^{\infty} x\, e^{-(a-ip)x}\, dx$$

$$= \left\{\frac{x\, e^{(a+ip)x}}{a+ip}\right\}_{-\infty}^{0} - \int_{-\infty}^{0} 1.\frac{e^{(a+ip)x}}{a+ip}\, dx + \left\{x\,.\frac{e^{-(a-ip)x}}{-(a-ip)}\right\}_{0}^{\infty} - \int_{0}^{\infty} 1.\frac{e^{-(a-ip)x}}{-(a-ip)}\, dx$$

$$= -\left[\frac{e^{(a+ip)x}}{(a+ip)^2}\right]_{-\infty}^{0} + \left[\frac{e^{-(a-ip)x}}{-(a-ip)^2}\right]_{0}^{\infty} = -\frac{1}{(a+ip)^2} + \frac{1}{(a-ip)^2} = \frac{4\,iap}{(a^2+p^2)^2}$$

(*ii*)
$$f(p) = \int_{-\infty}^{\infty} \frac{\sin ax}{x}\, e^{ipx}\, dx = \int_{-\infty}^{\infty} \frac{\sin ax}{x}(\cos px + i \sin px)\, dx$$

$$= 2\int_0^{\infty} \frac{\sin ax}{x}\cos px\, dx = \int_0^{\infty}\left\{\frac{\sin (a+p)\,x}{x} + \frac{\sin (a-p)\,x}{x}\right\} dx \qquad ...(1)$$

Case I. If $|p| < a$ then $a + p$ and $a - p$ both are positive and then (1) gives

$$f(p) = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Case II. If $|p| > a$ then for positive values of p, $(a + p)$ is positive whereas $(a - p)$ is negative and for negative values of p, $(a + p)$ is negative while $(a - p)$ is positive. Consequently, we get

$$f(p) = \frac{\pi}{2} - \frac{\pi}{2} \quad \text{or} \quad -\frac{\pi}{2} + \frac{\pi}{2} = 0$$

Hence, $f(p) = \begin{Bmatrix} \pi, & |p| < a \\ 0, & |p| > a \end{Bmatrix}$

(*iii*) $$f(p) = \int_{-\infty}^{\infty} F(x) \,.\, e^{ipx}\, dx = \int_{-a}^{0} \left(1 + \frac{x}{a}\right) e^{ipx}\, dx + \int_{0}^{a} \left(1 - \frac{x}{a}\right) e^{ipx}\, dx$$

$$= \int_{-a}^{a} e^{ipx}\, dx + \frac{1}{a}\int_{-a}^{0} x\, e^{ipx}\, dx - \frac{1}{a}\int_{0}^{a} x\, e^{ipx}\, dx \qquad ...(1)$$

$$= \left(\frac{e^{ipx}}{ip}\right)_{-a}^{a} + \frac{1}{a}\left[\left(x \,.\, \frac{e^{ipx}}{ip}\right)_{-a}^{0} - \int_{-a}^{0} 1 \,.\, \frac{e^{ipx}}{ip}\, dx\right] - \frac{1}{a}\left[\left(x \,.\, \frac{e^{ipx}}{ip}\right)_{0}^{a} - \int_{0}^{a} 1 \,.\, \frac{e^{ipx}}{ip}\, dx\right]$$

$$= \left(\frac{e^{ipa} - e^{-ipa}}{ip}\right) + \frac{1}{a}\left[a\frac{e^{-ipa}}{ip} - \frac{1}{ip}\left(\frac{e^{ipx}}{ip}\right)_{-a}^{0}\right] - \frac{1}{a}\left[a\frac{e^{ipa}}{ip} - \frac{1}{ip} \,.\left(\frac{e^{ipx}}{ip}\right)_{0}^{a}\right]$$

$$= \frac{1}{a}\left[\frac{1}{p^2}(1 - e^{-ipa}) - \frac{1}{p^2}(e^{ipa} - 1)\right]$$

$$= \frac{2}{ap^2} - \frac{1}{ap^2}\,[e^{-ipa} + e^{ipa}] = \frac{2}{ap^2}\,(1 - \cos ap)\,;\ p \neq 0$$

when $p = 0$,

$$f(p) = \int_{-a}^{a} 1 \,.\, dx + \frac{1}{a}\int_{-a}^{0} x\, dx - \frac{1}{a}\int_{0}^{a} x\, dx \qquad | \text{ From (1)}$$

$$= 2a + \frac{1}{a}\left(\frac{x^2}{2}\right)_{-a}^{0} - \frac{1}{a}\left(\frac{x^2}{2}\right)_{0}^{a} = 2a - \frac{a}{2} - \frac{a}{2} = 2a - a = a$$

Hence, $f(p) = \begin{Bmatrix} \dfrac{2}{ap^2}(1 - \cos ap)\,; & p \neq 0 \\ a; & p = 0 \end{Bmatrix}.$

Example 3. *Find the complex Fourier transform of dirac delta function* $\delta(x - a)$.

Sol. $$F\{\delta(x - a)\} = \int_{-\infty}^{\infty} \delta(x - a) \,.\, e^{ipx}\, dx$$

$$= \lim_{h \to 0} \int_{a}^{a+h} \frac{1}{h}\, e^{ipx}\, dx = \lim_{h \to 0} \frac{1}{h}\left(\frac{e^{ipx}}{ip}\right)_{a}^{a+h}$$

$$= \lim_{h \to 0} e^{ipa}\left(\frac{e^{iph} - 1}{iph}\right) = e^{ipa} \qquad \Big|\ \because \operatorname*{Lt}_{iph \to 0}\left(\frac{e^{iph} - 1}{iph}\right) = 1$$

Remark. For the function $\delta(t)$, $F[\delta(t)] = 1$.

Example 4. *Find the Fourier transform of* e^{-x^2}. *Hence find the Fourier transform of*

(*i*) $F(x) = e^{-ax^2}$, $(a > 0)$ (*ii*) $F(x) = e^{-x^2/2}$

(*iii*) $F(x) = e^{-4(x-3)^2}$ (*iv*) $F(x) = e^{-x^2} \cos 2x$.

Sol.
$$f(p) = \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{ipx}\, dx = \int_{-\infty}^{\infty} e^{-(x^2 - ipx)}\, dx$$
$$= \int_{-\infty}^{\infty} e^{-\left\{\left(x - \frac{ip}{2}\right)^2 + \frac{p^2}{4}\right\}}\, dx$$
$$= e^{-(p^2/4)} \int_{-\infty}^{\infty} e^{-\left(x - \frac{ip}{2}\right)^2}\, dx \qquad \left| \begin{array}{l} \text{Put } x - \frac{ip}{2} = z \\ \Rightarrow \quad dx = dz \end{array}\right.$$
$$= e^{-(p^2/4)} \int_{-\infty}^{\infty} e^{-z^2}\, dz$$

$$\Rightarrow \qquad F(e^{-x^2}) = 2e^{-(p^2/4)} \int_0^{\infty} e^{-z^2}\, dz = \sqrt{\pi}\, e^{-(p^2/4)} \qquad \text{...(1)}$$

(*i*) By change of scale property,
$$F(e^{-ax^2}) = \frac{1}{\sqrt{a}} \sqrt{\pi}\, e^{-\frac{1}{4}\left(\frac{p}{\sqrt{a}}\right)^2} = \sqrt{\frac{\pi}{a}}\, e^{-(p^2/4a)}$$

(*ii*) Comparing with the result of deduction (*i*), we get $a = \dfrac{1}{2}$

Hence, $$F(e^{-x^2/2}) = \sqrt{2\pi}\, e^{-(p^2/2)}$$

(*iii*) We have, from (1),
$$F(e^{-x^2}) = \sqrt{\pi}\, e^{-(p^2/4)}$$
$$\therefore \quad F(e^{-4x^2}) = F\{e^{-(2x)^2}\} = \frac{\sqrt{\pi}}{2} e^{-\frac{1}{4}\left(\frac{p}{2}\right)^2} = \frac{\sqrt{\pi}}{2} e^{-(p^2/16)} \qquad | \text{ By change of scale property}$$

Hence, $$F\{e^{-4(x-3)^2}\} = \frac{\sqrt{\pi}}{2} e^{3ip}\, e^{-(p^2/16)} = \frac{\sqrt{\pi}}{2} e^{[3ip - (p^2/16)]} \qquad | \text{ By shifting property}$$

(*iv*) We have, from (1),
$$F(e^{-x^2}) = \sqrt{\pi}\, e^{-(p^2/4)}$$
$$\therefore \quad F(e^{-x^2} \cos 2x) = \frac{\sqrt{\pi}}{2} \left[e^{-\frac{1}{4}(p+2)^2} + e^{-\frac{1}{4}(p-2)^2} \right]. \qquad | \text{ By Modulation theorem}$$

Example 5. *Find the inverse Fourier transform of* $f(p) = e^{-|p|y}$.

Sol.
$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|p|y} \cdot e^{-ipx}\, dp$$
$$= \frac{1}{2\pi} \left[\int_{-\infty}^{0} e^{py}\, e^{-ipx}\, dp + \int_0^{\infty} e^{-py}\, e^{-ipx}\, dp \right]$$
$$= \frac{1}{2\pi} \left[\left\{ \frac{e^{p(y-ix)}}{y - ix} \right\}_{-\infty}^{0} + \left\{ \frac{e^{-(y+ix)p}}{-(y+ix)} \right\}_0^{\infty} \right]$$
$$= \frac{1}{2\pi} \left[\frac{1}{y - ix} + \frac{1}{y + ix} \right] = \frac{y}{\pi(y^2 + x^2)}.$$

Example 6. *Find the Fourier transform of*

$$F(x) = \left.\begin{matrix} 1, & |x| < a \\ 0, & |x| > a \end{matrix}\right\}. \qquad \text{(U.P.T.U. 2014, 2010, 2008)}$$

Hence evaluate

(*i*) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ap \cos px}{p}\, dp$ (*ii*) $\displaystyle\int_{0}^{\infty} \frac{\sin p}{p}\, dp$.

Sol.
$$f(p) = \int_{-\infty}^{\infty} \mathrm{F}(x) \,.\, e^{ipx}\, dx = \int_{-a}^{a} 1 \,.\, e^{ipx}\, dx = 2\int_{0}^{a} \cos px\, dx$$
$$= \frac{2 \sin ap}{p},\ p \neq 0$$

For $p = 0$, we find $f(p) = 2a$

Taking Inverse Fourier transform of $f(p)$, we get

$$\mathrm{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p) \,.\, e^{-ipx}\, dp = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2 \sin ap}{p} e^{-ipx}\, dp$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ap\,(\cos px - i \sin px)}{p}\, dp$$

$$\Rightarrow \quad \left\{\begin{matrix} 1, & |x| < a \\ 0, & |x| > a \end{matrix}\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ap \cos px}{p}\, dp \qquad \text{| Second integral vanishes}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin ap \cos px}{p}\, dp = \left\{\begin{matrix} \pi, & |x| < a \\ 0, & |x| > a \end{matrix}\right\} \qquad ...(i)$$

Hence the first result.

Again, from (*i*), $\displaystyle 2\int_{0}^{\infty} \frac{\sin ap \cos px}{p}\, dp = \left\{\begin{matrix} \pi, & |x| < a \\ 0, & |x| > a \end{matrix}\right\}$

$$\Rightarrow \quad \int_{0}^{\infty} \frac{\sin ap \cos px}{p}\, dp = \left\{\begin{matrix} \pi/2, & |x| < a \\ 0, & |x| > a \end{matrix}\right\}$$

Putting $x = 0$ and $a = 1$, we get

$$\int_{0}^{\infty} \frac{\sin p}{p}\, dp = \frac{\pi}{2}. \qquad ...(ii)$$

Example 7. *Find the Fourier transform of* $F(x) = \begin{cases} 1 - x^2, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$

and use it to evaluate $\displaystyle\int_{0}^{\infty} \left(\frac{x \cos x - \sin x}{x^3}\right) \cos \frac{x}{2}\, dx$. (U.P.T.U. 2015)

Sol. Fourier transform of $\mathrm{F}(x)$ is given by

$$f(p) = \int_{-\infty}^{\infty} e^{ipx} \,.\, \mathrm{F}(x)\, dx = \int_{-1}^{1} e^{ipx}\,(1 - x^2)\, dx$$
$$= \int_{-1}^{1} \cos px\,(1 - x^2)\, dx + i\int_{-1}^{1} \sin px\,(1 - x^2)\, dx$$
$$= 2\int_{0}^{1} \cos px\,(1 - x^2)\, dx \qquad \text{| Using property of Definite Integral}$$

$$= 2\left[\left\{(1-x^2)\frac{\sin px}{p}\right\}_0^1 - \int_0^1 (-2x)\,.\,\frac{\sin px}{p}\,dx\right] = \frac{4}{p}\int_0^1 x \sin px\,dx$$

$$= \frac{4}{p}\left[\left\{x\left(\frac{-\cos px}{p}\right)\right\}_0^1 - \int_0^1 1\,.\left(\frac{-\cos px}{p}\right)dx\right] = \frac{4}{p}\left[\frac{-\cos p}{p} + \frac{1}{p}\left(\frac{\sin px}{p}\right)_0^1\right]$$

$$= \frac{4}{p^3}(\sin p - p\cos p)$$

By inversion formula for Fourier transform

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{p^3}(\sin p - p\cos p)\,.\,e^{-ipx}\,dp$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}(\cos px - i\sin px)\,.\left(\frac{\sin p - p\cos p}{p^3}\right)dp$$

$$= \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right).\cos px\,dp \quad | \text{ Using prop. of definite integral}$$

Put $x = \frac{1}{2}$, $\quad F\left(\frac{1}{2}\right) = \frac{4}{\pi}\int_0^{\infty}\frac{(\sin p - p\cos p)}{p^3}\,.\cos\frac{p}{2}\,dp$

$$\Rightarrow \quad \frac{3}{4} = -\frac{4}{\pi}\int_0^{\infty}\left(\frac{p\cos p - \sin p}{p^3}\right).\cos\frac{p}{2}\,dp \qquad ...(2)$$

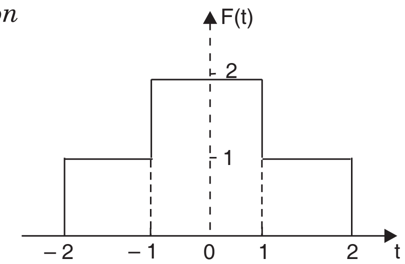
where $\quad F\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$

From eqn. (2), $\quad \int_0^{\infty}\frac{(p\cos p - \sin p)}{p^3}\,.\cos\frac{p}{2}\,dp = -\frac{3\pi}{16}$

Replacing p by x, we get

$$\int_0^{\infty}\frac{(x\cos x - \sin x)}{x^3}\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}.$$

Example 8. *Find the Fourier transform of the function shown in the adjoining figure.*



Sol. Here, $\quad F(t) = \begin{cases} 2, & \text{for } -1 < t < 1 \\ 1, & \text{for } -2 < t < -1 \\ 1, & \text{for } 1 < t < 2 \end{cases}$

Fourier transform is given by,

$$f(p) = \int_{-2}^{-1} 1\,.\,e^{ipt}\,dt + \int_{-1}^{1} 2\,.\,e^{ipt}\,dt + \int_{1}^{2} 1\,.\,e^{ipt}\,dt$$

$$= \left(\frac{e^{ipt}}{ip}\right)_{-2}^{-1} + 2\left(\frac{e^{ipt}}{ip}\right)_{-1}^{1} + \left(\frac{e^{ipt}}{ip}\right)_{1}^{2}$$

$$= \left(\frac{e^{-ip} - e^{-2ip}}{ip}\right) + 2\left(\frac{e^{ip} - e^{-ip}}{ip}\right) + \left(\frac{e^{2ip} - e^{ip}}{ip}\right)$$

$$= \left(\frac{e^{ip} - e^{-ip}}{ip}\right) + \left(\frac{e^{2ip} - e^{-2ip}}{ip}\right)$$

$$= \frac{1}{p}(2 \sin p) + \frac{1}{p} 2 \sin 2p = \frac{2}{p} \sin p\,(1 + 2 \cos p).$$

Example 9. *Find the Fourier sine transform of* $e^{-|x|}$. *Hence evaluate* $\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx$.

Sol. In the interval $(0, \infty)$, x is positive so that $e^{-|x|} = e^{-x}$.

Fourier sine transform of $f(x) = e^{-x}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin px\, dx = \int_0^\infty e^{-x} \sin px\, dx$$

$$= \left[\frac{e^{-x}}{1+p^2}(-\sin px - p \cos px)\right]_0^\infty = \frac{p}{1+p^2}$$

Using inversion formula for Fourier sine transform, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s\{f(x)\} \sin px\, dp \quad \text{or} \quad e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{p}{1+p^2} \sin px\, dp$$

Replacing x by m, we have $\quad e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{p}{1+p^2} \sin mp\, dp = \frac{2}{\pi}\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx$

Hence, $\quad \int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}.$

Example 10. *Find Fourier sine transform of* $\frac{e^{-ax}}{x}$, $a > 0$. *Hence find Fourier sine transform of* $\frac{1}{x}$. (G.B.T.U. 2011; U.P.T.U. 2008, 2015)

Sol. $\quad f_s(p) = \int_0^\infty F(x) \sin px\, dx = \int_0^\infty \frac{e^{-ax}}{x} . \sin px\, dx = I \quad \text{(say)} \qquad ...(1)$

$\therefore \quad \frac{dI}{dp} = \frac{d}{dp}\left(\int_0^\infty \frac{e^{-ax}}{x} . \sin px\, dx\right) \qquad$ | Differentiating (1) w.r.t. p

$$= \int_0^\infty \frac{e^{-ax}}{x} \frac{\partial}{\partial p}(\sin px)\, dx = \int_0^\infty \frac{e^{-ax}}{x} . x \cos px\, dx = \int_0^\infty e^{-ax} \cos px\, dx$$

$$= \left[\frac{e^{-ax}}{a^2+p^2}(-a \cos px + p \sin px)\right]_0^\infty = \frac{a}{a^2+p^2}$$

Integration w.r.t. p yields,

$$I = \tan^{-1}\left(\frac{p}{a}\right) + c \qquad ...(2)$$

Initially when $p = 0$, $I = 0 \quad \therefore \quad c = 0 \qquad$ | From (2)

$\therefore \quad$ From (2), $\quad I = \tan^{-1}\left(\frac{p}{a}\right)$

$\therefore \quad \int_0^\infty \frac{e^{-ax}}{x} . \sin px\, dx = \tan^{-1}\left(\frac{p}{a}\right)$

Take limit as $a \to 0$

$$\int_0^\infty \frac{1}{x} \sin px \, dx = \frac{\pi}{2}$$

Example 11. *Find Fourier cosine transform of the following functions:*

$$(i)\ F(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x < 1 \\ 0, & x > 1 \end{cases} \qquad \text{(U.P.T.U. 2009)}$$

$$(ii)\ F(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases} \qquad (iii)\ F(x) = \frac{e^{ax} + e^{-ax}}{e^{\pi x} + e^{-\pi x}} \ \text{ or } \ \frac{\cosh ax}{\cosh \pi x}, -\pi < a < \pi$$

$$(iv)\ F(x) = e^{-2x} + 4e^{-3x} \qquad (v)\ F(x) = \sin \frac{x^2}{2}.$$

Sol. (*i*)
$$f_c(p) = \int_0^\infty F(x) \cos px \, dx = \int_0^{1/2} x \cos px \, dx + \int_{1/2}^1 (1 - x) \cos px \, dx$$

$$= \left(\frac{x \sin px}{p}\right)_0^{1/2} - \int_0^{1/2} \frac{\sin px}{p} dx + \left\{(1-x)\frac{\sin px}{p}\right\}_{1/2}^1 - \int_{1/2}^1 (-1) \frac{\sin px}{p} dx$$

$$= \frac{1}{2p} \sin \frac{p}{2} + \left(\frac{\cos px}{p^2}\right)_0^{1/2} - \frac{1}{2p} \sin \frac{p}{2} - \frac{1}{p^2}\left(\cos px\right)_{1/2}^1$$

$$= \frac{1}{p^2}\left(\cos \frac{p}{2} - 1\right) - \frac{1}{p^2}\left(\cos p - \cos \frac{p}{2}\right)$$

$$= \frac{1}{p^2}\left(2 \cos \frac{p}{2} - 1 - \cos p\right)$$

(*ii*)
$$f_c(p) = \int_0^\infty F(x) \cos px \, dx$$

$$= \int_0^a \cos x \cos px \, dx = \frac{1}{2}\int_0^a [\cos (1 + p) x + \cos (1 - p)x] \, dx$$

$$= \frac{1}{2}\left[\frac{\sin (1+p)x}{1+p} + \frac{\sin (1-p)x}{1-p}\right]_0^a = \frac{1}{2}\left[\frac{\sin (1+p)a}{1+p} + \frac{\sin (1-p)a}{1-p}\right]$$

(*iii*)
$$f_c(p) = \int_0^\infty \left(\frac{e^{ax} + e^{-ax}}{e^{\pi x} + e^{-\pi x}}\right) \cos px \, dx = \int_0^\infty \left(\frac{e^{ax} + e^{-ax}}{e^{\pi x} + e^{-\pi x}}\right)\left(\frac{e^{ipx} + e^{-ipx}}{2}\right) dx$$

$$= \frac{1}{2}\left[\int_0^\infty \frac{e^{(a+ip)x} + e^{-(a+ip)x}}{e^{\pi x} + e^{-\pi x}} dx + \int_0^\infty \frac{e^{(a-ip)x} + e^{-(a-ip)x}}{e^{\pi x} + e^{-\pi x}} dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\sec\left(\frac{a+ip}{2}\right) + \frac{1}{2}\sec\left(\frac{a-ip}{2}\right)\right] = \frac{1}{4}\left[\frac{1}{\cos\left(\frac{a+ip}{2}\right)} + \frac{1}{\cos\left(\frac{a-ip}{2}\right)}\right]$$

$$= \frac{1}{4} \cdot \frac{2 \cos \frac{a}{2} \cos \frac{ip}{2}}{\cos\left(\frac{a+ip}{2}\right) \cos\left(\frac{a-ip}{2}\right)} = \frac{\cos \frac{a}{2} \cosh \frac{p}{2}}{\cos a + \cosh p} \qquad \Big| \because \cos ip = \cosh p$$

(*iv*) $$f_c(p) = \int_0^\infty (e^{-2x} + 4e^{-3x}) \cos px \, dx$$

$$= \int_0^\infty e^{-2x} \cos px \, dx + 4 \int_0^\infty e^{-3x} \cos px \, dx = \frac{2}{p^2+4} + \frac{12}{p^2+9}$$

(*v*) $$f_c(p) = \int_0^\infty \sin \frac{x^2}{2} \cos px \, dx = \frac{1}{2} \int_0^\infty \left\{ \sin\left(\frac{x^2}{2} + px\right) + \sin\left(\frac{x^2}{2} - px\right) \right\} dx$$

$$= \frac{1}{2} \int_0^\infty \sin\left(\frac{x^2}{2} + px\right) dx - \frac{1}{2} \int_0^{-\infty} \sin\left(\frac{x^2}{2} + px\right) dx = \frac{1}{2} \int_{-\infty}^\infty \sin\left(\frac{x^2}{2} + px\right) dx$$

$$= \text{Im. part of } \frac{1}{2} \int_{-\infty}^\infty e^{i\left(\frac{x^2}{2} + px\right)} dx = \text{Im. part of } \frac{1}{2} \int_{-\infty}^\infty e^{\frac{i}{2}(x^2+2px)} dx$$

$$= \text{Im. part of } \frac{1}{2} \int_{-\infty}^\infty e^{\frac{i}{2}\{(x+p)^2 - p^2\}} dx = \text{Im. part of } \frac{1}{2} e^{-ip^2/2} \int_{-\infty}^\infty e^{-\frac{(x+p)^2}{2i}} dx$$

$$= \text{Im. part of } \frac{1}{2} e^{-(ip^2/2)} \frac{\sqrt{\pi}}{\sqrt{\frac{1}{2i}}}$$

$$= \text{Im. part of } \sqrt{\frac{\pi}{2}} \left(\cos \frac{p^2}{2} - i \sin \frac{p^2}{2}\right)\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$$

$$= \sqrt{\frac{\pi}{2}} \left(\cos \frac{p^2}{2} - \sin \frac{p^2}{2}\right) \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{2} \left(\cos \frac{p^2}{2} - \sin \frac{p^2}{2}\right).$$

Example 12. *Find Fourier cosine transform of* $\frac{1}{1+x^2}$ *and hence find Fourier sine transform of* $\frac{x}{1+x^2}$. (M.T.U. 2014)

Sol. $$f_c(p) = \int_0^\infty \frac{1}{1+x^2} \cos px \, dx = \text{I} \quad \text{(say)} \qquad ...(1)$$

$\therefore$ $$\frac{d\text{I}}{dp} = \int_0^\infty \frac{-x \sin px}{1+x^2} dx = -\int_0^\infty \frac{(1+x^2-1)\sin px}{x(1+x^2)} dx$$

$$= -\int_0^\infty \frac{\sin px}{x} dx + \int_0^\infty \frac{\sin px}{x(1+x^2)} dx$$

$$\frac{d\text{I}}{dp} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin px}{x(1+x^2)} dx \qquad ...(2)$$

Again, $$\frac{d^2I}{dp^2} = \int_0^\infty \frac{x \cos px}{x(1+x^2)}\, dx = \int_0^\infty \frac{\cos px}{1+x^2}\, dx = I \qquad | \text{ From (1)}$$

$$\Rightarrow \quad \frac{d^2I}{dp^2} - I = 0 \qquad \text{...(3)}$$

Solution of (3) is, $$I = c_1 e^p + c_2 e^{-p} \qquad \text{...(4)}$$

$$\therefore \quad \frac{dI}{dp} = c_1 e^p - c_2 e^{-p} \qquad \text{...(5)} \quad | \text{ From (4)}$$

When $p = 0$, $$I = \int_0^\infty \frac{1}{1+x^2}\, dx = \frac{\pi}{2} \qquad | \text{ From (1)}$$

and $$\frac{dI}{dp} = -\frac{\pi}{2} \qquad | \text{ From (2)}$$

Applying to (4) and (5), we get

$$c_1 + c_2 = \pi/2 \quad \text{and} \quad c_1 - c_2 = -\pi/2$$

so that, $$c_1 = 0, \quad c_2 = \pi/2$$

$\therefore$ From (4), $$I = \frac{\pi}{2} e^{-p}$$

$$\Rightarrow \quad \int_0^\infty \frac{\cos px}{1+x^2}\, dx = \frac{\pi}{2} e^{-p}$$

Differentiating w.r.t. p, we get

$$\int_0^\infty \frac{-x \sin px}{1+x^2}\, dx = -\frac{\pi}{2} e^{-p} \quad \Rightarrow \quad \int_0^\infty \frac{x \sin px}{1+x^2}\, dx = \frac{\pi}{2} e^{-p}.$$

Example 13. *Find the Fourier cosine transform of* e^{-x^2}.

Sol. Fourier cosine transform of e^{-x^2} is given by

$$F_c\{e^{-x^2}\} = \int_0^\infty e^{-x^2} \cos px\, dx = I \text{ (say)} \qquad \text{...(1)}$$

Differentiating w.r.t. p, we have

$$\frac{dI}{dp} = -\int_0^\infty x e^{-x^2} \sin px\, dx = \frac{1}{2}\int_0^\infty (\sin px)(-2x\, e^{-x^2})\, dx$$

$$= \frac{1}{2}\left[\left\{\sin px\, e^{-x^2}\right\}_0^\infty - p\int_0^\infty \cos px\, e^{-x^2}\, dx\right] \qquad \text{(Integrating by parts)}$$

$$= -\frac{p}{2}\int_0^\infty e^{-x^2} \cos px\, dx = -\frac{p}{2} I$$

$$\frac{dI}{dp} = -\frac{p}{2}\, dp$$

Integrating, we have $\log I = -\dfrac{p^2}{4} + \log A$ or $I = A\, e^{-\frac{p^2}{4}}$ ...(2)

Now when $p = 0$, from (1), $I = \displaystyle\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

$\therefore$ From (2), $$\frac{\sqrt{\pi}}{2} = A$$

Hence, $$\text{I} = \text{F}_c\,\{e^{-x^2}\} = \frac{\sqrt{\pi}}{2}\,e^{-\frac{p^2}{4}}.$$

Example 14. *If* $f_c(p) = \frac{1}{2}\tan^{-1}\left(\frac{2}{p^2}\right)$, *then find F(x).* (M.T.U. 2014)

Sol.
$$\tan^{-1}\left(\frac{2}{p^2}\right) = \tan^{-1}\left[\frac{2}{(p^2-1)+1}\right] = \tan^{-1}\left(\frac{1}{p-1}\right) - \tan^{-1}\left(\frac{1}{p+1}\right)$$
$$\text{F}(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{2}\tan^{-1}\left(\frac{2}{p^2}\right)\cos px\,dp$$
$$= \frac{1}{\pi}\int_0^\infty \left\{\tan^{-1}\left(\frac{1}{p-1}\right) - \tan^{-1}\left(\frac{1}{p+1}\right)\right\}\cos px\,dp$$
$$= \frac{1}{\pi}\left[\int_0^\infty \tan^{-1}\left(\frac{1}{p-1}\right)\cos px\,dp - \int_0^\infty \tan^{-1}\left(\frac{1}{p+1}\right)\cos px\,dp\right]$$
$$= \text{I}_1 + \text{I}_2 \quad \text{(say)} \qquad \ldots(1)$$

where
$$\text{I}_1 = \frac{1}{\pi}\left[\left\{\tan^{-1}\left(\frac{1}{p-1}\right).\frac{\sin px}{x}\right\}_0^\infty - \int_0^\infty \frac{-1}{(p-1)^2}\cdot\frac{1}{\left\{1+\frac{1}{(p-1)^2}\right\}}\cdot\frac{\sin px}{x}\,dp\right]$$
$$= \frac{1}{\pi x}\int_0^\infty \frac{\sin px}{(p-1)^2+1}\,dp$$

Similarly, $$\text{I}_2 = \frac{-1}{\pi x}\int_0^\infty \frac{\sin px}{(p+1)^2+1}\,dp$$

From (1), $$\text{I} = \frac{1}{\pi x}\int_0^\infty \left\{\frac{\sin px}{(p-1)^2+1} - \frac{\sin px}{(p+1)^2+1}\right\}dp$$
$$= \frac{1}{2\pi x}\left[\frac{\pi}{1}e^{-x}\sin x - \frac{\pi}{1}e^{-x}\sin(-x)\right] = \frac{e^{-x}\sin x}{x}.$$

Example 15. *Solve the integral equations:*

(*i*) $\int_0^\infty f(x)\cos\lambda x\,dx = e^{-\lambda}$ (*ii*) $\int_0^\infty F(x)\sin tx\,dx = \begin{cases} 1, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}.$

Sol. (*i*) $$f(x) = \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\,d\lambda = \frac{2}{\pi}\cdot\frac{1}{1+x^2} \qquad \left|\ \because\ f_c(\lambda) = e^{-\lambda}\right.$$

(*ii*) $$\text{F}(x) = \frac{2}{\pi}\int_0^\infty f_s(t)\sin tx\,dt = \frac{2}{\pi}\left[\int_0^1 \sin tx\,dt + \int_1^2 2\sin tx\,dt\right]$$
$$= \frac{2}{\pi}\left[\left(\frac{-\cos tx}{x}\right)_0^1 - 2\left(\frac{\cos tx}{x}\right)_1^2\right]$$
$$= \frac{2}{\pi}\left[\frac{1}{x} - \frac{\cos x}{x} - 2\left(\frac{\cos 2x}{x} - \frac{\cos x}{x}\right)\right] = \frac{2}{\pi x}(1 + \cos x - 2\cos 2x).$$

Example 16. *Solve the integral equation*

$$\int_0^\infty F(x) \cos px \, dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$$

Hence prove that $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.

Sol. Let $\int_0^\infty F(x) \cos px \, dx = f_c(p)$, then $f_c(p) = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$

∴ By inversion formula for Fourier cosine transform, we have

$$F(x) = \frac{2}{\pi} \int_0^\infty f_c(p) \cos px \, dp = \frac{2}{\pi} \left[\int_0^1 (1-p) \cos px \, dp + \int_1^\infty 0 \, . \cos px \, dp \right]$$

$$= \frac{2}{\pi} \left[(1-p) \, . \frac{\sin px}{x} - (-1) \, . \frac{-\cos px}{x^2} \right]_0^1 = \frac{2}{\pi} \left[-\frac{\cos x}{x^2} + \frac{1}{x^2} \right] = \frac{2(1-\cos x)}{\pi x^2}.$$

Deduction. Since $\int_0^\infty F(x) \cos px \, dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$ where $F(x) = \frac{2(1-\cos x)}{\pi x^2}$

$$\therefore \qquad \frac{2}{\pi} \int_0^\infty \frac{1-\cos x}{x^2} \, . \cos px \, dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$$

When $p = 0$, we have $\frac{2}{\pi} \int_0^\infty \frac{1-\cos x}{x^2} dx = 1$ or $\int_0^\infty \frac{2 \sin^2 \frac{x}{2}}{x^2} dx = \frac{\pi}{2}$

Putting $x = 2t$ so that $dx = 2dt$, we get $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.

Example 17. *Taking the function* $F(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$

show that $\int_0^\infty \left(\frac{1-\cos p\pi}{p} \right) . \sin px \, dp = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$.

Sol. Consider $F(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$

Taking sine transform,

$$f_s(p) = \int_0^\infty F(x) \, . \sin px \, dx = \int_0^\pi 1 \, . \sin px \, dx = \frac{1-\cos p\pi}{p}$$

By inversion formula for Fourier sine transform.

$$\frac{2}{\pi} \int_0^\infty \left(\frac{1-\cos p\pi}{p} \right) \sin px \, dp = \begin{cases} 1, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

$$\Rightarrow \qquad \int_0^\infty \left(\frac{1-\cos p\pi}{p} \right) \sin px \, dp = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi. \end{cases}$$

Example 18. *Find Fourier sine and cosine transform of* $F(x) = x^n e^{-ax}$; $a > 0, n > -1$ *Hence find Fourier sine and cosine transforms of*

(*i*) x^{m-1} (*ii*) x^{-m}.

Sol. Here $F(x) = x^n e^{-ax}$

$$f_s(p) = \int_0^\infty x^n e^{-ax} . \sin px \, dx = \int_0^\infty x^n e^{-ax} . \left(\frac{e^{ipx} - e^{-ipx}}{2i}\right) dx$$

$$= \frac{1}{2i}\left[\int_0^\infty x^n \, e^{-(a-ip)x} \, dx - \int_0^\infty x^n \, e^{-(a+ip)x} \, dx\right]$$

$$= \frac{1}{2i}\left[\frac{\Gamma(n+1)}{(a-ip)^{n+1}} - \frac{\Gamma(n+1)}{(a+ip)^{n+1}}\right] \qquad \Bigg| \because \int_0^\infty e^{-zx} x^{n-1} dx = \frac{\Gamma(n)}{z^n}$$

Put $a = r\cos\theta$, $p = r\sin\theta$

so that, $a + ip = r(\cos\theta + i\sin\theta)$

$\therefore$ $(a + ip)^{n+1} = r^{n+1}\{\cos(n+1)\theta + i\sin(n+1)\theta\}$

and $a - ip = r(\cos\theta - i\sin\theta)$

$\therefore$ $(a - ip)^{n+1} = r^{n+1}\{\cos(n+1)\theta - i\sin(n+1)\theta\}$

Also, $r^2 = a^2 + p^2$ and $\tan\theta = \dfrac{p}{a}$

Now, $$f_s(p) = \frac{\Gamma(n+1)}{2i}\left[\frac{2i\, r^{n+1}\sin(n+1)\theta}{r^{2(n+1)}}\right]$$

$$= \frac{\Gamma(n+1)\sin(n+1)\theta}{r^{n+1}} = \frac{\Gamma(n+1)\sin(n+1)\theta}{(a^2+p^2)^{(n+1)/2}} \text{ where } \tan\theta = \frac{p}{a}$$

Similarly, $$f_c(p) = \frac{\Gamma(n+1)\cos(n+1)\theta}{(a^2+p^2)^{(n+1)/2}} \text{ where } \tan\theta = \frac{p}{a}$$

$\therefore$ We have the relations,

$$\int_0^\infty e^{-ax} . x^n \sin px \, dx = \frac{\Gamma(n+1)\sin(n+1)\theta}{(a^2+p^2)^{(n+1)/2}} \qquad \ldots(1)$$

and $$\int_0^\infty e^{-ax} x^n \cos px \, dx = \frac{\Gamma(n+1)\cos(n+1)\theta}{(a^2+p^2)^{(n+1)/2}} \qquad \ldots(2)$$

where $\tan\theta = \dfrac{p}{a}$

Put $a = 0$ and replace n by $m - 1$ in (1) and (2), we get

$$\int_0^\infty x^{m-1} \sin px \, dx = \frac{\Gamma(m)\sin m\pi/2}{p^m} \qquad \ldots(3)$$

and $$\int_0^\infty x^{m-1} \cos px \, dx = \frac{\Gamma(m)\cos m\pi/2}{p^m} \qquad \ldots(4)$$

Replace m by $m + 1$ in (3) and (4), we get

$$\int_0^\infty x^m \sin px \, dx = \frac{\Gamma(m+1)\sin(m+1)\pi/2}{p^{m+1}} = \frac{\Gamma(m+1)\cos m\pi/2}{p^{m+1}} \qquad \ldots(5)$$

and $$\int_0^\infty x^m \cos px\, dx = \frac{-\Gamma(m+1)\sin m\pi/2}{p^{m+1}} \qquad \text{...(6)}$$

Replacing m by $-m$ in (5) and (6), we get

$$\int_0^\infty x^{-m} \sin px\, dx = \frac{\Gamma(1-m)\cos m\pi/2}{p^{1-m}}$$

$$= \frac{\pi}{\sin m\pi}\cdot\frac{1}{\Gamma(m)}\cdot\frac{\cos m\pi/2}{p^{1-m}} \qquad \left|\ \because\ \Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi};\ 0 < m < 1\right.$$

$$= \frac{p^{m-1}}{\Gamma(m)}\cdot\frac{\pi\cos m\dfrac{\pi}{2}}{2\sin\dfrac{m\pi}{2}\cos m\dfrac{\pi}{2}} = \frac{\pi}{2}\cdot\frac{p^{m-1}}{\Gamma(m)}\operatorname{cosec}\frac{m\pi}{2}$$

Similarly, $\displaystyle\int_0^\infty x^{-m}\cos px\, dx = \frac{\pi}{2}\cdot\frac{p^{m-1}}{\Gamma(m)}\sec\frac{m\pi}{2}$

ASSIGNMENT

1. Find Fourier transform of $F(x) = \begin{cases} 0, & \text{for } x < a \\ 1, & \text{for } a < x < b \\ 0, & \text{for } x > b \end{cases}$.

2. (*i*) Show that the Fourier transform of

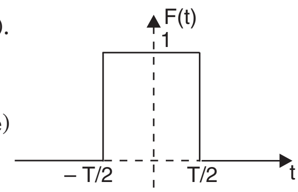
   $F(x) = \begin{cases} a - |x|, & \text{for } |x| < a \\ 0, & \text{for } |x| > a > 0 \end{cases}$ is $\dfrac{2}{p^2}$ $(1 - \cos ap)$. Hence show that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$.

   (*ii*) Find the Fourier transform of

   $F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ hence evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$. (*G.B.T.U. 2011*)

3. If $f(p)$ is the Fourier transform of $F(x)$, prove that $F[e^{iax}\,F(x)] = f(p + a)$.

4. Find the Fourier transform of the single gate function (rectangular pulse) shown in the adjoining figure.



5. Find the Fourier transform of the function $F(t)$ as shown in the figure:

   $\left[\textbf{Hint.}\quad F(t) = \begin{cases} \dfrac{At}{T}, & 0 < t < T \\ A, & T < t < 2T \end{cases}\right]$

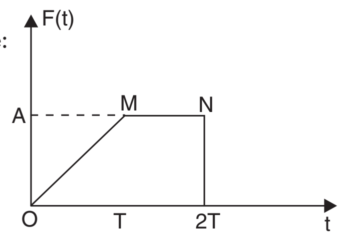


Fig. of Q. 5

6. Find Fourier transform of $F(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$.

7. Find Fourier sine and cosine transform of

   (*i*) $F(x) = e^{-x},\ x \ge 0$ (*ii*) $F(x) = 2e^{-5x} + 5e^{-2x}$

   (*iii*) $F(x) = \cosh x - \sinh x$ (*iv*) $F(x) = \begin{cases} 1, & 0 \le x < a \\ 0, & x > a \end{cases}$

   (*v*) $x\, e^{-ax},\ a > 0$.

8. Obtain Fourier cosine transform of $F(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$

Also find its Fourier sine transform.

9. Find Fourier sine transform of

(*i*) $F(x) = \begin{cases} 0, & 0 < x < a \\ x, & a \le x \le b \\ 0, & x > b \end{cases}$ (*ii*) $F(x) = \dfrac{1}{x(x^2 + a^2)}$

(*iii*) $F(x) = xe^{-x^2/2}$ (*iv*) $F(x) = \dfrac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}}$ or $\dfrac{\cosh ax}{\sinh \pi x}, -\pi < a < \pi$

(*v*) $\dfrac{e^{-ax} - e^{-bx}}{x}$

10. Find Fourier sine transform of $\dfrac{1}{e^{\pi x} - e^{-\pi x}}$ and deduce that f_s (cosec h πx) = $\dfrac{1}{2}$ tanh ($p/2$)

11. Find Fourier cosine transform of $F(x) = \operatorname{sech} \pi x$.

12. Find $F_s^{-1}\left(\dfrac{e^{-ap}}{p}\right)$ and hence evaluate $F_s^{-1}\left(\dfrac{1}{p}\right)$.

13. By taking e^{-ax} for $F(x)$; $a > 0$, $x > 0$; show that

$$\int_0^\infty \frac{\cos px}{a^2 + p^2}\, dp = \frac{\pi}{2a} e^{-ax} \quad \text{and} \quad \int_0^\infty \frac{p \sin px}{a^2 + p^2}\, dp = \frac{\pi}{2} e^{-ax}.$$

14. Find $F(x)$ if its Fourier sine transform is

(*i*) $\dfrac{p}{1 + p^2}$ (*ii*) $\dfrac{\pi}{2}$ (*iii*) $(2\pi p)^{1/2}$ (*iv*) $\begin{cases} \sin p, & 0 < p < \pi \\ 0, & p \ge \pi \end{cases}$.

15. Find Fourier cosine transform of $e^{-a^2x^2}$ and hence evaluate Fourier sine transform of $xe^{-a^2x^2}$.

16. Find Fourier sine and cosine transform of $\dfrac{1}{\sqrt{x}}$.

17. If $f(p)$ is the Fourier transform of $F(x)$, prove that $F[x^n F(x)] = (-i)^n \dfrac{d^n}{dp^n} \{f(p)\}$.

18. A certain function of time $F(t)$ has the following Fourier transform $f(p) = \dfrac{1}{p^2 + 1} e^{-\{2p^2/(p^2+1)\}}$

Using the properties of the Fourier transform, obtain the Fourier transforms of

(*i*) $F(2t)$ (*ii*) $F(t - 2)$

(*iii*) $F(t) \cos 2t$ (*iv*) $e^{2it} F(t)$.

19. State and prove the convolution theorem for the Fourier transform. Verify this theorem for the functions $f(t) = e^{-t}$ and $g(t) = \sin t$. (*U.P.T.U. 2010*)

Answers

1. $\dfrac{(e^{ibp} - e^{iap})}{ip}$ **2.** (*ii*) $\dfrac{2 \sin p}{p}, p \ne 0 ; \dfrac{\pi}{2}$

4. $T \sin c \left(\dfrac{pT}{2}\right)\left(\text{where } \sin c\, \theta = \dfrac{\sin \theta}{\theta}\right)$ **5.** $\dfrac{A}{Tp^2}(e^{ipT} - 1) + \dfrac{A}{ip} e^{2ipT}$

6. $\left(\frac{2a^2}{p}-\frac{4}{p^3}\right)\sin ap+\frac{4a}{p^2}\cos ap$

7. (*i*) $\frac{p}{1+p^2};\frac{1}{1+p^2}$ (*ii*) $p\left(\frac{2}{p^2+25}+\frac{5}{p^2+4}\right);10\left(\frac{1}{p^2+25}+\frac{1}{p^2+4}\right)$

(*iii*) $\frac{p}{1+p^2};\frac{1}{1+p^2}$ (*iv*) $\left(\frac{1-\cos ap}{p}\right);\frac{\sin ap}{p}$ (*v*) $\frac{2ap}{(a^2+p^2)^2};\frac{a^2-p^2}{(a^2+p^2)^2}$

8. $\frac{2}{p^2}\cos p\,(1-\cos p);\ \frac{2}{p^2}\sin p\,(1-\cos p)$

9. (*i*) $\frac{(a\cos ap-b\cos bp)}{p}+\frac{(\sin bp-\sin ap)}{p^2}$ (*ii*) $\frac{\pi}{2a^2}(1-e^{-ap})$

(*iii*) $\frac{\sqrt{\pi}}{2}\,p\,e^{-(p^2/2)}$ (*iv*) $\frac{\sinh p}{2(\cos a+\cosh p)}$ (*v*) $\tan^{-1}\left(\frac{p}{a}\right)-\tan^{-1}\left(\frac{p}{b}\right)$

10. $\frac{1}{4}\tanh\frac{p}{2}$ **11.** $\frac{1}{2}\operatorname{sech}\frac{p}{2}$ **12.** $\frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right);1$

14. (*i*) e^{-x} (*ii*) $\frac{1}{x}$ (*iii*) $\frac{1}{x\sqrt{x}}$ (*iv*) $\frac{2}{\pi}\cdot\frac{\sin\pi x}{(1-x^2)}$

15. $\frac{\sqrt{\pi}}{2a}e^{-(p^2/4a^2)};\ \frac{p\sqrt{\pi}}{4a^3}e^{-(p^2/4a^2)}$ **16.** $\sqrt{\frac{\pi}{2p}},\sqrt{\frac{\pi}{2p}}$

18. (*i*) $\frac{2}{p^2+4}e^{-\left(\frac{2p^2}{p^2+4}\right)}$ (*ii*) $e^{2ip}\cdot\frac{1}{p^2+1}e^{-\{2p^2/(p^2+1)\}}$

(*iii*) $\frac{1}{2}\left[\frac{1}{(p+2)^2+1}e^{-\left\{\frac{2(p+2)^2}{(p+2)^2+1}\right\}}+\frac{1}{(p-2)^2+1}e^{-\left\{\frac{2(p-2)^2}{(p-2)^2+1}\right\}}\right]$ (*iv*) $\frac{1}{(p+2)^2+1}e^{-\left\{\frac{2(p+2)^2}{(p+2)^2+1}\right\}}$.

2.11 FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

Let $\overline{u}(p,t)$ be the Fourier transform of the function $u(x, t)$.

Then $$\overline{u}(p,t)=\int_{-\infty}^{\infty}u(x,t)\,e^{ipx}\,dx$$

Suppose u and $\frac{\partial u}{\partial x}$ both vanish as $x\to\pm\infty$. Then the Fourier transform of $\frac{\partial^2 u}{\partial x^2}$ is given by

$$F\left\{\frac{\partial^2 u}{\partial x^2}\right\}=\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}e^{ipx}\,dx$$

$$=\left[\left\{e^{ipx}\frac{\partial u}{\partial x}-ipe^{ipx}\,.\,u\right\}_{-\infty}^{\infty}+\int_{-\infty}^{\infty}(ip)^2\,e^{ipx}\,.\,u\,dx\right]=-p^2\int_{-\infty}^{\infty}ue^{ipx}\,dx=-p^2\,\overline{u}$$

Hence, $$\boxed{F\left(\frac{\partial^2 u}{\partial x^2}\right)=-p^2\,\overline{u}}$$ where $\overline{u}=F(u)$

If $\overline{u}_s(p, t)$ and $\overline{u}_c(p, t)$ be the Fourier sine and cosine transforms of $u(x, t)$, then

$$\overline{u}_s(p, t) = \int_0^\infty u \sin px \, dx \quad \text{and} \quad \overline{u}(p, t) = \int_0^\infty u \cos px \, dx$$

The Fourier sine transform of $\dfrac{\partial^2 u}{\partial x^2}$ is given by

$$\mathrm{F}_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin px \, dx$$

$$= \left[\left\{\sin px \frac{\partial u}{\partial x} - p \cos px \, . \, u\right\}_0^\infty - \int_0^\infty p^2 \sin px \, . \, u \, dx\right]$$

$$= \left[p\left\{u\right\}_{x=0} - p^2 \int_0^\infty u \sin px \, dx\right] = p\left\{u\right\}_{x=0} - p^2 \overline{u}_s$$

Hence, $$\boxed{\mathrm{F}_s\left(\frac{\partial^2 u}{\partial x^2}\right) = p(u)_{x=0} - p^2 \overline{u}_s}$$ where $\overline{u}_s = \mathrm{F}_s(u)$

Similarly, we have,

$$\boxed{\mathrm{F}_c\left(\frac{\partial^2 u}{\partial x^2}\right) = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - p^2 \overline{u}_c}$$ where $\overline{u}_c = \mathrm{F}_c(u)$ *(U.P.T.U. 2008)*

In general, the Fourier transform of the n^{th} derivative of F(x) is given by

$$\boxed{\mathrm{F}\left(\frac{d^n \mathrm{F}}{dx^n}\right) = (-ip)^n \, \mathrm{F}\{\mathrm{F}(x)\}}$$

Note. s may also be used as a parameter in place of p.

2.12 CHOICE OF INFINITE FOURIER SINE OR COSINE TRANSFORM

For exclusion of $\dfrac{\partial^2 u}{\partial x^2}$ from a differential equation, we require

(*i*) $(u)_{x=0}$ in sine transform. (*ii*) $\left(\dfrac{\partial u}{\partial x}\right)_{x=0}$ in cosine transform.

2.13 APPLICATIONS OF FOURIER TRANSFORMS TO HEAT CONDUCTION (TRANSFER) EQUATIONS

In one dimensional heat transfer eqns, the partial differential equation can easily be transformed into an ordinary differential equation by applying Fourier transforms. The required solution is then obtained by solving this equation and inverting by means of the complex inversion formula. This is illustrated through the following examples.

EXAMPLES

Example 1. *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0, t > 0$ *subject to the conditions*

(*i*) $u = 0$ *when* $x = 0, t > 0$ (*ii*) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$ *when* $t = 0$

and (*iii*) $u(x, t)$ *is bounded.* (G.B.T.U. 2011; U.P.T.U. 2009, 2015)

Sol. Since $(u)_{x=0}$ is given, taking Fourier sine transform of both sides of the given equation, we have

$$\int_0^\infty \frac{\partial u}{\partial t} \sin px\, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin px\, dx$$

$$\Rightarrow \quad \frac{d}{dt}\left(\int_0^\infty u \sin px\, dx\right) = p(u)_{x=0} - p^2\, \overline{u}$$

$$\Rightarrow \quad \frac{d\overline{u}}{dt} + p^2 \overline{u} = 0 \quad \ldots(1) \qquad \text{where } \overline{u} = \int_0^\infty u \sin px\, dx$$

Solution to (1) is $\overline{u} = c_1 e^{-p^2 t}$, ...(2)

where c_1 is a constant.

Now, when $t = 0$, Fourier sine transform of $u(x, t)$

$$(\overline{u})_{t=0} = \int_0^\infty u(x, 0) \sin px\, dx = \int_0^1 1. \sin px\, dx + \int_1^\infty 0 . \sin px\, dx$$

$$= \left(\frac{-\cos px}{p}\right)_0^1 = \frac{1 - \cos p}{p}$$

$\therefore$ From (2), $(\overline{u})_{t=0} = c_1$

$$\Rightarrow \quad c_1 = \frac{1 - \cos p}{p}$$

$\therefore$ From (2), $\overline{u} = \left(\frac{1 - \cos p}{p}\right) e^{-p^2 t}$

Applying inverse Fourier sine transform, we have

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \left(\frac{1 - \cos p}{p}\right) e^{-p^2 t} \sin px\, dp$$

which is the required solution.

Example 2. *Determine the distribution of temperature in the semi-infinite medium* $x \geq 0$ *when the end* $x = 0$ *is maintained at zero temperature and the initial distribution of temperature is* $F(x)$.

Sol. Let $u(x, t)$ be the temperature at point x at any time t. Heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \; (x > 0, t > 0) \quad \ldots(1)$$

Subjected to the initial condition $u(x, 0) = F(x)$ ...(2)

and the boundary condition $u(0, t) = 0$ ...(3)

Taking Fourier sine transform of eqn. (1), we get

$$\int_0^\infty \frac{\partial u}{\partial t} \sin px \, dx = c^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin px \, dx$$

$$\Rightarrow \qquad \frac{d\overline{u}}{dt} = c^2 \,[p\,(u)_{x=0} - p^2\,\overline{u}\,] = -\,c^2p^2\overline{u} \qquad \Big|\because \; (u)_{x=0} = 0$$

$$\Rightarrow \qquad \frac{d\overline{u}}{dt} + c^2p^2\overline{u} = 0 \qquad \text{...(4)} \qquad \Big|\text{where } \overline{u} = \int_0^\infty u \sin px \, dx$$

Solution to (4) is $\quad \overline{u} = c_1 e^{-c^2p^2t}$ ...(5)

Taking Fourier sine transform of (2), we get

$$(\overline{u}\,)_{t=0} = \int_0^\infty \mathrm{F}(x) \sin px \, dx = f_s(p) \qquad |\text{ say}$$

From (5), $\quad (\overline{u}\,)_{t=0} = c_1 \;\Rightarrow\; c_1 = f_s(p)$

$\therefore$ From (5), $\quad \overline{u} = f_s(p)\, e^{-c^2p^2t}$

Now Taking its inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi} \int_0^\infty f_s(p)\, e^{-c^2p^2t} \sin px \, dp$$

Example 3. *Use Fourier sine transform to solve the equation* $\dfrac{\partial u}{\partial t} = 2\,\dfrac{\partial^2 u}{\partial x^2}$ *under the conditions*

(i) $u(0, t) = 0$ (ii) $u(x, 0) = e^{-x}$ (iii) $u(x, t)$ is bounded.

Sol. The given equation is

$$\frac{\partial u}{\partial t} = 2\,\frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

Taking Fourier sine transform on both sides of eqn. (1), we get

$$\int_0^\infty \frac{\partial u}{\partial t} \sin px \, dx = 2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin px \, dx$$

$$\Rightarrow \qquad \frac{d\overline{u}}{dt} = 2\,[p(u)_{x=0} - p^2\overline{u}\,]$$

$$\Rightarrow \qquad \frac{d\overline{u}}{dt} + 2p^2\overline{u} = 0 \qquad \Big|\text{where } \overline{u} = \int_0^\infty u \sin px \, dx$$

Its solution is $\quad \overline{u} = c_1 e^{-2p^2t}$ ...(2)

where c_1 is a constant.

At $t = 0$, $\quad (\overline{u}\,)_{t=0} = \displaystyle\int_0^\infty (u)_{t=0} \sin px \, dx = \int_0^\infty e^{-x} \sin px \, dx = \frac{p}{1+p^2}$ ...(3)

From (2), $\quad (\overline{u}\,)_{t=0} = c_1$ ...(4)

$\therefore$ From (3) and (4), $c_1 = \dfrac{p}{1+p^2}$

From (2), $\quad \overline{u} = \dfrac{p}{1+p^2}\, e^{-2p^2t}$ ...(5)

Taking inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{p}{1+p^2}\, e^{-2p^2t} \sin px \, dp.$$

Example 4. *The temperature u in the semi-infinite rod* $0 \le x < \infty$ *is determined by the differential equation* $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ *subject to conditions*

(*i*) $u = 0$ *when* $t = 0, \quad x \ge 0$ (*ii*) $\frac{\partial u}{\partial x} = -\mu$ *(a constant) when* $x = 0$ *and* $t > 0$

Making use of cosine transform, show that $u(x, t) = \frac{2\mu}{\pi} \int_0^\infty \frac{\cos px}{p^2} (1 - e^{-kp^2 t}) dp.$

Sol. Taking Fourier cosine transform on both sides of given equation, we get

$$\int_0^\infty \frac{\partial u}{\partial t} . \cos px \, dx = k \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos px \, dx$$

$$\Rightarrow \qquad \frac{d\overline{u}}{dt} = k \left[-\left(\frac{\partial u}{\partial x}\right)_{x=0} - p^2 \overline{u} \right]$$

$$= k\mu - kp^2 \overline{u} \qquad \Big| \text{ where } \overline{u} = \int_0^\infty u \cos px \, dx$$

$$\Rightarrow \qquad \frac{d\overline{u}}{dt} + kp^2 \overline{u} = k\mu \qquad \text{...(1)}$$

$$\text{I.F.} = e^{kp^2 t}$$

Solution to (1) is

$$\overline{u} . e^{kp^2 t} = \int k\mu . e^{kp^2 t} \, dt + c_1 = \frac{\mu}{p^2} e^{kp^2 t} + c_1$$

$$\Rightarrow \qquad \overline{u} = \frac{\mu}{p^2} + c_1 e^{-kp^2 t} \qquad \text{...(2)}$$

At $t = 0$, $$(\overline{u})_{t=0} = \int_0^\infty (u)_{t=0} \cos px \, dx = 0 \qquad \text{...(3)}$$

From (2), $$(\overline{u})_{t=0} = \frac{\mu}{p^2} + c_1 \qquad \text{...(4)}$$

$$\Rightarrow \qquad c_1 = -\frac{\mu}{p^2} \qquad \Big| \text{ From (3) and (4)}$$

$\therefore$ From (2), $$\overline{u} = \frac{\mu}{p^2} (1 - e^{-kp^2 t})$$

Taking inverse Fourier cosine transform, we get

$$u = \frac{2}{\pi} \int_0^\infty \frac{\mu}{p^2} (1 - e^{-kp^2 t}) \cos px \, dp.$$

Example 5. (*i*) *If the initial temperature of an infinite bar is given by*

$$\theta(x) = \begin{cases} \theta_0, & for\ |x| < a \\ 0, & for\ |x| > a \end{cases}$$

determine the temperature at any point x and at any instant t.

(*ii*) *If the initial temperature of an infinite bar is given by*

$$\mu(x, 0) = \begin{Bmatrix} 1, & for\ -c < x < c \\ 0, & otherwise \end{Bmatrix}$$

determine the temperature of an infinite bar at any point x and at any time $t > 0$.

Sol. (*i*) To determine the temperature $\theta(x, t)$, we have to solve the heat-flow equation

$$\frac{\partial \theta}{\partial t} = c^2 \frac{\partial^2 \theta}{\partial x^2}, \quad t > 0 \qquad \text{...(1)}$$

subject to the initial condition $\theta(x, 0) = \begin{cases} \theta_0, \text{ for } |x| < a \\ 0, \quad \text{for } |x| > a \end{cases}$ ...(2)

Taking Fourier transform of (1), we get

$$\int_{-\infty}^{\infty} \frac{\partial \theta}{\partial t} e^{ipx}\, dx = c^2 \int_{-\infty}^{\infty} \frac{\partial^2 \theta}{\partial x^2} e^{ipx}\, dx$$

or $$\frac{d}{dt}\int_{-\infty}^{\infty} \theta\, e^{ipx}\, dx = -c^2 p^2 \bar{\theta}$$

or $$\frac{d\bar{\theta}}{dt} = -c^2 p^2 \bar{\theta} \qquad \text{...(3)} \qquad \left|\ \text{where } \bar{\theta} = \int_{-\infty}^{\infty} \theta\, e^{ipx}\, dx\right.$$

Now taking the Fourier transform of (2), we get

$$\bar{\theta}(p, 0) = \int_{-\infty}^{\infty} \theta(x, 0)\, e^{ipx}\, dx = \int_{-a}^{a} \theta_0 e^{ipx}\, dx = \theta_0 \left[\frac{e^{ipx}}{ip}\right]_{-a}^{a}$$

$$= \theta_0 \left[\frac{e^{ipa} - e^{-ipa}}{ip}\right] = \frac{2\theta_0}{p}\left[\frac{e^{ipa} - e^{-ipa}}{2i}\right] = \frac{2\theta_0 \sin pa}{p} \qquad \text{...(4)}$$

From (3), $$\frac{d\bar{\theta}}{\bar{\theta}} = -c^2 p^2 dt$$

Integrating, $\log \bar{\theta} = -c^2 p^2 t + \log A$ or $\bar{\theta} = A\, e^{-c^2 p^2 t}$

Since $\bar{\theta} = \dfrac{2\theta_0 \sin pa}{p}$ when $t = 0$, from (4), we get $A = \dfrac{2\theta_0 \sin pa}{p}$

$\therefore$ $$\bar{\theta} = \frac{2\theta_0 \sin pa}{p} e^{-c^2 p^2 t}$$

Taking its inverse Fourier transform, we get

$$\theta(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\theta_0 \sin ap}{p} \,.\, e^{-c^2 p^2 t} \,.\, e^{-ipx}\, dp$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin ap}{p} \,.\, e^{-c^2 p^2 t} (\cos xp - i \sin xp)\, dp$$

$$= \frac{\theta_0}{\pi}\left[\int_{-\infty}^{\infty} \frac{\sin ap}{p} \,.\, e^{-c^2 p^2 t} \cos xp\, dp - i\int_{-\infty}^{\infty} \frac{\sin ap}{p} \,.\, e^{-c^2 p^2 t} \sin xp\, dp\right]$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin ap}{p} \,.\, e^{-c^2 p^2 t} \cos xp\, dp$$

(The second integral vanishes since its integrand is an odd function)

$$= \frac{2\theta_0}{\pi}\int_{0}^{\infty} \frac{\sin ap}{p} \,.\, e^{-c^2 p^2 t} \cos xp\, dp = \frac{\theta_0}{\pi}\int_{0}^{\infty} \frac{e^{-c^2 p^2 t}}{p} \,.\, 2 \sin ap \cos xp\, dp$$

$$= \frac{\theta_0}{\pi}\int_{0}^{\infty} e^{-c^2 p^2 t}\left(\frac{\sin (a + x)\, p + \sin (a - x) p}{p}\right) dp$$

$$= \frac{\theta_0}{\pi} \int_0^\infty e^{-v^2} \left\{ \sin \frac{(a+x)v}{c\sqrt{t}} + \sin \frac{(a-x)v}{c\sqrt{t}} \right\} \frac{dv}{v} \text{ where } v^2 = c^2p^2t$$

$$= \frac{\theta_0}{2} \left\{ erf\left(\frac{a+x}{2c\sqrt{t}}\right) + erf\left(\frac{a-x}{2c\sqrt{t}}\right) \right\}. \quad \left| \because \int_0^\infty e^{-t^2} \sin (ct/\sqrt{x}) . \frac{dt}{t} = \frac{\pi}{2} erf\left(\frac{c}{2\sqrt{x}}\right). \right.$$

(*ii*) To determine the temperature $\mu(x, t)$, we have to solve the heat-flow equation

$$\frac{\partial \mu}{\partial t} = k^2 \frac{\partial^2 \mu}{\partial x^2}, \; t > 0 \qquad ...(1)$$

subject to the initial condition $\mu(x, 0) = \left. \begin{matrix} 1, & \text{for } -c < x < c \\ 0, & \text{otherwise} \end{matrix} \right\}$ ...(2)

Taking Fourier transform of (1), we get

$$\int_{-\infty}^{\infty} \frac{\partial \mu}{\partial t} . e^{ipx} \, dx = k^2 \int_{-\infty}^{\infty} \frac{\partial^2 \mu}{\partial x^2} . e^{ipx} \, dx$$

$$\Rightarrow \quad \frac{d}{dt} \int_{-\infty}^{\infty} \mu \, e^{ipx} dx = -k^2p^2\bar{\mu}$$

$$\Rightarrow \quad \frac{d\bar{\mu}}{dt} = -k^2p^2\bar{\mu} \qquad \left| \text{ where } \bar{\mu} = \int_{-\infty}^{\infty} \mu \, e^{ipx} \, dx \right.$$

$$\Rightarrow \quad \frac{d\bar{\mu}}{\bar{\mu}} = -k^2p^2 \, dt$$

Integrating, we get

$$\log \bar{\mu} = -k^2p^2t + \log A \qquad | \text{ A is a constant}$$

or, $$\bar{\mu} = A \, e^{-k^2p^2t} \qquad ...(3)$$

Now, taking Fourier transform of (2), we get

$$\bar{\mu}(p, 0) = \int_{-\infty}^{\infty} \mu(x, 0) \, e^{ipx} \, dx$$

$$= \int_{-c}^{c} e^{ipx} \, dx = \frac{e^{ipc} - e^{-ipc}}{ip} = \frac{2}{p} \sin cp \qquad ...(4)$$

From (3), At $t = 0$, $\frac{2}{p} \sin cp = A$

$$\therefore \quad \bar{\mu} = \frac{2}{p} \sin cp \; e^{-k^2p^2t}$$

Taking its inverse Fourier transform, we get

$$\mu(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{p} \sin cp \; e^{-k^2p^2t} . e^{-ipx} \, dp$$

which is the required solution.

Example 6. *Use Fourier cosine transform to show that the steady temperature u in the semi-infinite solid y > 0 when the temperature on the surface y = 0 is kept at unity over the strip | x | < a and at zero outside the strip is*

$$\frac{1}{\pi} \left[\tan^{-1}\left(\frac{a+x}{y}\right) + \tan^{-1}\left(\frac{a-x}{y}\right) \right]$$

The results $\int_0^\infty e^{-px} x^{-1} \sin rx \, dx = \tan^{-1}\left(\frac{r}{p}\right)$ *(r , p > 0) may be assumed.*

Sol. Taking Fourier cosine transform of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, we have

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2} . \cos px \, dx + \int_0^\infty \frac{\partial^2 u}{\partial y^2} . \cos px \, dx = 0$$

$$-\left(\frac{\partial u}{\partial x}\right)_{x=0} - p^2 \bar{u} + \frac{d^2}{dy^2}(\bar{u}) = 0 \quad \text{where } \bar{u} = \int_0^\infty u \cos px \, dx$$

$$\Rightarrow \quad \frac{d^2 \bar{u}}{dy^2} - p^2 \bar{u} = 0 \qquad \because \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty \quad ...(1)$$

Its solution is $\bar{u} = c_1 e^{py} + c_2 e^{-py}$ ...(2)

But $\bar{u}$ is finite so $c_1 = 0$, otherwise $\bar{u} \to \infty$ as $y \to \infty$

$\therefore$ From (2), $\bar{u} = c_2 e^{-py}$ ...(3)

But $\bar{u} = \int_0^\infty u \cos px \, dx$

$$\therefore \quad (\bar{u})_{y=0} = \int_0^\infty (u)_{y=0} \cos px \, dx = \int_0^a 1 . \cos px \, dx = \frac{\sin pa}{p} \quad ...(4)$$

From (3), $(\bar{u})_{y=0} = c_2$

$$\therefore \quad c_2 = \frac{\sin pa}{p}$$

$$\therefore \text{ From (3),} \quad \bar{u} = \frac{\sin pa}{p} e^{-py}$$

Applying inverse Fourier cosine transform, we get

$$u = \frac{2}{\pi}\int_0^\infty \frac{\sin pa}{p} e^{-py} \cos px \, dp = \frac{1}{\pi}\int_0^\infty \frac{e^{-py}}{p}(2 \sin pa \cos px) \, dp$$

$$= \frac{1}{\pi}\int_0^\infty \frac{e^{-py}}{p}[\sin (a+x)p + \sin (a-x)p] \, dp$$

$$= \frac{1}{\pi}\left[\tan^{-1}\left(\frac{a+x}{y}\right) + \tan^{-1}\left(\frac{a-x}{y}\right)\right].$$

ASSIGNMENT

1. Apply appropriate Fourier transform to solve the partial differential equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}; \; x > 0, t > 0$$

subject to the conditions

(*i*) $V_x(0, t) = 0$ (*ii*) $V(x, 0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$ (*iii*) $V(x, t)$ is bounded.

2. Solve $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ for $x \ge 0$, $t \ge 0$ under the conditions

(*i*) $u(0, t) = u_0$, $t > 0$ (*ii*) $u(x, 0) = 0$, $x \ge 0$ (*iii*) $u(x, t)$ is bounded.

3. Using the method of Fourier transform, determine the displacement $y(x, t)$ of an infinite string given that the string is initially at rest and that the initial displacement is $f(x)$, $-\infty < x < \infty$. Show that the solution can also be put in the form $y(x, t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$.

4. Using Fourier transform, solve

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0; \quad V(x, 0) = f(x) \qquad \textit{(U.P.T.U. 2008, 2014)}$$

5. Using Fourier transforms, solve the following initial boundary value problem:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ t > 0; \ u(x, 0) = \left.\begin{matrix} 1, & -1 < x < 0 \\ -1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{matrix}\right\} \qquad \textit{(U.P.T.U. 2008)}$$

Answers

1. $V(x, t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \left(\frac{\sin p}{p} + \frac{\cos p - 1}{p^2}\right) e^{-p^2 t} \cos px \, dp$

2. $u(x, t) = \dfrac{2u_0}{\pi}\displaystyle\int_0^\infty \left(\frac{1 - e^{-kp^2 t}}{p}\right) \sin px \, dp.$

4. $V(x, t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^\infty F(p)\, e^{-p^2 t} \,.\, e^{-ipx}\, dp$ where $F(p) = \displaystyle\int_{-\infty}^\infty f(x)\, e^{ipx}\, dx$

5. $u(x, t) = \dfrac{1}{i\pi}\displaystyle\int_{-\infty}^\infty \frac{1}{p}(1 - \cos p)\, e^{-c^2 p^2 t} \,.\, e^{-ipx}\, dp$

2.14 FINITE FOURIER TRANSFORMS

The finite Fourier sine transform of $F(x)$, $0 < x < l$ is defined as

$$f_s(p) = \int_0^l F(x) \,.\, \sin \frac{p\pi x}{l}\, dx \,;\, p \in I$$

Similarly, the finite Fourier cosine transform of $F(x)$, $0 < x < l$ is defined as

$$f_c(p) = \int_0^l F(x) \,.\, \cos \frac{p\pi x}{l}\, dx \,;\, p \in I$$

Generally, the choice of π as the upper limit of integration in these transforms is found convenient and can easily be arranged by having suitable substitutions to actual problems, then

$$f_s(p) = \int_0^\pi F(x) \,.\, \sin px\, dx \quad \text{and} \quad f_c(p) = \int_0^\pi F(x) \,.\, \cos px\, dx.$$

Note. $f_s(p)$ is always zero when $p = 0$.

2.15 INVERSE FINITE FOURIER TRANSFORMS

Inversion formulae are given as follows:

(1) When upper limit is π

For sine transform:

$$F(x) = \frac{2}{\pi} \sum_{p=1}^{\infty} f_s(p) \sin px$$

For cosine transform:

$$F(x) = \frac{1}{\pi} f_c(0) + \frac{2}{\pi} \sum_{p=1}^{\infty} f_c(p) \cos px \text{ where } f_c(0) \text{ stands for } \int_0^{\pi} F(x)\, dx.$$

(2) When upper limit is *l*

For sine transform:

$$F(x) = \frac{2}{l} \sum_{p=1}^{\infty} f_s(p) \sin \frac{p\pi x}{l}$$

For cosine transform:

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{p=1}^{\infty} f_c(p) \cos \frac{p\pi x}{l} \text{ where } f_c(0) \text{ stands for } \int_0^{l} F(x)\, dx.$$

EXAMPLES

Example 1. *Find finite Fourier sine transform of* $F(x) = 1 - \dfrac{x}{\pi}$.

Sol. The finite Fourier sine transform of $F(x)$ is given by,

$$f_s(p) = \int_0^{\pi} \left(1 - \frac{x}{\pi}\right) \sin px\, dx$$

$$= \left\{\left(1 - \frac{x}{\pi}\right)\left(-\frac{\cos px}{p}\right)\right\}_0^{\pi} - \int_0^{\pi} \left(-\frac{1}{\pi}\right) \cdot \left(-\frac{\cos px}{p}\right) dx$$

$$= \frac{1}{p} - \frac{1}{\pi p}\left(\frac{\sin px}{p}\right)_0^{\pi} = \frac{1}{p}.$$

Example 2. *Find finite Fourier cosine transform of* $F(x) = -\dfrac{\cos k(\pi - x)}{k \sin k\pi}$.

Sol.

$$f_c(p) = \int_0^{\pi} \frac{-\cos k(\pi - x)}{k \sin k\pi} \cos px\, dx$$

$$= -\frac{1}{2k \sin k\pi} \int_0^{\pi} [\cos \{k(\pi - x) + px\} + \cos \{k(\pi - x) - px\}]\, dx$$

$$= -\frac{1}{2k \sin k\pi} \left[\frac{\sin (k\pi - kx + px)}{p - k} - \frac{\sin (k\pi - kx - px)}{p + k}\right]_0^{\pi}$$

$$= \frac{1}{2k}\left[\frac{1}{p - k} - \frac{1}{p + k}\right] = \frac{1}{p^2 - k^2}, \; k \neq 0, 1, 2, \ldots$$

Example 3. *Find F.F.S.T. and F.F.C.T. of* $F(x) = 2x$, $0 < x < 4$.

Sol. (*i*) $f_s(p) = \int_0^{4} 2x \cdot \sin \dfrac{p\pi x}{4}\, dx$ (Here $l = 4$)

$$= \left[2x \cdot \left\{\frac{-\cos \frac{p\pi x}{4}}{\left(\frac{p\pi}{4}\right)}\right\}\right]_0^4 - \int_0^4 2 \cdot \left\{\frac{-\cos \frac{p\pi x}{4}}{\left(\frac{p\pi}{4}\right)}\right\} dx = \begin{cases} \dfrac{32}{p\pi}(1 - \cos p\pi), & p \neq 0 \\ 0, & p = 0 \end{cases}$$

(*ii*) $$f_c(p) = \int_0^4 2x \,.\, \cos\frac{p\pi x}{4}\,dx$$

$$= \left[2x \,.\, \left\{\frac{\sin\frac{p\pi x}{4}}{\left(\frac{p\pi}{4}\right)}\right\}\right]_0^4 - \int_0^4 2 \,.\, \left\{\frac{\sin\frac{p\pi x}{4}}{\left(\frac{p\pi}{4}\right)}\right\} dx$$

$$= \frac{32}{p\pi}\sin p\pi - \frac{8}{p\pi}\left\{\frac{-\cos\frac{p\pi x}{4}}{\left(\frac{p\pi}{4}\right)}\right\}_0^4 = \frac{32}{p^2\pi^2}(\cos p\pi - 1),\ p \neq 0$$

When $p = 0$, $$f_c(p) = f_c(0) = \int_0^4 2x\,dx = 16.$$

Example 4. *Find* $F(x)$ *if* $f_c(p) = \begin{cases} \dfrac{\sin p\pi/2}{2p}, & p = 1, 2, \ldots \\ \pi/4, & p = 0 \end{cases}$ *where* $0 < x < 2\pi$.

Sol. $$F(x) = \frac{1}{2\pi}\cdot\frac{\pi}{4} + \frac{2}{2\pi}\sum_{p=1}^{\infty}\frac{\sin p\pi/2}{2p}\cdot\cos\frac{p\pi x}{2\pi}$$

$$= \frac{1}{2\pi}\cdot\frac{\pi}{4} + \frac{2}{2\pi}\sum_{p=1}^{\infty}\frac{\sin p\pi/2}{2p}\cos\frac{px}{2}$$

$$F(x) = \frac{1}{8} + \frac{1}{\pi}\sum_{p=1}^{\infty}\frac{\sin(p\pi/2)}{2p}\cos\frac{px}{2}.$$

ASSIGNMENT

1. Find finite Fourier sine and cosine transforms of

(*i*) $F(x) = x^2$, $0 < x < \pi$ (*ii*) $F(x) = 1$, $0 < x < \pi$ (*iii*) $F(x) = x$, $0 < x < \pi$.

2. Find finite Fourier cosine transform of $F(x) = \left(1 - \frac{x}{\pi}\right)^2$.

3. Find finite Fourier sine transform of

(*i*) $F(x) = \frac{x}{\pi}$ (*ii*) $F(x) = \sin nx$, $n \in I$ (*iii*) $F(x) = x(\pi^2 - x^2)$

(*iv*) $F(x) = x(\pi - x)$ (*v*) $F(x) = e^{cx}$ (*vi*) $\cos mx$.

4. Find finite Fourier cosine transform of

(*i*) $F(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$ (*ii*) $\sin nx$, $n \in I$ (*iii*) $F(x) = \left.\begin{matrix} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{matrix}\right\}$.

5. Find finite Fourier sine transform of $F(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \frac{\pi}{2} \le x < \pi \end{cases}$.

6. Find inverse finite Fourier sine transform of

(*i*) $f_s(p) = \frac{2\pi(-1)^{p-1}}{p^3}$, $p = 1, 2, \ldots (0 < x < \pi)$ (*ii*) $f_s(p) = \frac{1 - \cos p\pi}{p^2\pi^2}$, $0 < x < \pi$.

Answers

1. (*i*) (*a*) $f_s(p) = \begin{cases} -\dfrac{\pi^2(-1)^p}{p} + \dfrac{2}{p^3}[(-1)^p - 1], & p \neq 0 \\ 0, & p = 0 \end{cases}$ (*b*) $f_c(p) = \begin{cases} \dfrac{2\pi(-1)^p}{p^2}, & p \neq 0 \\ \dfrac{\pi^3}{3}, & p = 0 \end{cases}$

(*ii*) (*a*) $f_s(p) = \begin{cases} \dfrac{1}{p}\{1 - (-1)^p\}, & p \neq 0 \\ 0, & p = 0 \end{cases}$ (*b*) $f_c(p) = \begin{cases} 0, & \text{if } p \neq 0 \\ \pi, & \text{if } p = 0 \end{cases}$

(*iii*) (*a*) $f_s(p) = \begin{cases} \dfrac{\pi}{p}(-1)^{p+1}, & p \neq 0 \\ 0, & p = 0 \end{cases}$ (*b*) $f_c(p) = \begin{cases} \dfrac{\{(-1)^p - 1\}}{p^2}, & p \neq 0 \\ \dfrac{\pi^2}{2}, & p = 0 \end{cases}$

2. $f_c(p) = \begin{cases} \dfrac{2}{\pi p^2}, & p \neq 0 \\ \dfrac{\pi}{3}, & p = 0 \end{cases}$

3. (*i*) $\dfrac{(-1)^{p+1}}{p}$ (*ii*) $\begin{matrix} 0, & \text{if } p \neq n \\ \pi/2, & \text{if } p = n \end{matrix}\Big\}$ (*iii*) $\dfrac{6\pi}{p^3}(-1)^{p+1}$

(*iv*) $\dfrac{2}{p^3}[1 - (-1)^p]$ (*v*) $\dfrac{p}{p^2 + c^2}[1 - (-1)^p e^{\pi c}]$ (*vi*) $\dfrac{p}{p^2 - m^2}[1 - (-1)^p \cos m\pi]$

4. (*i*) $\dfrac{1}{p^2}$ (*ii*) $\dfrac{2n}{n^2 - p^2}$; $(n - p)$ is odd and 0 if even

(*iii*) $\dfrac{2}{p}\sin\dfrac{p\pi}{2}$; $p \neq 0$ and 0 if $p = 0$ **5.** $\dfrac{2}{p^2}\sin\dfrac{p\pi}{2}$

6. (*i*) $\dfrac{2}{\pi}\displaystyle\sum_{1}^{\infty} \dfrac{2\pi(-1)^{p-1}}{p^3}\sin px$ (*ii*) $\dfrac{2}{\pi^3}\displaystyle\sum_{1}^{\infty}\left(\dfrac{1 - \cos p\pi}{p^2}\right)\sin px.$

2.16 PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

M.A. Parseval (1755–1836), a French Mathematician, gave the following result.

If the Fourier transforms of F(x) and G(x) are $f(p)$ and $g(p)$ respectively, then

(*i*) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} f(p)\,\overline{g}(p)\,dp = \int_{-\infty}^{\infty} F(x)\,\overline{G}(x)\,dx$

(*ii*) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |f(p)|^2\,dp = \int_{-\infty}^{\infty} |F(x)|^2\,dx$

where bar implies the complex conjugate.

Proof. $\displaystyle\int_{-\infty}^{\infty} F(x)\,\overline{G}(x)\,dx = \int_{-\infty}^{\infty} F(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g}(p)\,e^{ipx}\,dp\right\}dx$ | Using inversion formula for Fourier transform

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g}(p)\left\{\int_{-\infty}^{\infty} F(x)\,e^{ipx}\,dx\right\}dp$$

| Changing order of integration

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\,\overline{g}(p)\,dp \qquad ...(1)$$ | By definition

Now, take G(x) = F(x) in (1), we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(p)\bar{f}(p)\, dp = \int_{-\infty}^{\infty} \mathrm{F}(x)\, \overline{\mathrm{F}}(x)\, dx$$

$$\Rightarrow \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(p)|^2\, dp = \int_{-\infty}^{\infty} |\mathrm{F}(x)|^2\, dx \qquad ...(2)$$

Hence the results.

Corollary 1. Following Parseval's identities for Fourier cosine and sine transforms can be proved as above:

(i) $\frac{2}{\pi}\int_0^{\infty} f_c(p)\; g_c(p)\, dp = \int_0^{\infty} \mathrm{F}(x)\; \mathrm{G}(x)\, dx$ $\quad (ii)$ $\frac{2}{\pi}\int_0^{\infty} |f_c(p)|^2\; dp = \int_0^{\infty} |\mathrm{F}(x)|^2\; dx$

(iii) $\frac{2}{\pi}\int_0^{\infty} f_s(p)\; g_s(p)\, dp = \int_0^{\infty} \mathrm{F}(x)\, \mathrm{G}(x)\, dx$ $\quad (iv)$ $\frac{2}{\pi}\int_0^{\infty} |f_s(p)|^2\; dp = \int_0^{\infty} |\mathrm{F}(x)|^2\, dx$.

EXAMPLES

Example 1. *Using Parseval's identity, show that* $\int_0^{\infty} \frac{x^2}{(a^2+x^2)(b^2+x^2)}\, dx = \frac{\pi}{2(a+b)}$

Hence find $\int_0^{\infty} \frac{x^2}{(x^2+1)^2}\, dx$. (U.P.T.U. 2015)

Sol. If F(x) = e^{-ax} then $f_s(p) = \frac{p}{a^2+p^2}$

$\therefore$ By Parseval's identity for sine transform,

$$\frac{2}{\pi}\int_0^{\infty} f_s(p)\, g_s(p)\, dp = \int_0^{\infty} \mathrm{F}(x)\, \mathrm{G}(x)\, dx$$

$$\Rightarrow \qquad \frac{2}{\pi}\int_0^{\infty} \frac{p}{a^2+p^2}\cdot\frac{p}{b^2+p^2}\, dp = \int_0^{\infty} e^{-ax}.\, e^{-bx}\, dx$$

$$\Rightarrow \qquad \int_0^{\infty} \frac{p^2}{(a^2+p^2)(b^2+p^2)}\, dp = \frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^{\infty} = \frac{\pi}{2(a+b)}$$

Thus,

$$\int_0^{\infty} \frac{x^2}{(a^2+x^2)(b^2+x^2)}\, dx = \frac{\pi}{2(a+b)}$$

For $a = b = 1$, we get

$$\int_0^{\infty} \frac{x^2}{(x^2+1)^2}\, dx = \frac{\pi}{4}$$

Example 2. *Using Parseval's identity, show that* $\int_0^{\infty} \frac{\sin ax}{x(a^2+x^2)}\, dx = \frac{\pi}{2}\frac{(1-e^{-a^2})}{a^2}$.

Sol. Let $F(x) = e^{-ax}$ and $G(x) = \begin{Bmatrix} 1, & 0 < x < a \\ 0, & x > a \end{Bmatrix}$

then $f_c(p) = \dfrac{a}{a^2 + p^2}$ and $g_c(p) = \dfrac{\sin ap}{p}$

Parseval's identity for Fourier cosine transform is

$$\frac{2}{\pi}\int_0^\infty f_c(p)\, g_c(p)\, dp = \int_0^\infty F(x)\, G(x)\, dx$$

$$\Rightarrow \quad \frac{2}{\pi}\int_0^\infty \frac{a \sin ap}{p\,(a^2 + p^2)}\, dp = \int_0^\infty e^{-ax} \,.\, 1\, dx = \frac{1 - e^{-a^2}}{a}$$

$$\therefore \quad \int_0^\infty \frac{\sin ax}{x(a^2 + x^2)}\, dx = \frac{\pi}{2a^2}(1 - e^{-a^2})$$

ASSIGNMENT

1. Using Parseval's identity, show that $\displaystyle\int_0^\infty \frac{dx}{(a^2 + x^2)(b^2 + x^2)} = \frac{\pi}{2ab\,(a + b)}$.

   Hence evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2 + 1)^2}$

2. If $F(x) = \begin{Bmatrix} 1, & |x| < a \\ 0, & |x| > a \end{Bmatrix}$, prove that $\displaystyle\int_0^\infty \frac{\sin^2 ax}{x^2}\, dx = \frac{\pi a}{2}$ using Parseval's identity.

3. Evaluate using Parseval's identity:

   (*i*) $\displaystyle\int_0^\infty \left(\frac{1 - \cos x}{x}\right)^2 dx$ (*ii*) $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^2}\, dx$ $\left[\textbf{Hint: } \text{Take } F(x) = \begin{matrix} 1, & 0 \le x \le 1 \\ 0, & x \ge 1 \end{matrix}\right]$

4. Prove that: $\displaystyle\int_0^\infty \left(\frac{\sin x}{x}\right)^4 dx = \pi/3$ $\left[\textbf{Hint: } \text{Take } F(x) = \begin{Bmatrix} 1 - |x|, & |x| < 1 \\ 0, & \text{otherwise} \end{Bmatrix}\right]$

5. Use Parseval's identity to prove that: $\displaystyle\int_0^\infty \frac{(x \cos x - \sin x)^2}{x^6}\, dx = \frac{\pi}{15}$

Answers

1. $\pi/4$ **3.** (*i*) $\pi/2$ (*ii*) $\pi/2$

2.17 THE Z-TRANSFORM

The Z-transform plays an important role in the field of Communication Engineering and Control Engineering at the stage of analysis and representation of discrete-time linear shift invariance system. When continuous signals are sampled, discrete-time functions arise. The application of Z-transform in discrete time systems is similar to that of the Laplace transform in continuous time systems.

2.18 DEFINITIONS

2.18.1. One-Sided Z-transform

Let $\{f(k)\}$ be a sequence defined for all positive integers 'k'. Then the Z-transform of $f(k)$ is defined as

$$Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k)\, z^{-k} \qquad ...(1)$$

where z is an arbitrary complex number and Z is an operator of Z-transform.

This is **one-sided Z-transform.**

2.18.2. Two-Sided Z-transform

If $\{f(k)\}$ is a sequence defined for $k = 0, \pm 1, \pm 2, ...$

then
$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} \qquad ...(2)$$

where z is an arbitrary complex number and Z is an operator of Z-transform.

This is **two sided Z-transform**.

Note **1.** If $f(k) = 0$ for $k < 0$ then $\{f(k)\}$ is called a casual sequence.

2. If $f(k)$ is a non-casual sequence, $f(k) = 0$ for $k \geq 0$, then its Z-transform is $F(z) = \sum_{-\infty}^{-1} f(k)\, z^{-k}$ and is also called **one sided Z-transform.**

3. The curly bracket { } represents sequence. Sequence $\{f(k)\}$ is an ordered list of real or complex numbers.

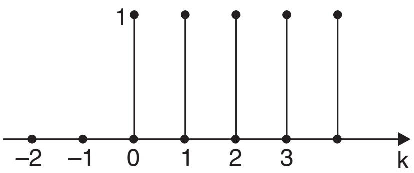
4. The infinite series on R.H.S. of (1) will be convergent only for certain values of z depending on sequence $\{f(x)\}$.

5. The inverse Z-transform of $Z\{f(k)\} = F(z)$ is defined as $Z^{-1}[F(z)] = \{f(k)\}$.

2.19 UNIT STEP AND UNIT IMPULSE SEQUENCES

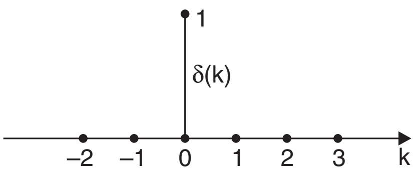
Unit step sequence is defined as

$$u(k) = \begin{Bmatrix} 1, & k \geq 0 \\ 0, & k < 0 \end{Bmatrix}$$



Unit impulse sequence is defined as

$$\delta(k) = \begin{Bmatrix} 1, & k = 0 \\ 0, & k \neq 0 \end{Bmatrix}$$



2.20 RELATION BETWEEN UNIT IMPULSE SEQUENCE AND UNIT STEP SEQUENCE

$$u(k) = \sum_{k=-\infty}^{\infty} \delta(k) \quad \text{and } \delta(k) = u(k) - u(k-1)$$

We have $$\delta(n-k) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

Also, $$f(n) = \sum_{k=-\infty}^{\infty} f(k)\,\delta\,(n-k).$$

2.21 Z-TRANSFORM OF UNIT IMPULSE FUNCTION

We know that $$\delta(k) = \left.\begin{matrix} 1, & k = 0 \\ 0, & k \neq 0 \end{matrix}\right\}$$

$$\therefore \qquad Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} \delta(k)\, z^{-k} = [\ldots\ldots + 0 + 0 + 1 + 0 + 0 + \ldots\ldots] = 1$$

Hence, $Z\{\delta(k)\} = 1$

2.22 Z-TRANSFORM OF DISCRETE UNIT STEP FUNCTION

We know that $$u(k) = \left.\begin{matrix} 1, & k \geq 0 \\ 0, & k < 0 \end{matrix}\right\}$$

$$\therefore \qquad Z\{u(k)\} = \sum_{k=-\infty}^{\infty} u(k)\, z^{-k} = \sum_{k=0}^{\infty} z^{-k}$$

$$= 1 + z^{-1} + z^{-2} + \ldots\ldots = \frac{1}{1-z^{-1}} = \frac{z}{z-1}\ ;\ |\ z\ | > 1.$$

2.23 Z-TRANSFORM FOR DISCRETE VALUES OF t

If $f(t)$ is a function defined for discrete values of t, where $t = n$T, $n = 0, 1, 2, \ldots\ldots$, T being the sampling period, then Z-transform of $f(t)$ is defined as

$$Z\{f(t)\} = \sum_{k=0}^{\infty} f(n\mathrm{T}) z^{-k} = \mathrm{F}(z).$$

Note. The important element of discrete-time systems is the samples in which a switch close to admit an input signal in every T seconds. A **samples** is a conversion device which converts a continuous signal into a sequence of pulses occurring at sampling instants 0, T, 2T, ... where T is the sampling period.

EXAMPLES

Example 1. *Find the Z-transform of the following sequences:*

(*i*) $f(k) = \{15, 10, 7, \underset{\uparrow}{4}, 1, -1, 0, 3, 6\}$ (*ii*) $f(k) = \{5, 6, 1, 2, -1, 0, 8, 4, 3\}$

(*iii*) $f(k) = \left\{\dfrac{1}{4^k}\right\}$ (*iv*) $f(k) = \left\{\dfrac{1}{2^k}\right\}, -3 \leq k \leq 3$

(*v*) $f(k) = \begin{cases} 5^k, & k < 0 \\ 2^k, & k \geq 0 \end{cases}$.

Sol. (*i*) The symbol ↑ is used to denote the term in zeroth position *i.e.,* when $k = 0$. k is an index of position of a term in a sequence.

$$Z\{f(k)\} = 15z^3 + 10z^2 + 7z + 4 + \frac{1}{z} - \frac{1}{z^2} + 0 + \frac{3}{z^4} + \frac{6}{z^5}$$

$$\Rightarrow \qquad F(z) = 15z^3 + 10z^2 + 7z + 4 + \frac{1}{z} - \frac{1}{z^2} + \frac{3}{z^4} + \frac{6}{z^5}$$

(*ii*) In case the symbol ↑ is not given, extreme left term is considered as zeroth term corresponding to $k = 0$. Here, the zeroth term is 5.

$$\therefore \qquad Z\{f(k)\} = 5 + \frac{6}{z} + \frac{1}{z^2} + \frac{2}{z^3} - \frac{1}{z^4} + 0 + \frac{8}{z^6} + \frac{4}{z^7} + \frac{3}{z^8}$$

$$\Rightarrow \qquad F(z) = 5 + \frac{6}{z} + \frac{1}{z^2} + \frac{2}{z^3} - \frac{1}{z^4} + \frac{8}{z^6} + \frac{4}{z^7} + \frac{3}{z^8}$$

$$(iii) \qquad Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} \frac{1}{4^k} z^{-k}$$

$$= \ldots\ldots + 64z^3 + 16z^2 + 4z + 1 + \frac{1}{4z} + \frac{1}{16z^2} + \frac{1}{64z^3} + \ldots\ldots$$

$$(iv) \qquad Z\{f(k)\} = \sum_{-3}^{3} f(k)\, z^{-k} \qquad \text{(since } -3 \le k \le 3)$$

$$= \sum_{-3}^{3} \frac{1}{2^k} z^{-k} = 8z^3 + 4z^2 + 2z + 1 + \frac{1}{2z} + \frac{1}{4z^2} + \frac{1}{8z^3}$$

$$(v) \qquad Z\{f(k)\} = \sum_{k=-\infty}^{-1} 5^k z^{-k} + \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$= [\ldots\ldots + 5^{-3}z^3 + 5^{-2}z^2 + 5^{-1}z] + \left[1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \ldots\ldots\right]$$

$$= \frac{5^{-1}z}{1 - 5^{-1}z} + \frac{1}{1 - (2/z)} = \frac{z}{5 - z} + \frac{z}{z - 2}\ ; \quad |z| < 5,\ |z| > 2.$$

Example 2. *Find the Z-transform of*

(*i*) $f(k) = \{a^{|k|}\}$ (*ii*) $f(k) = \dfrac{1}{k},\ k \ge 1$ (*iii*) $f(k) = \dfrac{1}{k(k+1)},\ k \ge 0$

(*iv*) $f(k) = \cos \dfrac{k\pi}{2},\ k \ge 0$ (*v*) $f(k) = \left.\begin{matrix} 0, & k > 0 \\ 1, & k \le 0 \end{matrix}\right\}$.

Sol. (*i*)

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k} = (\ldots\ldots + a^3z^3 + a^2z^2 + az) + \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$

$$= \frac{az}{1 - az} + \left(1 - \frac{a}{z}\right)^{-1} = \frac{az}{1 - az} + \frac{z}{z - a}. \qquad \left|\ \because\ (1 - x)^{-1} = \sum_{k=0}^{\infty} x^k\right.$$

(*ii*) $$Z\left(\frac{1}{k}\right) = \sum_{k=1}^{\infty} \frac{1}{k} z^{-k}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \ldots\ldots = -\log\left(1 - \frac{1}{z}\right) \text{ if } \left|\frac{1}{z}\right| < 1$$

$$= \log\left(\frac{z}{z-1}\right) \qquad \text{if } |z| > 1.$$

(*iii*) $$Z\left\{\frac{1}{k(k+1)}\right\} = Z\left(\frac{1}{k} - \frac{1}{k+1}\right) = Z\left(\frac{1}{k}\right) - Z\left(\frac{1}{k+1}\right)$$

$$= \log\left(\frac{z}{z-1}\right) - \sum_{k=0}^{\infty} \frac{1}{k+1} z^{-k} = \log\left(\frac{z}{z-1}\right) - \left(1 + \frac{1}{2z} + \frac{1}{3z^2} + \ldots\ldots\right)$$

$$= \log\left(\frac{z}{z-1}\right) - z\left\{\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots\ldots\right\}$$

$$= \log\left(\frac{z}{z-1}\right) - z\left\{-\log\left(1 - \frac{1}{z}\right)\right\}$$

$$= \log\left(\frac{z}{z-1}\right) - z\log\left(\frac{z}{z-1}\right) = (1-z)\log\left(\frac{z}{z-1}\right).$$

(*iv*) $$Z\left(\cos\frac{k\pi}{2}\right) = \sum_{k=0}^{\infty} \cos\frac{k\pi}{2} z^{-k}$$

$$= 1 - \frac{1}{z^2} + \frac{1}{z^4} - \ldots\ldots \infty = \left(1 + \frac{1}{z^2}\right)^{-1} = \frac{z^2}{z^2+1} \quad \text{if} \quad |z| > 1.$$

(*v*) $$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}; \quad \text{if} \quad |z| < 1.$$

Example 3. *Find the Z-transform of*

(*i*) $f(k) = \begin{Bmatrix} \dfrac{a^k}{k!}, & k \ge 0 \\ 0, & \textit{otherwise} \end{Bmatrix}$ (*ii*) $f(k) = \dfrac{1}{(k+1)!}$, $k \ge 0$.

Sol. (*i*) $$Z\{f(k)\} = \sum_{k=0}^{\infty} \frac{a^k}{k!} z^{-k} = \sum_{k=0}^{\infty} \frac{(az^{-1})^k}{k!} = e^{az^{-1}} = e^{a/z}.$$

(*ii*) $$Z\left[\frac{1}{(k+1)!}\right] = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} z^{-k} = 1 + \frac{1}{2!} z^{-1} + \frac{1}{3!} z^{-2} + \ldots\ldots$$

$$= z\left[z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \ldots\ldots\right]$$

$$= z\left[1 + z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \ldots\ldots - 1\right]$$

$$= z\left(e^{z^{-1}} - 1\right) = z(e^{1/z} - 1).$$

Example 4. *Find the Z-transform of*

(i) $u(k-1)$ *(ii)* $4^k\,\delta(k-1)\,;\,k \ge 0$

(iii) $\delta(k-n)\,;\,k \ge 0$ *(iv)* $\{{}^nC_k\}\,;\,0 \le k \le n.$ (M.T.U. 2014)

Sol. *(i)*
$$Z\{f(k)\} = \sum_{k=1}^{\infty} 1\,.\,z^{-k} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\ldots = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\ldots\right)$$
$$= \frac{1}{z}\,\frac{1}{\left(1 - \frac{1}{z}\right)} \quad \text{if} \quad \left|\frac{1}{z}\right| < 1$$
$$= \frac{1}{z-1} \quad \text{if} \quad |\,z\,| > 1.$$

(ii)
$$Z\{f(k)\} = \sum_{k=0}^{\infty} 4^k\,\delta(k-1)\,z^{-k} = \frac{4}{z}.$$

(iii)
$$Z\{f(k)\} = \sum_{k=0}^{\infty} \delta(k-n)\,z^{-k} = \frac{1}{z^n}\,,\ n \text{ is (+)ve integer.}$$

(iv)
$$Z\{f(k)\} = \sum_{k=0}^{n} {}^nC_k\,z^{-k} = 1 + {}^nC_1\,z^{-1} + {}^nC_2\,z^{-2} + \ldots\ldots + {}^nC_n\,z^{-n} = (1 + z^{-1})^n.$$

Example 5. *Determine the Z-transform of the sequence given by*

$$f(k) = \begin{Bmatrix} 2^k, & k < 0 \\ \left(\frac{1}{2}\right)^k, & k = 0, 2, 4, \ldots\ldots \\ \left(\frac{1}{3}\right)^k, & k = 1, 3, 5, \ldots\ldots \end{Bmatrix}$$

What is the region of convergence for the Z-transform F(z)?

Sol.
$$F(z) = \sum_{k=-\infty}^{\infty} f(k)\,z^{-k}$$
$$= \sum_{k=-\infty}^{-1} 2^k\,z^{-k} + \underset{(k\text{-even})}{\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-k}} + \underset{(k\text{-odd})}{\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k z^{-k}}$$
$$= \sum_{m=1}^{\infty} 2^{-m}\,z^m + \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^{2p} z^{-2p} + \sum_{q=0}^{\infty} \left(\frac{1}{3}\right)^{2q+1} z^{-(2q+1)}$$

where $m = -k$, $p = \dfrac{k}{2}, q = \dfrac{k-1}{2}$

$$= F_1(z) + F_2(z) + F_3(z)$$
$$= \frac{z/2}{1 - z/2} + \frac{z^2}{z^2 - \frac{1}{4}} + \frac{z/3}{z^2 - \frac{1}{9}}$$

Region of convergence for $F_1(z)$: $| z | < 2$

Region of convergence for $F_2(z)$: $| z | > \frac{1}{2}$

Region of convergence for $F_3(z)$: $| z | > \frac{1}{3}$

Hence, Region of convergence for $F(z)$: $\frac{1}{2} < | z | < 2$.

ASSIGNMENT

1. Determine the Z-transform of the following sequences :

(*i*) $f(k) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1, 2\}$ (*ii*) $f(k) = \{3, 1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

(*iii*) $f(k) = \{0, 0, 1, 2, 5, 4, 0, 1\}$ (*iv*) $f(k) = \{1, 2, 5, 4, 0, 1\}$

(*v*) $f(k) = \delta(k + n)$.

2. Find the Z-transform of

$$f(k) = \begin{Bmatrix} 0, & \text{for } k < 0 \\ 1, & \text{for } 0 \le k \le 5 \\ 2, & \text{for } 6 \le k \le 10 \\ 3, & \text{for } k > 10 \end{Bmatrix}.$$

3. Find the Z-transform of

(*i*) $u(k - 4)$ (*ii*) $\delta(k - 5)$ (*iii*) $\begin{cases} \left(\frac{1}{3}\right)^k & k \ge 0 \\ (-2)^k & k \le -1 \end{cases}$.

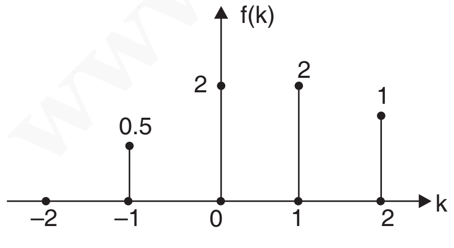
4. Determine Z-transform for the sequences given below :

$f(0) = 1$; $f(1) = 4.7$; $f(2) = 0$; $f(3) = 0$

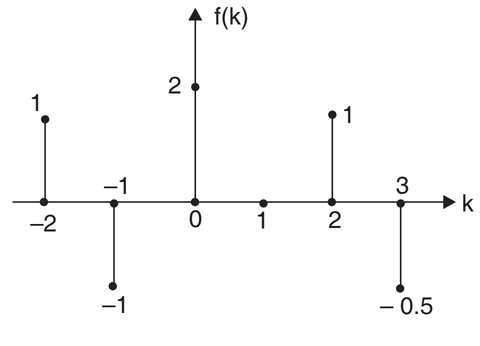
$f(4) = 0.75$; $f(5) = \sqrt{2}$; $f(k) = 0$; $k \ge 6$.

5. Express the signals shown below in terms of unit impulse functions and hence find the Z-transform.

(*i*)



(*ii*)



Answers

1. (*i*) $2z^2 + 4z + 5 + \frac{7}{z} + \frac{1}{z^3} + \frac{2}{z^4}$ (*ii*) $3z^3 + z^2 + 2z + 5 + \frac{7}{z} + \frac{1}{z^3}$

(*iii*) $z^{-2} + 2z^{-3} + 5z^{-4} + 4z^{-5} + z^{-7}$ (*iv*) $1 + 2z^{-1} + 5z^{-2} + 4z^{-3} + z^{-5}$ (*v*) z^n

2. $1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + 2(z^{-6} + z^{-7} + z^{-8} + \ldots\ldots + z^{-10}) + 3(z^{-11} + z^{-12} + \ldots\ldots)$

3. (*i*) $\dfrac{z^{-4}}{1-z^{-1}}$; $|z| > 1$ (*ii*) z^{-5} (*iii*) $\dfrac{z}{z-\frac{1}{3}} - \dfrac{z}{z+2}$

4. $F(z) = 1 + 4.7\,z^{-1} + 0.75\,z^{-4} + \sqrt{2}\,z^{-5}$

5. (*i*) $f(k) = 0.5\,\delta(k+1) + 2\delta(k) + 2\delta(k-1) + \delta(k-2)$; $F(z) = 0.5z + 2 + 2z^{-1} + z^{-2}$

(*ii*) $f(k) = \delta(k+2) - \delta(k+1) + 2\delta(k) + \delta(k-2) - 0.5\,\delta(k-3)$; $F(z) = z^2 - z + 2 + z^{-2} - 0.5\,z^{-3}$.

2.24 PROPERTIES OF Z-TRANSFORMS

2.24.1. Linearity Property

$$\boxed{\mathbf{Z\{a\,f(k) \pm bg(k)\} = a\,Z\{f(k)\} \pm b\,Z\,\{g(k)\}}}$$

Proof. $Z\{a\,f(k) \pm b\,g(k)\} = \sum_{k=-\infty}^{\infty} \{a\,f(k) \pm b\,g(k)\}z^{-k}$ | by definition

$$= a\sum_{k=-\infty}^{\infty} f(k)z^{-k} \pm b\sum_{k=-\infty}^{\infty} g(k)z^{-k} = a\,Z\{f(k)\} \pm b\,Z\{g(k)\}.$$

Remark. If $Z\{f(k)\} = F(z)$ and $Z\{g(k)\} = G(z)$, then $Z^{-1}[a\,F(z) \pm b\,G(z)] = a\,Z^{-1}\{F(z)\} \pm b\,Z^{-1}\{G(z)\}$ where a and b are constants and Z^{-1} is inverse Z-transform operator.

2.24.2. Change of Scale Property

$$\boxed{\textbf{If} \quad \mathbf{Z\{f(k)\} = F(z),\ then} \quad \mathbf{Z\{\,|a^k f(k)|\,\} = F\left(\frac{z}{a}\right)}}$$

Proof. $$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k}$$

Substituting $\dfrac{z}{a}$ for z, we get

$$F\left(\frac{z}{a}\right) = \sum_{k=-\infty}^{\infty} f(k)\left(\frac{z}{a}\right)^{-k} = \sum_{k=-\infty}^{\infty} a^k f(k)z^{-k} = Z\{\,|a^k f(k)|\,\}.$$

Remark (*i*) $$Z\{a^k\,U(k)\} = \frac{z/a}{\frac{z}{a}-1} = \frac{z}{z-a} \text{ if } |z| > |a|$$

(*ii*) $$Z\{a^k\} = \frac{z}{z-a} \text{ since } Z(1) = \frac{z}{z-1}.$$

2.24.3. Multiplication by k^n

$$\boxed{\textbf{If} \quad \mathbf{Z\{f(k)\} = F(z),\ then\ Z\{k^n f(k)\} = \left(-z\frac{d}{dz}\right)^n F(z)}}$$

Proof. $$Z\{k\,f(k)\} = \sum_{k=-\infty}^{\infty} k\,f(k)z^{-k}$$

$$= -z \sum_{k=-\infty}^{\infty} f(k)(-kz^{-k-1}) = -z \sum_{k=-\infty}^{\infty} f(k) \frac{d}{dz}(z^{-k}) = -z \frac{d}{dz} \mathrm{F}(z)$$

In general, $\mathrm{Z}\{|k^n f(k)|\} = \left(-z \frac{d}{dz}\right)^n \mathrm{F}(z)$.

Remark. $\mathrm{Z}\{(k\, a^k)\} = -z \frac{d}{dz} [\mathrm{Z}(a^k)] = -z \frac{d}{dz}\left(\frac{z}{z-a}\right) = \frac{az}{(z-a)^2}$.

Note. Since $\mathrm{Z}(1) = \frac{z}{z-1}$, we have the following results.

(*i*) $\mathrm{Z}(k) = \frac{z}{(z-1)^2}$ (*ii*) $\mathrm{Z}(k^2) = \frac{z(z+1)}{(z-1)^3}$

(*iii*) $\mathrm{Z}(k^3) = \frac{z^3 + 4z^2 + z}{(z-1)^4}$ (*iv*) $\mathrm{Z}(k^4) = \frac{z^4 + 11z^3 + 11z^2 + z}{(z-1)^5}$.

2.24.4. Shifting Property

$$\boxed{\textbf{If } \mathbf{Z\{f(k)\} = F(z), \text{ then } Z\{f(k \pm n)\} = z^{\pm n} F(z)}}$$

Proof.

$$\mathrm{Z}\{f(k \pm n)\} = \sum_{k=-\infty}^{\infty} f(k \pm n) z^{-k} = z^{\pm n} \sum_{k=-\infty}^{\infty} f(k \pm n) z^{-(k \pm n)}$$

$$= z^{\pm n} \sum_{k=-\infty}^{\infty} f(r) z^{-r} \qquad (r = k \pm n)$$

$$= z^{\pm n} \mathrm{F}(z).$$

Corollary 1. For casual sequence,

$$\mathrm{Z}\{f(k-n)\} = z^{-n} \mathrm{F}(z)$$

Also, $\mathrm{Z}\{f(k+1)\} = z\, \mathrm{F}(z) - z\, f(0)$

and $\mathrm{Z}\{f(k+2)\} = z^2\, \mathrm{F}(z) - z^2 f(0) - z\, f(1)$

$$= z^2\left\{\mathrm{F}(z) - f(0) - \frac{f(1)}{z}\right\} \text{ and so on.}$$

Corollary 2. $\mathrm{Z}^{-1}[z^{-n}\, \mathrm{F}(z)] = f(k-n) = \mathrm{Z}^{-1}[\mathrm{F}(z)]_{k \to k-n}$.

2.24.5. Division by k

$$\boxed{\textbf{If } \mathbf{Z\{f(k)\} = F(z), \text{ then } Z\left\{\frac{f(k)}{k}\right\} = -\int^z z^{-1} F(z)\, dz}}$$

Proof.

$$\mathrm{Z}\left\{\frac{f(k)}{k}\right\} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{k} z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} f(k)\left(\frac{1}{k} z^{-k}\right) = -\sum_{k=-\infty}^{\infty} f(k) \int^z z^{-k-1}\, dz$$

$$= -\int^z \sum_{k=-\infty}^{\infty} f(k)\, z^{-k-1} dz$$

$$= -\int^{z} z^{-1}\left(\sum_{k=-\infty}^{\infty} f(k)\, z^{-k}\right) dz = -\int^{z} z^{-1}\, \mathrm{F}(z)\, dz.$$

2.24.6. Initial Value Theorem

$$\boxed{\textbf{If}\quad \mathbf{Z\{f(k)\} = F(z),\ k \geq 0,\ then\ f(0) = \underset{z \to \infty}{Lt}\ F(z)}}$$

Proof. $$\mathrm{Z}\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$\Rightarrow \qquad \mathrm{F}(z) = f(0) + f(1)\, z^{-1} + f(2) z^{-2} + \ldots\ldots$$

Taking limit as $z \to \infty$, we get

$$f(0) = \underset{z \to \infty}{\mathrm{Lt}}\ \mathrm{F}(z).$$

2.24.7. Final Value Theorem

$$\boxed{\textbf{If}\quad \mathbf{Z\{f(k)\} = F(z)\ ;\ k \geq 0,\ then\ \underset{k \to \infty}{Lt}\ f(k) = \underset{z \to 1}{Lt}\ (z-1)\, F(z)}}$$

Proof. $$\mathrm{Z}\{f(k+1) - f(k)\} = \sum_{k=0}^{\infty} \{f(k+1) - f(k)\}\, z^{-k}$$

$$z\, \mathrm{F}(z) - f(0) - \mathrm{F}(z) = \underset{n \to \infty}{\mathrm{Lt}} \sum_{k=0}^{n} \{f(k+1) - f(k)\}\, z^{-k}$$

$$\underset{z \to 1}{\mathrm{Lt}}\ (z-1)\mathrm{F}(z) = f(0) + \underset{z \to 1}{\mathrm{Lt}}\ .\ \underset{n \to \infty}{\mathrm{Lt}} \sum_{k=0}^{n} \{f(k+1) - f(k)\}\, z^{-k}$$

$$= f(0) + \underset{n \to \infty}{\mathrm{Lt}} \sum_{k=0}^{n} \underset{z \to 1}{\mathrm{Lt}} \{f(k+1) - f(k)\}\, z^{-k}$$

| Changing the order of limits

$$= \underset{n \to \infty}{\mathrm{Lt}} \left[f(0) + \sum_{k=0}^{n} \{f(k+1) - f(k)\} \right]$$

$$= \underset{n \to \infty}{\mathrm{Lim}}\ f(n+1) = \underset{n \to \infty}{\mathrm{Lim}}\ f(n) = \underset{k \to \infty}{\mathrm{Lim}}\ f(k).$$

2.24.8. Differentiation Property

Let $\mathrm{Z}[\{f(k)\}] = \mathrm{F}(z)$. An infinite series can be differentiated term by term within its region of convergence. $\mathrm{F}(z)$ may be treated as a function of z^{-1}.

$$\mathrm{F}(z) = \sum_{k=0}^{\infty} f(k)\, z^{-k} = \sum_{k=0}^{\infty} f(k)\, (z^{-1})^{k}$$

Differentiating on both sides w.r.t. z^{-1}

$$\frac{d}{dz^{-1}}\, \mathrm{F}(z) = \sum_{k=0}^{\infty} kf(k)\, (z^{-1})^{k-1} \qquad \ldots(1)$$

$$z^{-1}\, \frac{d\mathrm{F}(z)}{dz^{-1}} = \sum_{k=0}^{\infty} kf(k)\, z^{-k} = \mathrm{Z}\{kf(k)\}$$

$$\therefore \qquad \boxed{Z\{kf(k)\} = z^{-1} \frac{dF(z)}{dz^{-1}}} \qquad ...(2)$$

Differentiating (1) w.r.t. z^{-1} again, we get

$$\frac{d^2F(z)}{d(z^{-1})^2} = \sum_{k=0}^{\infty} k(k-1) f(k) (z^{-1})^{k-2}$$

$$z^{-2} \frac{d^2F(z)}{d(z^{-1})^2} = \sum_{k=0}^{\infty} k(k-1) f(k) z^{-k} = Z\{k(k-1) f(k)\}$$

$$\therefore \qquad \boxed{Z[k(k-1) f(k)] = z^{-2} \frac{d^2F(z)}{d(z^{-1})^2}} \qquad ...(3)$$

2.24.9. Convolution Theorem *(U.P.T.U. 2015)*

The convolution of two sequences $\{f(n)\}$ and $\{g(n)\}$ is defined as

$$w(n) = \sum_{k=-\infty}^{\infty} f(k)\, g(n-k) = f * g$$

If it is one sided (right) sequence, take $f(k) = 0$, $g(k) = 0$ for $k < 0$, then

$$w(n) = \sum_{k=0}^{\infty} f(k)\, g(n-k) = f * g.$$

Statement. If $w(n)$ is the convolution of two sequences $f(n)$ and $g(n)$, then

$$\boxed{Z\{w(n)\} = W(z) = Z\{f(n)\} \,.\, Z\{g(n)\} = F(z)\, G(z)} \qquad (U.P.T.U.\ 2015)$$

Proof.

$$W(z) = Z\{w(n)\} = Z\left[\sum_{k=0}^{\infty} f(k) g(n-k)\right] = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\infty} f(k) g(n-k)\right] z^{-n}$$

$$= \sum_{k=0}^{\infty} f(k) \left[\sum_{n=0}^{\infty} g(n-k) z^{-n}\right]$$

(by changing the order of summation)

$$= \sum_{k=0}^{\infty} f(k) \left[\sum_{p=0}^{\infty} g(p) z^{-(p+k)}\right], \qquad \text{(putting } n-k=p\text{)}$$

$$= \left[\sum_{k=0}^{\infty} f(k) z^{-k}\right] \left[\sum_{p=0}^{\infty} g(p) z^{-p}\right] = F(z)\, G(z).$$

Note. This result will be true only for the values of z inside the region of convergence.

2.24.10. Another form of Convolution Theorem

If $Z\{f(t)\} = F(z)$, $Z\{g(t)\} = G(z)$, then the convolution product is

$$w(t) = \sum_{k=0}^{\infty} f(kT)\, g(nT - kT) = f * g$$

and

$$Z\{w(t)\} = W(z) = Z\{f(t)\}\, Z\{g(t)\} = F(z)\, G(z).$$

Proof. (Here we are dealing with one sided Z-transform only)

$$F(z) = \sum_{m=0}^{\infty} f(mT)\, z^{-m} \;;\; G(z) = \sum_{n=0}^{\infty} g(nT)\, z^{-n}$$

$$F(z)\, G(z) = \left\{ \sum_{m=0}^{\infty} f(mT) z^{-m} \right\} \left\{ \sum_{n=0}^{\infty} g(nT) z^{-n} \right\}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(mT)\, g(nT)\, z^{-m-n} = \sum_{n=0}^{\infty} \left(\sum_{p=0}^{n} f(pT) g\{(n-p)T\} \right) z^{-n}$$

$$= Z \left\{ \sum_{p=0}^{\infty} f(pT) g\{(n-p)T\} \right\} z^{-n} = Z\{f * g\}.$$

2.24.11. Time Reversal Property

If $F(z)$ is the Z-transform of $f(k)$, then $Z\{f(-k)\} = F(z^{-1})$

2.24.12. Correlation Property

If $Z\{f_1(k)\} = F_1(z)$ and $Z\{f_2(k)\} = F_2(z)$, then $R_{f_1 f_2}(z) = F_1(z).\, F_2(z^{-1})$

and cross-correlation sequence $r_{f_1 f_2}(l) = Z^{-1}[R_{f_1 f_2}(z)]$.

2.25 SOME IMPORTANT Z-TRANSFORM RESULTS

S. No.	*$\{f(k)\}$, $k \geq 0$*	*$F(z) = Z\{f(k)\}$*
1.	$\delta(k)$	1
2.	$U(k)$ or 1	$\dfrac{z}{z-1}$
3.	k	$\dfrac{z}{(z-1)^2}$, $\lvert z \rvert > 1$
4.	$k^n f(k)$	$\left(-z \dfrac{d}{dz}\right)^n F(z)$
5.	k^n	$\left(-z \dfrac{d}{dz}\right)^n \dfrac{z}{z-1}$, $\lvert z \rvert > 1$
6.	a^k or $a^k U(k)$	$\dfrac{z}{z-a}$; $\lvert z \rvert > \lvert a \rvert$
7.	$\dfrac{1}{k}$	$\log\left(\dfrac{z}{z-1}\right)$, $\lvert z \rvert > 1$
8.	$\dfrac{1}{k+1}$	$z \log\left(\dfrac{z}{z-1}\right)$

9.	$\frac{1}{k!}$	$e^{1/z}$
10.	$\delta(k-n)$	z^{-n}
11.	$f(0)$	$\underset{z \to \infty}{\text{Lt}}\ F(z)$
12.	$\underset{k \to \infty}{\text{Lt}}\ f(k)$	$\underset{z \to 1}{\text{Lt}}\ (z-1)\,F(z)$
13.	$r_{f_1 f_2}(l) = \sum_{k=-\infty}^{\infty} f_1(k)\, f_2(k-l)$	$R_{f_1 f_2}(z) = F_1(z)\, F_2(z^{-1})$
14.	$h(k) = f(k) * g(k)$	$F(z)\,.\,G(z)$
15.	$f(-k)$	$F(z^{-1})$

EXAMPLES

Example 1. *Determine the Z-transform of*

$$f(k) = \delta(k+1) + 3\,\delta(k) + 6\delta(k-3) - \delta(k-4).$$

Sol. By linearity property, we have

$$F(z) = Z\{f(k)\} = Z\{\delta(k+1)\} + 3\,Z\{\delta(k)\} + 6\,Z\{\delta(k-3)\} - Z\{\delta(k-4)\}$$
$$= z + 3 + 6z^{-3} - z^{-4}.$$

Example 2. *Find the Z-transform of*

(*i*) $\{a^{k+3}\}$ (*ii*) $\{k^2\}$ (*iii*) $\left\{\frac{(k+1)(k+2)}{2}\right\}$

(*iv*) $\{ab^k\}$; $a \neq 0, b \neq 0$ (*v*) $\{k(k-1)\}$.

Sol. (*i*)
$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} a^{k+3} z^{-k} = a^3 Z(a^k) \quad \ldots(1)$$

We know that, $Z(1) = \dfrac{z}{z-1}$

$\therefore$ $Z(a^k) = \dfrac{z/a}{(z/a)-1} = \dfrac{z}{z-a}$ (By change of scale property)

$\therefore$ From (1), $Z\{f(k)\} = \dfrac{za^3}{z-a}$.

(*ii*)
$$Z(k^2) = \left(-z\frac{d}{dz}\right)^2 Z(1) = \left(-z\frac{d}{dz}\right)\left[\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right)\right]$$
$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = \frac{z(z+1)}{(z-1)^3}.$$

(*iii*)
$$Z\left\{\frac{(k+1)(k+2)}{2}\right\} = \frac{1}{2} Z(k^2 + 3k + 2)$$

$$= \frac{1}{2}\,[Z(k^2) + 3\,Z(k) + Z(2)] = \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + 3\left\{-z\,\frac{d}{dz}\,Z(1)\right\} + \frac{2z}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} - 3z\,\frac{d}{dz}\left(\frac{z}{z-1}\right) + \frac{2z}{z-1}\right] = \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}\right].$$

(*iv*) $Z(ab^k) = a\,Z(b^k) = a\left(\frac{z}{z-b}\right)$ if $|\,z\,| > |\,b\,|$.

(*v*) $Z\{k(k-1)\} = Z(k^2 - k) = Z(k^2) - Z(k) = \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2} = \frac{2z}{(z-1)^3}$.

Example 3. *Find the Z-transform of f(k) = u(– k).*

Sol. We know that $Z\{u(k)\} = \frac{z}{z-1}$

By time reversal property, we have

$$Z\{u(-k)\} = \frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z}\ ;\ |\,z\,| < 1.$$

Example 4. *Find the Z-transform of*

(*i*) $\{{}^{k+n}C_n\}$ *or* $\{{}^{k+n}C_k\}$; $k \geq 0$ (*ii*) $\{{}^{k+n}C_n\, a^k\}$; $k \geq 0$.

Sol. (*i*) $Z\{{}^{k+n}C_n\} = Z\{{}^{k+n}C_k\}$

$$= \sum_{k=0}^{\infty} {}^{k+n}C_k\, z^{-k} = 1 + {}^{n+1}C_1\, z^{-1} + {}^{n+2}C_2\, z^{-2} + \ldots\ldots$$

$$= 1 + (n+1)\,z^{-1} + \frac{(n+2)(n+1)}{2\,!}\, z^{-2} + \ldots\ldots$$

$$= 1 + (-n-1)(-z^{-1}) + \frac{(-n-1)(-n-2)}{2\,!}\,(-z^{-1})^2 + \ldots\ldots$$

$$= (1 - z^{-1})^{-n-1} = (1 - z^{-1})^{-(n+1)} = \left(\frac{z}{z-1}\right)^{n+1}.$$

(*ii*) $Z\{{}^{k+n}C_n\, a^k\} = \left(\frac{z/a}{z/a - 1}\right)^{n+1}$ | Change of scale property using (*i*)

$$= \left(\frac{z}{z-a}\right)^{n+1}.$$

Example 5. *Find the Z-transform of*

(*i*) *sin* αk, $k \geq 0$ (U.P.T.U. 2009, 2015) (*ii*) *sin* $(3k + 5)$, $k \geq 0$.

Sol. (*i*) $Z(\sin \alpha k) = \sum_{k=0}^{\infty} \sin \alpha k\, z^{-k} = \frac{1}{2i} \sum_{k=0}^{\infty} (e^{i\alpha k} - e^{-i\alpha k})\, z^{-k}$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} (e^{i\alpha}\, z^{-1})^k - \sum_{k=0}^{\infty} (e^{-i\alpha}\, z^{-1})^k\right]$$

$$= \frac{1}{2i}\,[(1 - e^{i\alpha}\, z^{-1})^{-1} - (1 - e^{-i\alpha}\, z^{-1})^{-1}]$$

$$= \frac{1}{2i}\left[\frac{1}{1-e^{i\alpha}z^{-1}} - \frac{1}{1-e^{-i\alpha}z^{-1}}\right] = \frac{1}{2i}\left[\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right]$$

$$= \frac{1}{2i}\left[\frac{z(e^{i\alpha}-e^{-i\alpha})}{z^2 - z(e^{i\alpha}+e^{-i\alpha})+1}\right] = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}.$$

(ii) $\mathrm{Z}\{\sin(3k+5)\} = \sum_{k=0}^{\infty} \sin(3k+5)\, z^{-k} = \frac{1}{2i}\sum_{k=0}^{\infty}\{e^{i(3k+5)} - e^{-i(3k+5)}\}\, z^{-k}$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} e^{5i}(e^{3i}z^{-1})^k - \sum_{k=0}^{\infty} e^{-5i}(e^{-3i}z^{-1})^k\right]$$

$$= \frac{1}{2i}\,[e^{5i}(1-e^{3i}z^{-1})^{-1} - e^{-5i}(1-e^{-3i}z^{-1})^{-1}]$$

$$= \frac{e^{5i}}{2i}\left(\frac{1}{1-e^{3i}z^{-1}}\right) - \frac{e^{-5i}}{2i}\left(\frac{1}{1-e^{-3i}z^{-1}}\right)$$

$$= \frac{e^{5i}}{2i}\left(\frac{z}{z-e^{3i}}\right) - \frac{e^{-5i}}{2i}\left(\frac{z}{z-e^{-3i}}\right) = \frac{1}{2i}\left[\frac{e^{5i}z(z-e^{-3i}) - e^{-5i}z(z-e^{3i})}{(z-e^{3i})(z-e^{-3i})}\right]$$

$$= \frac{1}{2i}\left[\frac{(e^{5i}-e^{-5i})z^2 - z(e^{2i}-e^{-2i})}{1 - z(e^{3i}+e^{-3i}) + z^2}\right] = \frac{z^2\sin 5 - z\sin 2}{z^2 - 2z\cos 3 + 1}\ ; \quad |z| > 1.$$

Example 6. *Find the Z-transform of*

(i) *$c^k \cosh(\alpha k)$, $k \ge 0$* $\qquad$ (ii) *$c^k \cos(\alpha k)$, $k \ge 0$*

(iii) *$\cosh\left(\frac{k\pi}{2}+\alpha\right)$, $k \ge 0$.* (U.P.T.U. 2008)

Sol. (i) $\mathrm{Z}\{\cosh(\alpha k)\} = \sum_{k=0}^{\infty}\left(\frac{e^{\alpha k}+e^{-\alpha k}}{2}\right) z^{-k} = \frac{1}{2}\left[\sum_{k=0}^{\infty}(e^{\alpha}z^{-1})^k + \sum_{k=0}^{\infty}(e^{-\alpha}z^{-1})^k\right]$

$$= \frac{1}{2}\left[(1-e^{\alpha}z^{-1})^{-1} + (1-e^{-\alpha}z^{-1})^{-1}\right] = \frac{1}{2}\left[\frac{z}{z-e^{\alpha}} + \frac{z}{z-e^{-\alpha}}\right]$$

$$= \frac{z}{2}\cdot\left[\frac{2z-(e^{\alpha}+e^{-\alpha})}{z^2 - z(e^{\alpha}+e^{-\alpha})+1}\right] = \frac{z(z-\cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$$

By change of scale property,

$$\mathrm{Z}\{c^k\cosh(\alpha k)\} = \frac{\frac{z}{c}\left(\frac{z}{c}-\cosh\alpha\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cosh\alpha + 1} = \frac{z(z - c\cosh\alpha)}{z^2 - 2cz\cosh\alpha + c^2}.$$

(ii) $\qquad$ $\mathrm{Z}\{\cos(\alpha k)\} = \sum_{k=0}^{\infty}\left(\frac{e^{i\alpha k}+e^{-i\alpha k}}{2}\right) z^{-k} = \frac{1}{2}\left[\sum_{k=0}^{\infty}(e^{i\alpha}z^{-1})^k + \sum_{k=0}^{\infty}(e^{-i\alpha}z^{-1})^k\right]$

$$= \frac{1}{2}\left[(1-e^{i\alpha}z^{-1})^{-1} + (1-e^{-i\alpha}z^{-1})^{-1}\right] = \frac{1}{2}\left[\frac{z}{z-e^{i\alpha}} + \frac{z}{z-e^{-i\alpha}}\right]$$

$$= \frac{z}{2}\left(\frac{2z - 2\cos\alpha}{z^2 - 2z\cos\alpha + 1}\right) = \frac{z(z - \cos\alpha)}{z^2 - 2z\cos\alpha + 1}$$

By change of scale property,

$$Z\{c^k \cos(\alpha k)\} = \frac{\left(\frac{z}{c}\right)\left(\frac{z}{c} - \cos\alpha\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cos\alpha + 1} = \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}$$

$$(iii)\ Z\left\{\cosh\left(\frac{k\pi}{2} + \alpha\right)\right\} = \sum_{k=0}^{\infty} \cosh\left(\frac{k\pi}{2} + \alpha\right) z^{-k} = \frac{1}{2}\sum_{k=0}^{\infty}\left\{e^{\left(\frac{k\pi}{2}+\alpha\right)} + e^{-\left(\frac{k\pi}{2}+\alpha\right)}\right\} z^{-k}$$

$$= \frac{1}{2}\left[e^{\alpha}\sum_{0}^{\infty}(e^{\pi/2} z^{-1})^k + e^{-\alpha}\sum_{0}^{\infty}(e^{-\pi/2} z^{-1})^k\right]$$

$$= \frac{1}{2}[e^{\alpha}(1 - e^{\pi/2} z^{-1})^{-1} + e^{-\alpha}(1 - e^{-\pi/2} z^{-1})^{-1}]$$

$$= \frac{1}{2}\left[e^{\alpha}\cdot\left(\frac{z}{z - e^{\pi/2}}\right) + e^{-\alpha}\left(\frac{z}{z - e^{-\pi/2}}\right)\right] = \frac{z}{2}\left[\frac{e^{\alpha}(z - e^{-\pi/2}) + e^{-\alpha}(z - e^{\pi/2})}{(z - e^{\pi/2})(z - e^{-\pi/2})}\right]$$

$$= \frac{z}{2}\left[\frac{2z\cosh\alpha - \cosh(\alpha - \pi/2)}{z^2 - (2\cosh\pi/2)z + 1}\right] = \frac{z\left[z\cosh\alpha - \cosh\left(\frac{\pi}{2} - \alpha\right)\right]}{z^2 - 2z\cosh\pi/2 + 1}.$$

Example 7. *Find the Z-transform of* $\left\{\cos\left(\frac{k\pi}{8} + \alpha\right)\right\}$*; $k \geq 0$.*

Sol. $Z\left\{\cos\left(\frac{k\pi}{8} + \alpha\right)\right\} = \sum_{k=0}^{\infty}\cos\left(\frac{k\pi}{8} + \alpha\right) z^{-k} = \sum_{k=0}^{\infty}\left(\cos\frac{k\pi}{8}\cos\alpha - \sin\frac{k\pi}{8}\sin\alpha\right) z^{-k}$

$$= \cos\alpha\, Z\left(\cos\frac{k\pi}{8}\right) - \sin\alpha\, Z\left(\sin\frac{k\pi}{8}\right)$$

$$= \cos\alpha\,.\left\{\frac{z(z - \cos\pi/8)}{z^2 - 2z\cos\frac{\pi}{8} + 1}\right\} - \sin\alpha\,.\left(\frac{z\sin\pi/8}{z^2 - 2z\cos\frac{\pi}{8} + 1}\right)$$

| From Ex. 5 and 6

$$= \frac{\left(z^2 - z\cos\frac{\pi}{8}\right)\cos\alpha - z\sin\frac{\pi}{8}\sin\alpha}{z^2 - 2z\cos\frac{\pi}{8} + 1} = \frac{z^2\cos\alpha - z\cos\left(\frac{\pi}{8} - \alpha\right)}{z^2 - 2z\cos\frac{\pi}{8} + 1}.$$

Example 8. *If* $F(z) = \frac{z(z - \cos aT)}{z^2 - 2z\cos aT + 1}$*, find f(0).*

Sol. From initial value theorem, $f(0) = \underset{z \to \infty}{\text{Lt}}\ F(z)$

$$\therefore \qquad f(0) = \underset{z \to \infty}{\text{Lt}} \frac{z(z - \cos a\text{T})}{z^2 - 2z \cos a\text{T} + 1} = 1 \text{ by L' Hospital rule.}$$

Example 9. *If* $F(z) = \dfrac{z}{z - e^{-T}}$, *find* $\underset{t \to \infty}{Lt} f(t)$.

Sol. From final value theorem, we have

$$\underset{t \to \infty}{\text{Lt}} f(t) = \underset{z \to 1}{\text{Lt}} (z - 1)\text{F}(z) = \underset{z \to 1}{\text{Lt}} (z - 1) \frac{z}{z - e^{-\text{T}}} = 0.$$

Example 10. *Find* $Z^{-1} \left\{\dfrac{1}{z+1}\right\}$ *given* $Z^{-1} \left\{\dfrac{z}{z+1}\right\} = (-1)^k$.

Sol.
$$\text{Z}^{-1} \left\{\frac{1}{z+1}\right\} = \text{Z}^{-1} \left\{z^{-1} . \frac{z}{z+1}\right\}$$
$$= \text{Z}^{-1} \left\{\frac{z}{z+1}\right\}_{k \to k-1} = \left\{\left(-1\right)^k\right\}_{k \to k-1} = \{(-1)^{k-1}\},\ k = 1, 2, 3, \ldots\ldots$$

Example 11. *Find* $Z^{-1} \left(\dfrac{3}{3z - 1}\right)$.

Sol.
$$\text{Z}^{-1} \left\{\frac{3}{3z-1}\right\} = \text{Z}^{-1} \left\{\frac{1}{z - \frac{1}{3}}\right\} = \text{Z}^{-1} \left[z^{-1} \left\{\frac{z}{z - \frac{1}{3}}\right\}\right]$$
$$= \text{Z}^{-1} \left\{\frac{z}{z - \frac{1}{3}}\right\}_{k \to k-1} = \left(\frac{1}{3}\right)^{k-1} \quad \text{or} \quad \left(\frac{1}{3}\right)^{k-1} u(k-1).$$

Example 12. *Using differentiation property, find the Z-transform of*
(*i*) $k\, a^k\, u(k)$ (*ii*) $k(k-1)\, a^k\, u(k)$.

Sol. (*i*)
$$\text{Z}\{ka^k\, u(k)\} = z^{-1} \frac{d}{dz^{-1}} \left(\frac{z}{z-a}\right)$$
$$= z^{-1} \frac{d}{dz^{-1}} (1 - az^{-1})^{-1} = z^{-1} . \frac{a}{(1 - az^{-1})^2} = \frac{az^{-1}}{(1 - az^{-1})^2}$$

(*ii*)
$$\text{Z}\,\{k(k-1)\, a^k\, u(k)\} = z^{-2} \frac{d^2}{d(z^{-1})^2} (1 - az^{-1})^{-1} = \frac{2a^2 z^{-2}}{(1 - az^{-1})^3}.$$

Note. If $a = 1$, then $\text{Z}\{k\, u(k)\} = \dfrac{z^{-1}}{(1 - z^{-1})^2}$ and $\text{Z}\{k(k-1)\, u(k)\} = \dfrac{2z^{-2}}{(1 - z^{-1})^3}$.

Example 13. *Find the Z-transform of* $f * g$ *where*
(*i*) $f(n) = u(n), g(n) = 2^n u(n)$
(*ii*) $f(n) = 3^n u(n), g(n) = 4^n u(n)$ *using convolution theorem.*

Sol. (*i*)
$$\text{F}(z) = \text{Z}\{u(n)\} = \sum_{n=0}^{\infty} 1 . z^{-n} = \frac{z}{z-1} \text{ if } |\, z\,| > 1$$

$$G(z) = Z\{2^n u(n)\} = \sum_{n=0}^{\infty} 2^n z^{-n} = \frac{z}{z-2} \text{ if } |z| > |2|$$

By convolution theorem

$$Z\{f * g\} = Z\{w(n)\} = W(z) = F(z).\, G(z)$$

$$= \frac{z}{z-1} \cdot \frac{z}{z-2} = \frac{z^2}{(z-1)(z-2)} \text{ if } |z| > |2|.$$

(*ii*)
$$F(z) = Z\{3^n u(n)\} = \frac{z}{z-3} \text{ if } |z| > |3|$$

$$G(z) = Z\{4^n u(n)\} = \frac{z}{z-4} \text{ if } |z| > |4|$$

By convolution theorem

$$Z\{f * g\} = Z\{w(n)\} = W(z) = F(z)\,.\, G(z)$$

$$= \frac{z}{z-3} \cdot \frac{z}{z-4} = \frac{z^2}{(z-3)(z-4)} \text{ if } |z| > |4|.$$

Example 14. *Compute the convolution f(k) of the two sequences :*

$$f_1(k) = \{4, -2, 1\} \quad \text{and} \quad f_2(k) = \begin{Bmatrix} 1, & 0 \le k \le 5 \\ 0, & \text{otherwise} \end{Bmatrix}.$$

Sol.
$$F_1(z) = Z\{f_1(k)\} = 4 - 2z^{-1} + z^{-2}$$

$$F_2(z) = Z\{f_2(k)\} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$\therefore \quad F(z) = F_1(z)\,.\,F_2(z) = 4 + 2z^{-1} + 3z^{-2} + 3z^{-3} + 3z^{-4} + 3z^{-5} - z^{-6} + z^{-7}$$

Taking inverse Z-transform, we get

$$f(k) = \{\underset{\uparrow}{4}, 2, 3, 3, 3, 3, -1, 1\}.$$

Example 15. *Determine the cross-correlation sequence* $r_{f_1 f_2}(l)$ *of the sequences :*

$$f_1(k) = \{1, 2, 3, 4\} \quad \text{and} \quad f_2(k) = \{4, 3, 2, 1\}.$$

Sol. Cross-correlation sequence can be obtained using the correlation property of Z-transform.

$$F_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

$$F_2(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}$$

$$\Rightarrow \quad F_2(z^{-1}) = 4 + 3z + 2z^2 + z^3$$

$$R_{f_1 f_2}(z) = F_1(z)\,.\,F_2(z^{-1}) = (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})\,(4 + 3z + 2z^2 + z^3)$$

$$= (z^3 + 4z^2 + 10z + 20 + 25z^{-1} + 24z^{-2} + 16z^{-3})$$

$$\therefore \quad r_{f_1 f_2}(l) = Z^{-1}\{R_{f_1 f_2}(z)\} = \{1, 4, 10, \underset{\uparrow}{20}, 25, 24, 16\}.$$

Example 16. *If* $F(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$, *evaluate f(2) and f(3).*

Sol.
$$F(z) = \frac{1}{z^2} \cdot \left[\frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4}\right]$$

By initial value theorem, $f(0) = \underset{z \to \infty}{\text{Lt}}\,.\, F(z) = 0$

Similarly,

$$f(1) = \underset{z \to \infty}{\text{Lt}} \; z\{F(z) - f(0)\} = 0$$

$$f(2) = \underset{z \to \infty}{\text{Lt}} \; z^2\{F(z) - f(0) - z^{-1}f(1)\} = 2 - 0 - 0 = 2$$

$$f(3) = \underset{z \to \infty}{\text{Lt}} \; z^3\{F(z) - f(0) - z^{-1}f(1) - z^{-2} f(2)\} = \underset{z \to \infty}{\text{Lt}} \; z^3\{F(z) - 2z^{-2}\}$$

$$= \underset{z \to \infty}{\text{Lt}} \; z^3 \cdot \left[\frac{2z^2 + 5z + 14}{(z-1)^4} - \frac{2}{z^2}\right]$$

$$= \underset{z \to \infty}{\text{Lt}} \; z^3 . \left\{\frac{13z^3 + 2z^2 + 8z - 2}{z^2\,(z-1)^4}\right\} = 13.$$

ASSIGNMENT

1. Find the Z-transform (one sided) of the following sequences $\{f(k)\}$ where $f(k)$ is

(*i*) $\left(\frac{1}{4}\right)^k u(k)$ (*ii*) $(\cos\theta + i\sin\theta)^k$ (*iii*) $(-1)^k u(k)$

(*iv*) $3^k \sin\frac{k\pi}{2}$ (*v*) $2^k \cos\frac{k\pi}{2}$ (*vi*) $4^k + \left(\frac{1}{2}\right)^k + u(k-3)$.

2. Find the Z-transform of $\{f(k)\}$ where $f(k) = k\,2^k$

3. Show that

(*i*) $Z(\cosh k\theta) = \dfrac{z(z-\cosh\theta)}{z^2 - 2z\cosh\theta + 1}$ (*ii*) $Z(\sinh k\,\theta) = \dfrac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$.

4. Show that

(*i*) $Z(e^{-ak}\cos k\theta) = \dfrac{ze^a\,(ze^a - \cos\theta)}{z^2e^{2a} - 2ze^a\cos\theta + 1}$ (*ii*) $Z(e^{-ak}\sin k\theta) = \dfrac{ze^a\sin\theta}{z^2e^{2a} - 2ze^a\cos\theta + 1}$.

5. Using $Z(n^2) = \dfrac{z(z+1)}{(z-1)^3}$, show that $Z(n+1)^2 = \dfrac{z^2(z+1)}{(z-1)^3}$.

6. Given $Z(u_n) = \dfrac{2z^2 + 3z + 4}{(z-3)^3}$, $|\,z\,| > 3$, show that $u_1 = 2$, $u_2 = 21$, $u_3 = 139$.

7. Using $Z(k) = \dfrac{z}{(z-1)^2}$, show that $Z(k\cos k\theta) = \dfrac{(z^3 + z)\cos\theta - 2z^2}{(z^2 - 2z\cos\theta + 1)^2}$.

8. Find the convolutions of

(*i*) $k(k-1) * 3^k$ (*ii*) $3^k * \cos k\theta$ (*iii*) $\cos\frac{k\pi}{2} * \sin\frac{k\pi}{2}$.

9. Prove that $Z\{k^n\} = -z\,\dfrac{d}{dz}\,[Z\{k^{n-1}\}]$.

10. Evaluate the Z-transform of the sequence $\{f(k)\} = \displaystyle\sum_{k=0}^{\infty} 2^k \sum_{k=0}^{\infty} 3^k$.

Answers

1. (*i*) $\dfrac{4z}{4z-1}$ (*ii*) $\dfrac{z}{z - e^{i\theta}}$ (*iii*) $\dfrac{z}{z+1}$ (*iv*) $\dfrac{3z}{z^2 + 9}$

(v) $\dfrac{z^2}{z^2+4}$ (vi) $\dfrac{z}{z-4}+\dfrac{2z}{2z-1}+\dfrac{1}{z^2(z-1)}$

2. $\dfrac{2z}{(z-2)^2}$ **8.** (i) $\dfrac{2z^2}{(z-1)^3(z-3)}$ (ii) $\dfrac{z^2\,(z-\cos\theta)}{(z-3)(z^2-2z\cos\theta+1)}$

(iii) $\dfrac{z^3}{(z^2+1)^2}$ **10.** $\dfrac{z^2}{(z-2)(z-3)}$.

2.26 INVERSE Z-TRANSFORM

Inverse Z-transform is a process for determining the sequence which generates given Z-transform. If F(z) is the Z-transform of the sequence $\{f(k)\}$, then $\{f(k)\}$ is called the inverse Z-transform of F(z). The operator for inverse Z-transform is Z^{-1}.

If $Z\{f(k)\} = F(z)$, then $Z^{-1}[F(z)] = \{f(k)\}$.

2.27 METHODS OF FINDING INVERSE Z-TRANSFORMS

We have the following methods of finding inverse Z-transforms:

(1) Convolution method
(2) Long division method
(3) Partial fractional method
(4) Residue method (or Inverse Integral method)
(5) Power series method.

2.27.1. Convolution Method

We know that $Z\{f * g\} = F(z)\,G(z)$

$$\therefore \quad Z^{-1}\{F(z)\,G(z)\} = f * g = \sum_{m=0}^{k} f(m)\,g(k-m).$$

EXAMPLES

Example 1. *Find* $Z^{-1}\left\{\dfrac{z^2}{(z-a)(z-b)}\right\}$ *using convolution theorem.*

Sol. We know that $Z^{-1}\{F(z)\,G(z)\} = f * g$

Let $F(z) = \dfrac{z}{(z-a)}$ $\therefore$ $f(k) = Z^{-1}\{F(z)\} = Z^{-1}\left\{\dfrac{z}{z-a}\right\} = a^k$

$G(z) = \dfrac{z}{z-b}$ $\therefore$ $g(k) = Z^{-1}\{G(z)\} = Z^{-1}\left\{\dfrac{z}{z-b}\right\} = b^k$

$$Z^{-1}\{F(z)\,G(z)\} = f * g = a^k * b^k$$

$$= \sum_{m=0}^{k} a^m b^{k-m} = b^k \sum_{m=0}^{k}\left(\frac{a}{b}\right)^m \quad \text{(a G.P.)}$$

$$= b^k\left\{\frac{\left(\frac{a}{b}\right)^{k+1}-1}{\frac{a}{b}-1}\right\} = \frac{a^{k+1}-b^{k+1}}{a-b}.$$

Example 2. *Using Convolution theorem, evaluate* $Z^{-1}\left\{\dfrac{z^2}{(z-1)(z-3)}\right\}$.

Sol. We know that

$$Z^{-1}\{F(z)\,.\,G(z)\} = f * g$$

Let $\quad F(z) = \dfrac{z}{z-1} \quad \therefore \quad f(k) = (1)^k$

$\quad G(z) = \dfrac{z}{z-3} \quad \therefore \quad g(k) = (3)^k$

Now, $\quad Z^{-1}\{F(z)\,.\,G(z)\} = (1)^k * (3)^k$

$$= \sum_{m=0}^{k} 1^m\, 3^{k-m} = 3^k \sum_{m=0}^{k} \left(\frac{1}{3}\right)^m \qquad \text{(a G.P.)}$$

$$= 3^k \left\{\frac{\left(\frac{1}{3}\right)^{k+1} - 1}{\frac{1}{3} - 1}\right\} = \frac{(1)^{k+1} - (3)^{k+1}}{1-3} = \left\{\frac{1}{2}(3^{k+1} - 1)\right\}.$$

2.27.2. Long Division Method

Since Z-transform is defined by the series $F(z) = \sum_{k=0}^{\infty} f(k)\, z^{-k}$ (one sided), to find the inverse Z-transform *i.e.*, $Z^{-1}[F(z)]$, expand $F(z)$ in the proper power series and collect the coefficient of z^{-k} to get $f(k)$.

EXAMPLES

Example 1. *Find inverse Z-transform of*

(*i*) $\dfrac{10z}{z^2 - 3z + 2}$ $\qquad$ (*ii*) $\dfrac{2(z^3 - z)}{(z^2+1)^2}$.

Sol. (*i*) $F(z) = \dfrac{10z}{z^2 - 3z + 2} = \dfrac{10z^{-1}}{1 - 3z^{-1} + 2z^{-2}}$

By actual division,

$$\begin{array}{r|l}
 & 10z^{-1} + 30z^{-2} + 70z^{-3} + \ldots\ldots \\
\hline
1 - 3z^{-1} + 2z^{-2} & 10z^{-1} \\
 & 10z^{-1} - 30z^{-2} + 20z^{-3} \\
\cline{2-2}
 & 30z^{-2} - 20z^{-3} \\
 & 30z^{-2} - 90z^{-3} + 60z^{-4} \\
\cline{2-2}
 & 70z^{-3} - 60z^{-4} \\
 & 70z^{-3} - 210z^{-4} + 140z^{-5} \\
\cline{2-2}
 & + 150z^{-4} - 140z^{-5}
\end{array}$$

$\therefore \qquad F(z) = 10z^{-1} + 30z^{-2} + 70z^{-3} + \ldots\ldots$

Now comparing the quotient with

$$\sum_{k=0}^{\infty} f(k)\, z^{-k} = f(0) + f(1)z^{-1} + f(2)\, z^{-2} + \ldots\ldots$$

We get the sequence $f(k)$ as $\quad f(0) = 0,\ f(1) = 10,\ f(2) = 30,\ f(3) = 70, \ldots\ldots$

i.e., we can get $\quad f(k) = 10(2^k - 1),\ k = 0, 1, 2, \ldots\ldots$

(*ii*) $$F(z) = \frac{2z(z^2-1)}{(z^2+1)^2} = \frac{2z^{-1} - 2z^{-3}}{1 + 2z^{-2} + z^{-4}}$$

By actual division, we get $\quad F(z) = 2z^{-1} - 6z^{-3} + 10z^{-5} - 14z^{-7} + \ldots\ldots$

Comparing the quotient with $\displaystyle\sum_{k=0}^{\infty} f(k)\, z^{-k} = f(0) + f(1)z^{-1} + f(2)z^{-2} + \ldots\ldots$

We get $\quad f(0) = 0, f(1) = 2, f(2) = 0, f(3) = -6, f(4) = 0, f(5) = 10, f(6) = 0, \ldots\ldots$

In general $\quad f(k) = 2k \sin \dfrac{k\pi}{2},\ k = 0, 1, 2, \ldots\ldots$

Example 2. *Find the inverse Z-transform of*

$$F(z) = \frac{4z}{z-a}$$

for $\quad$ (*i*) $|z| > |a|$ $\qquad$ (*ii*) $|z| < |a|$.

Sol. (*i*) For $|z| > |a|$, we have

$$\frac{4z}{z-a} = \frac{4z}{z}\left(1 - \frac{a}{z}\right)^{-1} = 4\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k \quad \text{if } \left|\frac{a}{z}\right| < 1$$

$$= \sum_{k=0}^{\infty} 4a^k z^{-k}, \quad \text{where } |z| > |a|$$

$$Z^{-1}\left(\frac{4z}{z-a}\right) = \{4a^k\}.$$

(*ii*) For $|z| < |a|$, we have

$$\frac{4z}{z-a} = -\frac{4z}{a}\left(1 - \frac{z}{a}\right)^{-1}$$

$$= -\frac{4z}{a}\left(1 + \frac{z}{a} + \frac{z^2}{a^2} + \frac{z^3}{a^3} + \ldots\ldots\right), \quad \text{if } |z| < |a|$$

$$= -\frac{4z}{a} - \frac{4z^2}{a^2} - \frac{4z^3}{a^3} - \frac{4z^4}{a^4} - \ldots..$$

$$Z^{-1}\left(\frac{4z}{z-a}\right) = \{f(k)\}$$

where $\quad \{f(k)\} = \left\{\ldots\ldots -\dfrac{4}{a^4}, \dfrac{-4}{a^3}, \dfrac{-4}{a^2}, \dfrac{-4}{a}\right\}.$

Example 3. *Find the inverse Z-transform of* (U.P.T.U. 2014)

$$F(z) = \frac{1}{(z-3)(z-2)} \quad \text{for}$$

(*i*) $|z| < 2$ (*ii*) $2 < |z| < 3$ (*iii*) $|z| > 3$.

Sol.
$$F(z) = \frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2}$$

(*i*) **For | z | < 2.**

$$F(z) = -\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{3}(1 + 3^{-1}z + 3^{-2}z^2 + 3^{-3}z^3 + \ldots) + \frac{1}{2}(1 + 2^{-1}z + 2^{-2}z^2 + 2^{-3}z^3 + \ldots)$$

$$= -(3^{-1} + 3^{-2}z + 3^{-3}z^2 + \ldots) + (2^{-1} + 2^{-2}z + 2^{-3}z^2 + \ldots)$$

Here coeff. of z^{-k} (if $k > 0$) = 0

Coeff. of z^{-k} (if $k < 0$) $= -\dfrac{1}{3^{-k+1}} + \dfrac{1}{2^{-k+1}} = -3^{k-1} + 2^{k-1}$

Now, $Z^{-1}\{F(z)\} = \{f(k)\} = \{-3^{k-1}\} + \{2^{k-1}\}$.

(*ii*) **For 2 < | z | < 3.**

$$F(z) = -\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\frac{z^3}{3^3}+\ldots\ldots\right) - \frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\ldots\ldots\right)$$

$$= -(3^{-1} + 3^{-2}z + 3^{-3}z^2 + 3^{-4}z^3 + \ldots\ldots) - (z^{-1} + 2z^{-2} + 2^2z^{-3} + \ldots\ldots)$$

Here, coeff. of z^{-k} (if $k > 0$) $= -2^{k-1}$

Coeff. of z^{-k} (if $k \le 0$) $= -3^{k-1}$

Now, $$Z^{-1}\{F(z)\} = \{f(k)\} = \begin{Bmatrix} -2^{k-1}, & k > 0 \\ -3^{k-1}, & k \le 0 \end{Bmatrix}$$

(*iii*) **For | z | > 3.**

$$F(z) = \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}(1 + 3z^{-1} + 3^2z^{-2} + \ldots\ldots) - \frac{1}{z}(1 + 2z^{-1} + 2^2z^{-2} + \ldots\ldots)$$

$\therefore$ $$Z^{-1}\{F(z)\} = \{f(k)\} = \begin{cases} 3^{k-1} - 2^{k-1}, & k \ge 1 \\ 0, & k \le 0 \end{cases}.$$

2.27.3. Partial Fractional Method

Here we split the given F(z) into partial fractions whose inverse transforms can be written directly.

Example. *Find the inverse Z-transform of*

(*i*) $\dfrac{z}{z^2+7z+10}$ (*ii*) $\dfrac{z^3-20z}{(z-2)^3(z-4)}$ (*iii*) $\dfrac{8z^2}{(2z-1)(4z-1)}$.

Sol. (*i*) $$\frac{F(z)}{z} = \frac{1}{z^2+7z+10} = \frac{1}{(z+2)(z+5)} = \frac{A}{z+2} + \frac{B}{z+5}$$

$$= \frac{1}{3}\frac{1}{z+2} - \frac{1}{3}\cdot\frac{1}{z+5} \qquad \left(\because \; A = \frac{1}{3}, B = -\frac{1}{3}\right)$$

$$\therefore \quad F(z) = \frac{1}{3}\frac{z}{z+2} - \frac{1}{3}\frac{z}{z+5}$$

$$\therefore \quad f(k) = Z^{-1}\{F(z)\} = \frac{1}{3} Z^{-1}\left\{\frac{z}{z+2}\right\} - \frac{1}{3} Z^{-1}\left\{\frac{z}{z+5}\right\}$$

$$= \left\{\frac{1}{3}(-2)^k - \frac{1}{3}(-5)^k\right\} \qquad \left[\because \; Z(a^k) = \frac{z}{z-a}\right]$$

(*ii*) $$F(z) = \frac{z^3-20z}{(z-2)^3(z-4)} \quad \text{or} \quad \frac{F(z)}{z} = \frac{z^2-20}{(z-2)^3(z-4)}$$

Now $$\frac{F(z)}{z} = \frac{z^2-20}{(z-2)^3(z-4)} = \frac{A+Bz+Cz^2}{(z-2)^3} + \frac{D}{z-4}$$

$$\Rightarrow \quad D = -\frac{1}{2}, A = 6, B = 0, C = \frac{1}{2}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{6+\frac{1}{2}z^2}{(z-2)^3} + \frac{\left(-\frac{1}{2}\right)}{z-4}$$

or $$F(z) = \frac{1}{2}\left\{\frac{12z+z^3}{(z-2)^3} - \frac{z}{z-4}\right\} = \frac{1}{2}\left\{\frac{z(z-2)^2+4z^2+8z}{(z-2)^3} - \frac{z}{z-4}\right\}$$

$$= \frac{1}{2}\left\{\frac{z}{z-2} + 2\cdot\frac{2z^2+4z}{(z-2)^3} - \frac{z}{z-4}\right\}$$

Now $$f(k) = Z^{-1}\{F(z)\} = \frac{1}{2}\{2^k + 2k^2 2^k - 4^k\}, \qquad \left[\because \; Z^{-1}\left\{\frac{az^2+a^2z}{(z-a)^3}\right\} = k^2a^k\right]$$

$$= \{2^{k-1} + 2^k \cdot k^2 - 2^{2k-1}\}$$

(*iii*) $$F(z) = \frac{8z^2}{(2z-1)(4z-1)} = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$\frac{F(z)}{z} = \frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{4}} = \frac{2}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{4}}$$

$$\therefore \quad F(z) = \frac{2z}{z-(1/2)} - \frac{z}{z-(1/4)}$$

$$f(k) = Z^{-1}\{F(z)\} = \left\{2\left(\frac{1}{2}\right)^k - \left(\frac{1}{4}\right)^k\right\}, \; k = 0, 1, 2, \ldots\ldots$$

2.27.4. Inverse Integral Method (or Residue Method)

By using the theory of complex variables, it can be shown that the inverse Z-transform is given by $f(k) = \frac{1}{2\pi i} \oint_C F(z)\, z^{k-1}\, dz$, where C is the circle (may be even closed contour) which contains all the isolated singularities of F(z) and containing the origin of the z-plane in the region of convergence. Hence by Cauchy's Residue theorem,

$f(k)$ = sum of the residues of the singularities of F(z).

EXAMPLES

Example 1. *By Residue method, find the inverse Z-transform of*

(*i*) $\dfrac{z}{z^2+7z+10}$ (*ii*) $\dfrac{z}{z^2-2z+2}$ (*iii*) $\dfrac{z^2+z}{(z-1)(z^2+1)}$

(*iv*) $\dfrac{2z}{z^3-z^2+z-1}$ (*v*) $\dfrac{z(z+1)}{(z-1)^3}$ (*vi*) $\dfrac{z(z^2-1)}{(z^2+1)^2}$.

Sol. (*i*) $$F(z) = \frac{z}{z^2+7z+10}$$

$$f(k) = \frac{1}{2\pi i}\int_C z^{k-1}\, F(z)\, dz = \text{sum of residues}$$

$$= \frac{1}{2\pi i}\int_C z^{k-1}\, \frac{z}{z^2+7z+10}\, dz = \frac{1}{2\pi i}\int_C \frac{z^k}{(z+2)(z+5)}\, dz$$

Poles are $z = -2, -5$. These are simple poles.

$$\text{Residue (at } z=-2) = \operatorname*{Lt}_{z \to -2} (z+2)\, \frac{z^k}{(z+2)(z+5)} = \frac{(-2)^k}{3}$$

$$\text{Residue (at } z=-5) = \operatorname*{Lt}_{z \to -5} (z+5)\, \frac{z^k}{(z+2)(z+5)} = \frac{(-5)^k}{-3}$$

$$\therefore \quad f(k) = \text{sum of residues} = \frac{(-2)^k}{3} + \left\{\frac{(-5)^k}{-3}\right\} = \frac{1}{3}\{(-2)^k - (-5)^k\}.$$

(*ii*) $$F(z) = \frac{z}{z^2-2z+2}$$

$$f(k) = \frac{1}{2\pi i}\int_C z^{k-1}\, F(z)\, dz = \text{sum of the residues}$$

$$= \frac{1}{2\pi i}\int_C z^{k-1} \cdot \frac{z}{z^2-2z+2}\, dz = \frac{1}{2\pi i}\int_C \frac{z^k}{z^2-2z+2}\, dz$$

The poles are given by $z^2 - 2z + 2 = 0 \quad \therefore \quad z = 1 \pm i$ both are simple poles.

Residue at $z = 1 + i$ is

$$\operatorname*{Lt}_{z \to 1+i} [z-(1+i)]\, \frac{z^k}{[z-(1+i)][z-(1-i)]} = \frac{(1+i)^k}{2i}$$

Similarly, residue at $z = 1 - i$ is

$$\underset{z \to 1-i}{\text{Lt}} [z - (1-i)] \frac{z^k}{[z-(1+i)][z-(1-i)]} = \frac{(1-i)^k}{-2i}$$

$\therefore \quad f(k) = \text{sum of the residues} = \dfrac{(1+i)^k}{2i} + \dfrac{(1-i)^k}{-2i}$

$$= \left\{\frac{1}{2i}[(1+i)^k - (1-i)^k]\right\} \qquad ...(1)$$

We know that $\quad (1+i)^k = (\sqrt{2})^k \left\{\cos\dfrac{k\pi}{4} + i \sin\dfrac{k\pi}{4}\right\}$

$$(1-i)^k = (\sqrt{2})^k \left\{\cos\frac{k\pi}{4} - i \sin\frac{k\pi}{4}\right\}$$

$\therefore \quad (1+i)^k - (1-i)^k = (\sqrt{2})^k \left\{2i \sin\dfrac{k\pi}{4}\right\}$

Substituting in (1), we get $\quad f(k) = \left\{(\sqrt{2})^k \sin\dfrac{k\pi}{4}\right\}$

(*iii*) $\quad \mathrm{F}(z) = \dfrac{z^2+z}{(z-1)(z^2+1)}$

$$f(k) = \frac{1}{2\pi i}\int_{\mathrm{C}} z^{k-1}\,\mathrm{F}(z)\,dz = \text{sum of the residues}$$

$$= \frac{1}{2\pi i}\int_{\mathrm{C}} z^{k-1} \frac{z^2+z}{(z-1)(z^2+1)}\,dz = \frac{1}{2\pi i}\int_{\mathrm{C}} z^k \frac{(z+1)}{(z-1)(z^2+1)}\,dz$$

Poles are given by $z = 1, \pm i$

Residue (at $z = 1$) $= \underset{z \to 1}{\text{Lt}} (z-1)\, z^k \dfrac{(z+1)}{(z-1)(z^2+1)} = 1$

Residue (at $z = i$) $= \underset{z \to i}{\text{Lt}} (z-i) \dfrac{z^k (z+1)}{(z-1)(z-i)(z+i)} = \dfrac{i^k (i+1)}{(i-1)(2i)} = \dfrac{i^k(1+i)}{-2(1+i)} = -\dfrac{1}{2}\, i^k$

Similarly, residue (at $z = -i$) is $= -\dfrac{1}{2}(-i)^k$

$\therefore \quad f(k) = \text{sum of the residues}$

$$= 1 - \frac{1}{2} i^k - \frac{1}{2}(-i)^k = 1 - \frac{1}{2}\{i^k + (-i)^k\} \qquad ...(1)$$

$$i^k = \cos\frac{k\pi}{2} + i \sin\frac{k\pi}{2}\ ;\ (-i)^k = \cos\frac{k\pi}{2} - i\sin\frac{k\pi}{2}$$

$i^k + (-i)^k = 2\cos\dfrac{k\pi}{2}$. Substituting in (1), we get

$$f(k) = 1 - \frac{1}{2}\left\{2\cos\frac{k\pi}{2}\right\} = \left\{1 - \cos\frac{k\pi}{2}\right\}.$$

(*iv*) $$F(z) = \frac{2z}{z^3 - z^2 + z - 1} = \frac{2z}{(z-1)(z^2+1)}$$

Poles are $z = 1, z = \pm i$

Residue (at $z = 1$) $= \underset{z \to 1}{\text{Lt}} (z-1) \dfrac{2z^k}{(z-1)(z^2+1)} = 1$

Residue (at $z = i$) $= \underset{z \to i}{\text{Lt}} (z-i) \dfrac{2z^k}{(z-1)(z-i)(z+i)} = \dfrac{2i^k}{(i-1)(2i)} = -\dfrac{i^k}{1+i}$

Residue (at $z = -i$) $= -\dfrac{(-i)^k}{1-i}$

$$f(k) = \text{sum of the residues} = 1 - \left\{\frac{i^k}{1+i} + \frac{(-i)^k}{1-i}\right\}$$

$$= 1 - 2 \text{ R.P. of } \left(\frac{i^k}{1+i}\right) = 1 - 2\text{R.P. of } \frac{e^{ik\pi/2}}{\sqrt{2}\, e^{i\pi/4}}$$

$$= 1 - \sqrt{2} \text{ R.P. of } e^{ik\pi/2 - i\pi/4}$$

$$= 1 - \sqrt{2} \cos\left(\frac{k\pi}{2} - \frac{\pi}{4}\right) = \left\{1 - \left(\cos\frac{k\pi}{2} + \sin\frac{k\pi}{2}\right)\right\}.$$

(*v*) $$F(z) = \frac{z(z+1)}{(z-1)^3}$$

$z = 1$ is a pole of order 3

$\therefore$ Residue (at $z = 1$) $= \dfrac{1}{2!} \underset{z \to 1}{\text{Lt}} \dfrac{d^2}{dz^2} \left\{\dfrac{(z-1)^3 \,.\, z^k (z+1)}{(z-1)^3}\right\}$

$$= \frac{1}{2!} \underset{z \to 1}{\text{Lt}} \{k(k+1)z^{k-1} + k(k-1)z^{k-2}\}$$

$$= \frac{1}{2} \{k^2 + k + k^2 - k\} = k^2$$

$\therefore$ $f(k)$ = sum of the residues = $\{k^2\}$, $k = 0, 1, 2, \ldots\ldots$

(*vi*) $$F(z) = \frac{z(z^2-1)}{(z^2+1)^2}$$

$z = \pm i$ are poles and each is a pole of order 2.

Residue (at $z = i$) $= \underset{z \to i}{\text{Lt}} \dfrac{d}{dz} \left\{(z-1)^2 \dfrac{z^k (z^2-1)}{(z-i)^2 (z+i)^2}\right\} = \underset{z \to i}{\text{Lt}} \dfrac{d}{dz} \left\{\dfrac{z^k (z^2-1)}{(z+i)^2}\right\}$

$$= \underset{z \to i}{\text{Lt}} \frac{(z+i)^2 [z^k \,.\, 2z + kz^{k-1}(z^2-1)] - z^k (z^2-1) \,.\, 2(z+i)}{(z+i)^4} = \frac{k}{2} i^{k-1}$$

Similarly, R_2 = Residue of $f(z)$ (at $z = -i$) $= \dfrac{k}{2}(-i)^{k-1}$

$\therefore$ $f(k)$ = sum of the residues $= \left\{\dfrac{k}{2} [i^{k-1} + (-i)^{k-1}]\right\}$, $k = 0, 1, 2, \ldots..$

Example 2. *Using residue's method, evaluate* $Z^{-1}\left\{\dfrac{8z-z^3}{(4-z)^3}\right\}$.

Sol. Here, $$F(z) = \frac{(8z-z^3)}{(4-z)^3}$$

Poles are given by, $z = 4$ (pole of order 3)

Residue of F(z) (at $z = 4$) is

$$R = \frac{1}{(3-1)!}\left[\frac{d^2}{dz^2}\left\{(z-4)^3\, z^{k-1}\,.\,\frac{8z-z^3}{(4-z)^3}\right\}\right]_{z=4}$$

$$= \frac{1}{2}\left[\frac{d^2}{dz^2}\{(z^3-8z)z^{k-1}\}\right]_{z=4} = \frac{1}{2}\left[\frac{d^2}{dz^2}(z^{k+2}-8z^k)\right]_{z=4}$$

$$= \frac{1}{2}\,[(k+2)(k+1)z^k - 8k(k-1)z^{k-2}]_{z=4}$$

$$= \frac{1}{2}\,[(k+2)(k+1)\,(4)^k - \frac{1}{2}\,k(k-1)\,(4)^k]$$

$$= \frac{4^k}{2}\left[k^2+3k+2-\frac{1}{2}(k^2-k)\right] = (k^2+7k+4)\,(4)^{k-1}$$

$$\therefore \quad Z^{-1}\left\{\frac{8z-z^3}{(4-z)^3}\right\} = R = \{(k^2+7k+4)\,(4)^{k-1}\}.$$

Example 3. *Using residue's method, show that*

$$Z^{-1}\left\{\frac{3z^2+2}{(5z-1)(5z+2)}\right\} = \frac{53}{75}\,(.2)^k + \frac{31}{75}\,(-.4)^k.$$

Sol. Here, $$F(z) = \frac{3z^2+2}{(5z-1)(5z+2)}$$

Poles are given by $(5z-1)(5z+2) = 0$

$\Rightarrow \quad z = \dfrac{1}{5}, \dfrac{-2}{5}$ which are simple poles. Consider a contour $|\, z \,| = 1$

Residue (at $z = \dfrac{1}{5}$) is $$R_1 = \underset{z\to\frac{1}{5}}{\text{Lt}}\left[\left(z-\frac{1}{5}\right)z^{k-1}\,.\,\frac{3z^2+2}{(5z-1)(5z+2)}\right]$$

$$= \underset{z\to\frac{1}{5}}{\text{Lt}}\left[\frac{1}{5}\,.\,\frac{1}{(5z+2)}\,z^{k-1}\,(3z^2+2)\right]$$

$$= \frac{1}{15}\left(\frac{1}{5}\right)^{k-1}\left(\frac{3}{25}+2\right) = \frac{53}{75}\,(.2)^k$$

Residue (at $z = -2/5$) is

$$R_2 = \underset{z\to-2/5}{\text{Lt}}\left(z+\frac{2}{5}\right).\,z^{k-1}\,.\,\frac{3z^2+2}{(5z-1)(5z+2)}$$

$$= \underset{z \to -2/5}{\text{Lt}} \left[\frac{1}{5} \cdot \frac{1}{(5z-1)} z^{k-1} (3z^2 + 2) \right]$$

$$= \frac{1}{5} \cdot \left(-\frac{1}{3} \right) \cdot \left(\frac{-2}{5} \right)^{k-1} \left(\frac{12}{25} + 2 \right) = \frac{(-1)^k \cdot 2^{k-1}}{(5)^{k+2}} \cdot \frac{62}{3}$$

$$= \frac{62}{75} (-1)^k \cdot 2^{k-1} (.2)^k = \frac{31}{75} (-.4)^k$$

Hence, $Z^{-1} \{F(z)\} = \{f(k)\} = \text{sum of residues} = R_1 + R_2 = \frac{53}{75} (.2)^k + \frac{31}{75} (-.4)^k.$

2.27.5. Power Series Method

In this method we find the inverse Z-transform by expanding F(z) in power series as illustrated in the following example.

Example. *Find* $Z^{-1} \left\{ \log \left(\frac{z}{z+1} \right) \right\}$ *by power series method.*

Sol. $F(z) = \log \left\{ \frac{z}{z+1} \right\},$

Let $z = \frac{1}{y}$ then $F(z) = \log \left\{ \frac{1/y}{1/y + 1} \right\} = -\log(1+y)$

$$= -y + \frac{1}{2} y^2 - \frac{1}{3} y^3 + \ldots\ldots = -\frac{1}{z} + \frac{1}{2z^2} - \frac{1}{3z^3} + \cdots\cdots + \frac{(-1)^k}{k} z^{-k}$$

$$\therefore \quad f(k) = z^{-1} \{F(z)\} = \begin{cases} 0, & \text{for } k = 0 \\ \dfrac{(-1)^k}{k}, & \text{otherwise} \end{cases}.$$

2.28 SOME IMPORTANT INVERSE Z-TRANSFORM RESULTS

S.No.	*F(z)*	*{f(k)}* *where* $\lvert z \rvert > \lvert a \rvert, k > 0$	*{f(k)}* *where* $\lvert z \rvert < \lvert a \rvert, k < 0$
1.	$\frac{1}{z-a}$	$a^{k-1}\, U(k-1)$	$-a^{k-1}\, U(-k)$
2.	$\frac{z}{z-a}$	$a^k\, U(k)$ or a^k	$-a^k$
3.	$\frac{z^2}{(z-a)^2}$	$(k+1)\, a^k$	$-(k+1)\, a^k$
4.	$\frac{1}{(z-a)^2}$	$(k-1)\, ak^{-2}\, U(k-2)$	$-(k-1)\, ak^{-2}\, U(-k+1)$
5.	$\frac{z^n}{(z-a)^n}$	$\frac{1}{(n-1)!} (k+1) \ldots\ldots (k+n-1)\, a^k\, U(k)$	$-\frac{1}{(n-1)!} (k+1) \ldots\ldots (k+n-1)\, a^k$

2.29 DIFFERENCE EQUATIONS

The inherent discrete nature of some physical phenomena gives rise to work with discrete functions. The mathematical models in which a variable can have only discrete set of values, give a chance to study difference equations. Difference calculus also forms the basis of difference equations which arise in the theory of probability, in the study of electrical networks, in statistical problems and in all situations where sequential relation exists at various discrete values of the independent variable.

2.30 DEFINITION

A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Or

An equation which connects various differences of an unknown function is called a difference equation.

A difference equation is a relationship of the form

$$\mathrm{F}\left[x, y, \frac{\Delta y}{\Delta x}, \frac{\Delta^2 y}{\Delta x^2}, \ldots\ldots, \frac{\Delta^n y}{\Delta x^n}\right] = 0 \qquad \ldots(1)$$

Let $y = f(x)$, then

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

$$\vdots$$

$$\frac{\Delta^n y}{\Delta x^n} = \frac{f(x+nh) - \ldots\ldots + \ldots\ldots}{h^n}\ldots,$$

where h is the interval of differencing. Hence eqn. (1) can be rewritten as

$$\phi\,[x, f(x), f(x+h), f(x+2h), \ldots\ldots, f(x+nh)] = 0 \qquad \ldots(2)$$

where $f(x) = y$ is an unknown function.

e.g.

$$\Delta y_k + 2y_k = 0 \qquad \ldots(3)$$

$$\Delta^2 y_k + 3\Delta y_k + y_k = 0 \qquad \ldots(4)$$

If we put $\Delta \equiv \mathrm{E} - 1$, where E is an operator called shift operator such that $\mathrm{E}\{f(x)\} = f(x+1)$, then eqns. (3) and (4) can be rewritten as

$$(\mathrm{E} - 1)y_k + 2y_k = 0$$

$$\Rightarrow \quad (\mathrm{E} + 1)y_k = 0 \qquad \ldots(5)$$

and

$$(\mathrm{E} - 1)^2 y_k + 3(\mathrm{E} - 1)y_k + y_k = 0$$

$$\Rightarrow \quad \mathrm{E}^2 y_k + \mathrm{E}y_k - y_k = 0 \qquad \ldots(6)$$

Eqns. (3) and (4) may also be put as

$$y_{k+1} + y_k = 0;\ y_{k+2} + y_{k+1} - y_k = 0.$$

2.31 ORDER OF A DIFFERENCE EQUATION

The order of a difference equation is defined as the difference between the largest and the smallest arguments for the function involved divided by h, the interval of differencing.

Consider the difference equation

$$y_{k+2} + y_{k+1} - y_k = 0$$

$$\text{Order} = \frac{(k+2)-k}{1} = 2.$$

Note. While finding the order of a difference equation, it must always be expressed in a form free of Δ's.

2.32 DEGREE OF A DIFFERENCE EQUATION

The degree of a difference equation is defined to be the highest power of $f(x)$.

2.33 SOLUTION OF A DIFFERENCE EQUATION

A solution of a difference equation is any function which satisfies the given equation. The general solution of a difference equation is defined as the solution which involves as many arbitrary constants as the order of the difference equation.

The particular solution is a solution obtained from the general solution by assigning particular values to periodic constants.

Consider a difference equation

$$y_{h+1} - 2y_h = 0 \; ; \; h = 0, 1, 2, \ldots\ldots \qquad \ldots(1)$$

Let

$$y_h = 2^h, \; h = 0, 1, 2, \ldots.. \qquad \ldots(2)$$

The function y_h defined by (2) satisfies the difference equation (1) so it is called a solution of eqn. (1). Generally eqn. (1) is satisfied by

$$y_h = c \,.\, 2^h \text{ for any constant } c \qquad \ldots(3)$$

The function y_h given by eqn. (2) is a particular solution of eqn. (1) while function y_h in eqn. (3) is the general solution of eqn. (1).

Remark. A difference equation may have no solution just as in case of algebraic equation *e.g.* $(y_{h+1} - y_h)^2 + {y_h}^2 = -1$ is satisfied for no real-valued function y.

We will now proceed to solve difference equations with the help of Z-transforms.

2.34 APPLICATIONS OF Z-TRANSFORMS TO DIFFERENCE EQUATIONS

Z-transform is useful in solving difference equations.

The given difference equation can be converted to the form $\bar{y} = \phi(z)$ by taking Z-transform on it, provided the initial values of y are known. Using inversion, we can get the value of y_k which is the solution of the given difference equations.

2.35 PROVE THAT

$$\mathbf{Z(y_{k+n}) = z^n \left(\bar{y} - y_0 - \frac{y_1}{z} - \ldots - \frac{y_{n-1}}{z^{n-1}}\right), \text{ where } Z(y_k) = \bar{y}.}$$

Proof.

$$\text{LHS} = \text{Z}\,(y_{k+n}) = \sum_{k=0}^{\infty} y_{k+n}\, z^{-k} = z^n \sum_{k=0}^{\infty} y_{k+n}\, z^{-(n+k)}$$

Setting $m = n + k$, we get

$$z(y_{k+n}) = z^n \sum_{m=n}^{\infty} y_m\, z^{-m} = z^n \left[\sum_{m=0}^{\infty} y_m\, z^{-m} - \sum_{m=0}^{n-1} y_m\, z^{-m}\right]$$

$$= z^n\left[\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2} - \ldots\ldots - \frac{y_{n-1}}{z^{n-1}}\right].$$

Note. For $n = 1, 2, 3, \ldots\ldots$, we have

$$Z(y_{k+1}) = z(\bar{y} - y_0)$$

$$Z(y_{k+2}) = z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right)$$

$$Z(y_{k+3}) = z^3\left(\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2}\right) \text{ and so on ...}$$

Note. If $Z(y_k) = \bar{y}$, then $Z(y_{k-n}) = z^{-n}\,\bar{y}$.

EXAMPLES

Example 1. *Solve by Z-transform:* $y_{k+1} + y_k = 1$ *if* $y_0 = 0$.

Sol. Take Z-transform on both sides, we get

$$Z(y_{k+1}) + Z(y_k) = Z(1)$$

$$\therefore \quad z(\bar{y} - y_0) + \bar{y} = \frac{z}{z-1}$$

$$\Rightarrow \quad \bar{y}(z+1) = \frac{z}{z-1} \qquad (\because \; y_0 = 0)$$

$$\therefore \quad \bar{y} = \frac{z}{(z-1)(z+1)} = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z+1}\right]$$

Take inverse Z-transform,

$$\therefore \quad y_k = \frac{1}{2}\left[Z^{-1}\left(\frac{z}{z-1}\right) - Z^{-1}\left(\frac{z}{z+1}\right)\right] = \frac{1}{2}\,\{1 - (-1)^k\}.$$

Example 2. *Solve by Z-transform:* $y_{k+2} - 3y_{k+1} + 2y_k = 0;\ y_0 = 0,\ y_1 = 1$.

Sol. Take Z-transform on both sides,

$$Z(y_{k+2}) - 3Z(y_{k+1}) + 2Z(y_k) = Z(0)$$

$$\Rightarrow \quad z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) - 3z\,(\bar{y} - y_0) + 2\bar{y} = 0$$

$$(z^2 - 3z + 2)\,\bar{y} = z$$

$$\therefore \quad \bar{y} = \frac{z}{z^2 - 3z + 2} = \frac{z}{(z-1)(z-2)} = \frac{z}{z-2} - \frac{z}{z-1}$$

Take inverse Z-transform, we get

$$y_k = Z^{-1}\left(\frac{z}{z-2} - \frac{z}{z-1}\right) = \{2^k - 1\} \qquad \text{where } k = 0, 1, 2, \ldots.$$

Example 3. *Solve the following difference equations using Z-transform*

$$y_{k+1} - 2y_k = 1;\ k \ge 0,\ y_0 = 1.$$

Sol. Taking Z-transform on both sides, we get

$$Z(y_{k+1}) - 2\,Z(y_k) = Z(1)$$

$$\Rightarrow \qquad z(\bar{y} - y_0) - 2\bar{y} = \frac{z}{z-1}$$

$$\Rightarrow \qquad (z-2)\,\bar{y} = \frac{z}{z-1} + z = \frac{z^2}{z-1} \qquad \Big|\ \because\ y_0 = 1$$

$$\Rightarrow \qquad \bar{y} = \frac{z^2}{(z-1)(z-2)}$$

$$\Rightarrow \qquad \frac{\bar{y}}{z} = \frac{z}{(z-1)(z-2)} = \frac{2}{z-2} - \frac{1}{z-1}$$

or

$$\bar{y} = \frac{2z}{z-2} - \frac{z}{z-1}$$

Taking inverse Z-transform, we get

$$y_k = 2\,Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z-1}\right) = 2\,(2^k) - 1 = \{2^{k+1} - 1\}.$$

Example 4. *Solve using Z-transform :* $y_{x+2} - 2\cos\alpha\, y_{x+1} + y_x = 0$, $y_0 = 0$, $y_1 = 1$.

Sol. Take Z-transform on both sides,

$$z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) - 2\cos\alpha\,.\,z\,(\bar{y} - y_0) + \bar{y} = 0$$

$$\Rightarrow \qquad z^2\left(\bar{y} - \frac{1}{z}\right) - 2z\cos\alpha\,.\,\bar{y} + \bar{y} = 0$$

$$\bar{y} = \frac{z}{z^2 - 2z\cos\alpha + 1} = \frac{z}{(z - e^{i\alpha})(z - e^{-i\alpha})}$$

$$\bar{y} = \frac{-i}{2\sin\alpha}\left(\frac{z}{z - e^{i\alpha}}\right) + \frac{i}{2\sin\alpha}\,.\left(\frac{z}{z - e^{-i\alpha}}\right)$$

$\therefore$ Taking inverse Z-transform, we get

$$y_x = \frac{-i}{2\sin\alpha}\,[(e^{i\alpha})^x - (e^{-i\alpha})^x]$$

$$= \frac{-i}{2\sin\alpha}\,[(\cos\alpha x + i\sin\alpha x) - (\cos\alpha x - i\sin\alpha x)]$$

$$= \left\{\frac{\sin\alpha x}{\sin\alpha}\right\}.$$

Example 5. *Solve using Z-transform :* $y_k + \dfrac{1}{4}\, y_{k-1} = \delta(k) + \dfrac{1}{3}\,\delta(k-1)$

where $\delta(k)$ *is unit impulse sequence.*

Sol. Taking Z-transform on both sides, we get

$$Z(y_k) + \frac{1}{4}\,Z(y_{k-1}) = Z\{\delta(k)\} + \frac{1}{3}\,Z\{\delta(k-1)\}$$

$$\Rightarrow \qquad \bar{y} + \frac{1}{4}\,z^{-1}\,\bar{y} = 1 + \frac{1}{3}\,z^{-1}$$

$$\Rightarrow \qquad \bar{y} = \frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{4}z^{-1}} = \frac{z+\frac{1}{3}}{z+\frac{1}{4}}$$

There is only one simple pole at $z = -\frac{1}{4}$.

$$\text{Residue } \left(\text{at } z = \frac{-1}{4}\right) \text{ is } = \left[\left(z+\frac{1}{4}\right) . z^{k-1} . \frac{z+\frac{1}{3}}{\left(z+\frac{1}{4}\right)}\right]_{z=-\frac{1}{4}}$$

$$= \left[z^{k-1} . \left(z+\frac{1}{3}\right)\right]_{z=-\frac{1}{4}} = \left(\frac{1}{3}-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} = \frac{1}{12} . \left(-\frac{1}{4}\right)^{k-1}$$

$$\therefore \qquad y_k = \text{Residue} = \left\{\frac{1}{12}\left(-\frac{1}{4}\right)^{k-1}\right\}.$$

Example 6. *Solve by Z-transform* $y_{k+1} + \frac{1}{4} y_k = \left(\frac{1}{4}\right)^k$ *; $k \geq 0$, $y_0 = 0$.* (U.P.T.U. 2009)

Sol. Take Z-transform on both sides,

$$Z(y_{k+1}) + \frac{1}{4} Z(y_k) = Z\left[\left(\frac{1}{4}\right)^k\right]$$

$$\Rightarrow \qquad z(\bar{y} - y_0) + \frac{1}{4}\bar{y} = \frac{z}{z-\frac{1}{4}}$$

$$\therefore \qquad \bar{y} = \frac{z}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)}$$

$$\frac{\bar{y}}{z} = 2\left[\frac{1}{z-\frac{1}{4}} - \frac{1}{z+\frac{1}{4}}\right]$$

$$\therefore \qquad \bar{y} = 2\left(\frac{z}{z-\frac{1}{4}} - \frac{z}{z+\frac{1}{4}}\right)$$

Take inverse Z-transform on both sides,

$$y_k = 2\left\{\left(\frac{1}{4}\right)^k - \left(-\frac{1}{4}\right)^k\right\}.$$

Example 7. *Solve:* $y_k + \frac{1}{25} y_{k-2} = \left(\frac{1}{5}\right)^k \cos\frac{k\pi}{2}$ *($k \geq 0$) by residue method.*

Sol. Take Z-transform on both sides, we get

$$Z\left(y_k + \frac{1}{25} y_{k-2}\right) = Z\left[\left(\frac{1}{5}\right)^k \cos k\pi/2\right]$$

$$\Rightarrow \quad \bar{y} + \frac{1}{25} z^{-2}\, \bar{y} = \frac{z^2}{z^2 + \frac{1}{25}}$$

$\because$ $Z(c^k \cos \alpha k) = \dfrac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}$

Here $\alpha = \pi/2, c = 1/5$

$$\therefore \quad \bar{y} = \frac{z^2}{\left(z^2 + \frac{1}{25}\right)\left(1 + \frac{1}{25} z^{-2}\right)} = \frac{z^4}{\left(z^2 + \frac{1}{25}\right)^2}$$

There are two poles of II order at $z = \dfrac{i}{5}$ and $z = \dfrac{-i}{5}$.

$$\text{Residue } \left(\text{at } z = \frac{i}{5}\right) = \left[\frac{d}{dz}\left\{\left(z - \frac{i}{5}\right)^2 . z^{k-1} . \frac{z^4}{\left(z^2 + \frac{1}{25}\right)^2}\right\}\right]_{z = \frac{i}{5}}$$

$$= \left[\frac{d}{dz} . \left\{\frac{z^{k+3}}{\left(z + \frac{i}{5}\right)^2}\right\}\right]_{z = \frac{i}{5}} = \left[\frac{\left(z + \frac{i}{5}\right)^2 (k+3)\, z^{k+2} - z^{k+3}\, 2\left(z + \frac{i}{5}\right)}{\left(z + \frac{i}{5}\right)^4}\right]_{z = \frac{i}{5}}$$

$$= \left(\frac{k+2}{4}\right) . \left(\frac{1}{5}\right)^k . i^k = \left(\frac{k+2}{4}\right) . \left(\frac{1}{5}\right)^k \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^k$$

$$= \frac{(k+2)}{4 . (5^k)} . \left(\cos k\frac{\pi}{2} + i \sin k\frac{\pi}{2}\right)$$

$$\text{Residue } \left(\text{at } z = \frac{-i}{5}\right) = \frac{(k+2)}{4 . (5^k)} \left(\cos\frac{k\pi}{2} - i \sin\frac{k\pi}{2}\right)$$

$$y_k = \text{sum of residues} = \frac{(k+2)}{2 . (5^k)} \cos \frac{k\pi}{2} .$$

Example 8. *Solve by Z-transform the difference equation*

$$y_{k+2} + 6y_{k+1} + 9y_k = 2^k \; ; (y_0 = y_1 = 0).$$ (U.P.T.U. 2010, 2014)

Sol. Taking Z-transform on both sides, we get

$$Z(y_{k+2}) + Z(6y_{k+1}) + Z(9y_k) = Z(2^k)$$

$$\Rightarrow \quad z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) + 6z\,(\bar{y} - y_0) + 9\bar{y} = \frac{z}{z-2}$$

$$\Rightarrow \quad (z^2 + 6z + 9)\bar{y} = \frac{z}{z-2}$$

$$\Rightarrow \quad \bar{y} = \frac{z}{(z-2)(z+3)^2}$$

Poles are given by $(z-2)(z+3)^2 = 0 \Rightarrow z = 2, -3$

There are two poles out of which one is simple and other is double.

$$\text{Residue (at } z = 2) = \left[(z-2)\, z^{k-1} . \frac{z}{(z-2)(z+3)^2}\right]_{z=2} = \left[\frac{z^k}{(z+3)^2}\right]_{z=2} = \frac{2^k}{25}$$

Residue (at $z = -3$) $= \dfrac{1}{(2-1)!}\left[\dfrac{d^{2-1}}{dz^{2-1}}\left\{(z+3)^2 \,.\, z^{k-1}\dfrac{z}{(z-2)(z+3)^2}\right\}\right]_{z=-3}$

$$= \left[\frac{d}{dz}\left\{\frac{z^k}{(z-2)}\right\}\right]_{z=-3} = \left[\frac{(z-2)\,.\,kz^{k-1} - z^k}{(z-2)^2}\right]_{z=-3}$$

$$= \frac{-5k(-3)^{k-1} - (-3)^k}{25} = -\frac{1}{5}\,k(-3)^{k-1} - \frac{(-3)^k}{25}$$

Hence $f(k)$ = Sum of the residues $= \left\{\dfrac{2^k}{25} - \dfrac{k}{5}\,(-3)^{k-1} - \dfrac{(-3)^k}{25}\right\}$.

Example 9. *Using Z-transform, solve the following difference equation:*

$$y_{k+2} + 4y_{k+1} + 3y_k = 3^k, \text{ given } \quad y_0 = 0 \text{ and } y_1 = 1.$$

Sol. Taking Z-transform on both sides, we get

$$Z(y_{k+2}) + 4Z(y_{k+1}) + 3Z(y_k) = Z(3^k)$$

$$\Rightarrow \quad z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) + 4z(\bar{y} - y_0) + 3\bar{y} = \frac{z}{z-3}$$

$$\Rightarrow \quad (z^2 + 4z + 3)\bar{y} - z = \frac{z}{z-3}$$

$$\Rightarrow \quad \bar{y} = \frac{z}{(z-3)(z+1)(z+3)} + \frac{z}{(z+1)(z+3)}$$

Taking inverse Z-transform,

$$y_k = Z^{-1}\left\{\frac{z}{(z-3)(z+1)(z+3)}\right\} + Z^{-1}\left\{\frac{z}{(z+1)(z+3)}\right\}$$

$$= Z^{-1}\{P(z)\} + Z^{-1}\{Q(z)\}$$

Residue of P(z) at ($z = 3$) $= \underset{z\to 3}{\text{Lt}}\ (z-3).\,z^{k-1}\,.\,\dfrac{z}{(z-3)(z+1)(z+3)} = \dfrac{3^k}{24}$

Residue of P(z) at ($z = -1$) $= \underset{z\to -1}{\text{Lt}}\ (z+1)\,.\,z^{k-1}\,.\,\dfrac{z}{(z-3)(z+1)(z+3)} = \dfrac{(-1)^k}{(-8)}$

Residue of P(z) at ($z = -3$) $= \underset{z\to -3}{\text{Lt}}\ (z+3)\,.\,z^{k-1}\,.\,\dfrac{z}{(z-3)(z+1)(z+3)} = \dfrac{(-3)^k}{12}$

Residue of Q(z) at ($z = -1$) $= \underset{z\to -1}{\text{Lt}}\ (z+1)\,.\,z^{k-1}\,.\,\dfrac{z}{(z+1)(z+3)} = \dfrac{(-1)^k}{2}$

Residue of Q(z) at ($z = -3$) $= \underset{z\to -3}{\text{Lt}}\ (z+3)\,.\,z^{k-1}\,.\,\dfrac{z}{(z+1)(z+3)} = \dfrac{(-3)^k}{-2}$

$\therefore \quad y_k$ = sum of residues $= \left\{\dfrac{3^k}{24} - \dfrac{(-1)^k}{8} + \dfrac{(-3)^k}{12}\right\} + \left\{\dfrac{(-1)^k}{2} - \dfrac{(-3)^k}{2}\right\}$

$$= \left\{\frac{3^k}{24} + \frac{3}{8}(-1)^k - \frac{5}{12}(-3)^k\right\}.$$

Example 10. *Use Z-transform to solve the difference equation:*

$$y_{k+2} - 2y_{k+1} + y_k = 3k + 5.$$

Sol. We have,

$$y_{k+2} - 2y_{k+1} + y_k = 3k + 5. \quad ...(1)$$

Taking Z-transform on both sides of the given equation, we get

$$z^2\left(\overline{y} - y_0 - \frac{y_1}{z}\right) - 2z\,(\overline{y} - y_0) + \overline{y} = \text{Z}(3k + 5) = 3\,.\,\frac{z}{(z-1)^2} + 5\,.\,\frac{z}{z-1} \quad ...(2)$$

Let $y_0 = a$ and $y_1 = b$ then, (2) becomes

$$z^2\left(\overline{y} - a - \frac{b}{z}\right) - 2z(\overline{y} - a) + \overline{y} = \frac{3z}{(z-1)^2} + \frac{5z}{z-1}$$

$$(z^2 - 2z + 1)\overline{y} - az^2 + 2az - bz = \frac{3z}{(z-1)^2} + \frac{5z}{z-1}$$

$$\Rightarrow \quad (z-1)^2\overline{y} = \frac{3z}{(z-1)^2} + \frac{5z}{z-1} + az^2 + (b - 2a)\,z$$

$$\overline{y} = \frac{3z}{(z-1)^4} + \frac{5z}{(z-1)^3} + \frac{az^2}{(z-1)^2} + (b-2a)\frac{z}{(z-1)^2} \quad ...(3)$$

Taking inverse Z-transform, we get

$$y_k = \text{Z}^{-1}\left[\frac{3z}{(z-1)^4}\right] + \text{Z}^{-1}\left[\frac{5z}{(z-1)^3}\right] + \text{Z}^{-1}\left[\frac{az^2 + (b-2a)\,z}{(z-1)^2}\right] \quad ...(4)$$

Now, let $\quad \text{F}(z) = \dfrac{3z}{(z-1)^4}$

Pole is $z = 1$ of order 4.

$$\text{Residue (at } z = 1) = \frac{1}{3\,!}\left[\frac{d^3}{dz^3}\left\{(z-1)^4\,.\,z^{k-1}\,.\,\frac{3z}{(z-1)^4}\right\}\right]_{z=1} = \frac{1}{6}\left[\frac{d^3}{dz^3}(3z^k)\right]_{z=1}$$

$$= \frac{1}{2}\left[k(k-1)(k-2)z^{k-3}\right]_{z=1} = \frac{k(k-1)(k-2)}{2}$$

$$\therefore \quad \text{Z}^{-1}[\text{F}(z)] = \frac{k(k-1)(k-2)}{2}$$

Let $\quad \text{G}(z) = \dfrac{5z}{(z-1)^3}$

Pole is $z = 1$ of order 3.

$$\text{Residue (at } z = 1) = \frac{1}{2\,!}\left[\frac{d^2}{dz^2}\left\{(z-1)^3\,.\,\frac{5z}{(z-1)^3}\,.\,z^{k-1}\right\}\right]_{z=1} = \frac{1}{2}\left[\frac{d^2}{dz^2}(5z^k)\right]_{z=1}$$

$$= \frac{5}{2}\,[k(k-1)z^{k-2}]_{z=1} = \frac{5k\,(k-1)}{2}$$

$$\therefore \quad \text{Z}^{-1}[\text{G}(z)] = \frac{5}{2}\,k(k-1)$$

Let $\quad \text{H}(z) = \dfrac{az^2 + (b-2a)z}{(z-1)^2}$

Pole is $z = 1$ of order 2.

$$\text{Residue (at } z = 1) = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{z^{k-1}(z-1)^2 \cdot \frac{\{az^2+(b-2a)z\}}{(z-1)^2}\right\}\right]_{z=1}$$

$$= \left(\frac{d}{dz}\left[z^{k-1}\{az^2+(b-2a)z\}\right]\right)_{z=1} = \left[\frac{d}{dz}(az^{k+1}+(b-2a)z^k)\right]_{z=1}$$

$$= [a(k+1)z^k + (b-2a)\,kz^{k-1}]_{z=1} = a(k+1) + (b-2a)k$$

$$= a + bk - ak = a(1-k) + bk$$

$$\therefore \qquad Z^{-1}[H(z)] = a(1-k) + bk$$

Now, from (4),

$$y_k = \frac{k(k-2)(k-1)}{2} + \frac{5}{2}\,k(k-1) + bk - a(k-1)$$

$$= \frac{k(k-1)}{2}\,(k+3) + bk - a(k-1)$$

$$= \frac{k(k-1)(k+3)}{2} + (b-a)k + a$$

$$= \left\{\frac{k(k-1)(k+3)}{2} + C_0 + C_1 k\right\}.$$

where, $C_0 = a$ and $C_1 = b - a$.

Example 11. *Using the Z-transform, solve the following difference equation:*

$$6y_{k+2} - y_{k+1} - y_k = 0,\ y(0) = 0,\ y(1) = 1.$$

Sol. Taking Z-transform on both sides of given equation, we get

$$Z(6y_{k+2}) - Z(y_{k+1}) - Z(y_k) = Z(0)$$

$$\Rightarrow \qquad 6z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) - z(\bar{y} - y_0) - \bar{y} = 0$$

$$6z^2\left(\bar{y} - \frac{1}{z}\right) - z\bar{y} - \bar{y} = 0$$

$$(6z^2 - z - 1)\bar{y} = 6z$$

or
$$\bar{y} = \frac{6z}{6z^2 - z - 1} = \frac{6z}{(3z+1)(2z-1)} = \frac{z}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

or
$$\frac{\bar{y}}{z} = \frac{6}{5}\left[\left(\frac{1}{z-\frac{1}{2}}\right) - \left(\frac{1}{z+\frac{1}{3}}\right)\right] \quad \text{or} \quad \bar{y} = \frac{6}{5}\left(\frac{z}{z-\frac{1}{2}} - \frac{z}{z+\frac{1}{3}}\right)$$

Taking inverse Z-transform on both sides, we get

$$y_k = \frac{6}{5}\left[Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) - Z^{-1}\left(\frac{z}{z+\frac{1}{3}}\right)\right] = \frac{6}{5}\left\{\left(\frac{1}{2}\right)^k - \left(-\frac{1}{3}\right)^k\right\}.$$

Example 12. *Using the Z-transform, solve the following difference equation:*

$$y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k = u(k);\ y(0) = y(1) = y(2) = 0.$$

Sol. We have,

$$y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k = u(k) \quad \ldots(1)$$

Taking Z-transform on both sides of equation (1), we get

$$z^3\left(\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2}\right) - 3z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) + 3z\,(\bar{y} - y_0) - \bar{y} = \frac{z}{z-1}$$

$$\Rightarrow \quad (z^3 - 3z^2 + 3z - 1)\,\bar{y} = \frac{z}{z-1}$$

$$\Rightarrow \quad \bar{y} = \frac{z}{(z-1)^4}$$

Taking inverse Z-transform, we get

$$y_k = Z^{-1}\left[\frac{z}{(z-1)^4}\right]$$

Now, $\quad F(z) = \dfrac{z}{(z-1)^4}$

Pole is $z = 1$ (Pole of order 4)

Residue at this pole $= \dfrac{1}{3!}\left[\dfrac{d^3}{dz^3}\left\{(z-1)^4 \cdot z^{k-1} \cdot \dfrac{z}{(z-1)^4}\right\}\right]_{z=1} = \dfrac{1}{6}\left[\dfrac{d^3}{dz^3}(z^k)\right]_{z=1}$

$$= \frac{1}{6}[k(k-1)(k-2)\,z^{k-3}]_{z=1} = \frac{k(k-1)(k-2)}{6}$$

Hence, $\quad y_k = \left\{\dfrac{k(k-1)(k-2)}{6}\right\},\ k \geq 3.$

Example 13. *Using the Z-transform, solve the following difference equation:*

$$y_k - \left(\frac{1}{3}\right) y_{k-1} = \left\{\left(\frac{1}{3}\right)^k\right\},\ k \geq 0,\ y(0) = 0.$$

Sol. Taking Z-transform on both sides, we get

$$Z(y_k) - \frac{1}{3}\,Z(y_{k-1}) = Z\left\{\left(\frac{1}{3}\right)^k\right\}$$

$$\Rightarrow \quad \bar{y} - \frac{1}{3}\,z^{-1}\bar{y} = \frac{z}{z - \frac{1}{3}}$$

$$\Rightarrow \quad \left(1 - \frac{1}{3}z^{-1}\right)\bar{y} = \frac{z}{z - \frac{1}{3}} \quad \Rightarrow \quad \bar{y} = \frac{z^2}{\left(z - \frac{1}{3}\right)^2} \quad \ldots(1)$$

Taking inverse Z-transform on both sides, we get

$$y_k = Z^{-1}\left[\frac{z^2}{\left(z - \frac{1}{3}\right)^2}\right] \quad \ldots(2)$$

Here, $$F(z) = \frac{z^2}{\left(z - \frac{1}{3}\right)^2}$$

Poles are $z = \frac{1}{3}$ which are of order 2.

Residue at this pole $$= \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{\left(z-\frac{1}{3}\right)^2 . z^{k-1} . \frac{z^2}{\left(z-\frac{1}{3}\right)^2}\right\}\right]_{z=\frac{1}{3}}$$

$$= \left[\frac{d}{dz}(z^{k+1})\right]_{z=\frac{1}{3}} = \left[(k+1)z^k\right]_{z=\frac{1}{3}} = \frac{k+1}{3^k}$$

$\therefore$ From (2), $$y_k = \left\{\frac{k+1}{3^k}\right\}.$$

Example 14. *Using the Z-transform, solve the following difference equation:*

$$y_k + \left(\frac{1}{16}\right) y_{k-2} = \left\{\left(\frac{1}{4}\right)^k \cos\left(\frac{k\pi}{2}\right)\right\},\ k \geq 0.$$

Sol. $$y_k + \frac{1}{16}\, y_{k-2} = \left(\frac{1}{4}\right)^k \cos\frac{k\pi}{2},\ k \geq 0 \quad \text{...(1)}$$

Taking Z-transform on both sides of equation (1), we get

$$\bar{y} + \frac{1}{16} z^{-2} \bar{y} = \frac{z^2}{z^2 + \frac{1}{16}}$$

$$\left(1 + \frac{1}{16} z^{-2}\right) \bar{y} = \frac{z^2}{z^2 + \frac{1}{16}}$$

$$\bar{y} = \frac{z^4}{\left(z^2 + \frac{1}{16}\right)\left(z^2 + \frac{1}{16}\right)} = \frac{z^4}{\left(z^2 + \frac{1}{16}\right)^2}$$

There are two poles of II order at $z = \frac{i}{4}$ and $z = \frac{-i}{4}$. Consider a contour $|\, z \,| = \frac{1}{2}$

Residue $\left(\text{at } z = \frac{i}{4}\right) = \left[\frac{d}{dz}\left\{\left(z - \frac{i}{4}\right)^2 . z^{k-1} . \frac{z^4}{\left(z^2 + \frac{1}{16}\right)^2}\right\}\right]_{z=\frac{i}{4}} = \left[\frac{d}{dz}\left\{\frac{z^{k+3}}{\left(z + \frac{i}{4}\right)^2}\right\}\right]_{z=\frac{i}{4}}$

$$= \left[\frac{\left(z + \frac{i}{4}\right)^2 . (k+3)\, z^{k+2} - z^{k+3} . 2\left(z + \frac{i}{4}\right)}{\left(z + \frac{i}{4}\right)^4}\right]_{z=\frac{i}{4}}$$

$$= \left[\frac{\left(z + \frac{i}{4}\right) . (k+3)\, z^{k+2} - 2z^{k+3}}{\left(z + \frac{i}{4}\right)^3} \right]_{z = \frac{i}{4}}$$

$$= \frac{\frac{i}{2} . (k+3) \left(\frac{i}{4}\right)^{k+2} - 2\left(\frac{i}{4}\right)^{k+3}}{\left(\frac{i}{2}\right)^3} = \frac{i^3 \left(\frac{k+3}{32}\right) \frac{i^k}{4^k} - \frac{2}{64}\, i^3\, \frac{i^k}{4^k}}{i^3\, \frac{1}{8}}$$

$$= \left(\frac{k+2}{4}\right) \left(\frac{i}{4}\right)^k = \frac{k+2}{(4)^{k+1}} \left(\cos \frac{k\pi}{2} + i \sin \frac{k\pi}{2}\right)$$

Residue $\left(\text{at } z = -\frac{i}{4}\right)$ is $= \frac{k+2}{(4)^{k+1}} \left(\cos \frac{k\pi}{2} - i \sin \frac{k\pi}{2}\right)$

$\therefore \quad y_k$ = Sum of residues $= \left\{\frac{(k+2)}{(4)^{k+1}} . 2 \cos \frac{k\pi}{2}\right\}$.

Example 15. *Solve using Z-transform*: $y_{k+1} - 5y_k = \begin{Bmatrix} \sin k; & k \geq 0 \\ 0; & k < 0 \end{Bmatrix}$; *given that* $y_0 = 0$.

Sol. Take Z-transform, we get

$$z(\overline{y} - y_0) - 5\overline{y} = \frac{z \sin 1}{z^2 - 2z \cos 1 + 1} \qquad \text{...(1)}$$

$$\Rightarrow \qquad (z-5)\overline{y} = \frac{z \sin 1}{z^2 - 2z \cos 1 + 1} \qquad | \because \quad y_0 = 0$$

$$\therefore \qquad \overline{y} = \frac{z \sin 1}{(z-5)(z^2 - 2z \cos 1 + 1)}$$

Taking inverse Z-transform, we get

$$y_k = Z^{-1} \left[\frac{z \sin 1}{(z-5)(z-e^i)(z-e^{-i})}\right]$$

Here $\qquad F(z) = \frac{z \sin 1}{(z-5)(z-e^i)(z-e^{-i})}$

Consider a contour $|\ z\ | = 6$

Residue of F(z) at ($z = 5$)

$$= \underset{z \to 5}{\text{Lt}} (z-5) . z^{k-1} . \frac{z \sin 1}{(z-5)(z-e^i)(z-e^{-i})} = \frac{5^k \sin 1}{26 - 10 \cos 1}$$

Residue of F(z) at ($z = e^i$)

$$= \underset{z \to e^i}{\text{Lt}} (z - e^i)\, z^{k-1} . \frac{z \sin 1}{(z-5)(z-e^i)(z-e^{-i})}$$

$$= \frac{(e^i)^k \sin 1}{(e^i - 5)(e^i - e^{-i})} = \frac{e^{ik}}{2i\,(e^i - 5)}$$

Residue of F(z) at ($z = e^{-i}$)

$$= \underset{z \to e^{-i}}{\text{Lt}} (z + e^{-i}) \cdot z^{k-1} \cdot \frac{z \sin 1}{(z-5)(z-e^{i})(z-e)^{-i}}$$

$$= \frac{(e^{-i})^k \sin 1}{(e^{-i}-5)(e^{-i}-e^{i})} = \frac{e^{-ik}}{2i\,(5-e^{-i})}$$

$f(k)$ = sum of residues

$$= \frac{5^k \sin 1}{26 - 10\cos 1} + \frac{e^{ik}}{2i\,(e^{i}-5)} + \frac{e^{-ik}}{2i\,(5-e^{-i})}$$

$$= \frac{5^k \sin 1}{26 - 10\cos 1} + \frac{1}{2i}\left[\frac{\cos k + i \sin k}{\cos 1 + i \sin 1 - 5} + \frac{\cos k - i \sin k}{5 - \cos 1 + i \sin 1}\right]$$

$$= \frac{5^k \sin 1}{26 - 10\cos 1} - \frac{\cos k \sin 1}{26 - 10\cos 1} - \frac{(5-\cos 1)\sin k}{26 - 10\cos 1}$$

$$f(k) = \left\{\text{A}(5)^k - \text{A}\cos k - \text{A}\left(\frac{5-\cos 1}{\sin 1}\right)\sin k\right\} \quad \text{where,} \quad \text{A} = \frac{\sin 1}{26 - 10\cos 1}.$$

Example 16. *Solve by Z-transform:* $y_{k+1} = 7y_k + 10\,x_k$

$x_{k+1} = y_k + 4x_k\,;\ y_0 = 3,\ x_0 = 2.$

Sol. $y_{k+1} = 7y_k + 10x_k$

Taking Z-transforms, we get

$$z(\bar{y} - y_0) = 7\bar{y} + 10\bar{x}$$

$$\Rightarrow \quad (7-z)\bar{y} + 10\bar{x} = -3z \qquad ...(1)$$

Applying Z-transform to $x_{k+1} = y_k + 4x_k$, we get

$$z(\bar{x} - x_0) = 4\bar{x} + \bar{y}$$

$$\Rightarrow \quad \bar{y} - (z-4)\bar{x} = -2z \qquad ...(2)$$

Eliminating $\bar{y}$ from (1) and (2), we get

$$(z^2 - 11z + 18)\bar{x} = 2z^2 - 11z$$

$$\bar{x} = \frac{2z^2 - 11z}{(z-2)(z-9)} = \frac{z}{z-9} + \frac{z}{z-2}$$

$\therefore$ $x_k = \{9^k + 2^k\}$ (Take inverse Z-transform)

The equation is $x_{k+1} = y_k + 4x_k$

$\therefore$ $y_k = x_{k+1} - 4x_k = (9)^{k+1} + (2)^{k+1} - 4(9^k + 2^k) = 5 \cdot 9^k - 2 \cdot 2^k$

Hence the solution is

$$x_k = \{9^k + 2^k\} \text{ and } y_k = \{5 \cdot 9^k - 2 \cdot 2^k\}.$$

ASSIGNMENT

1. Find the inverse Z-transform of:

(i) $\dfrac{z^2+z}{(z-1)^2}$ (ii) $\dfrac{z^2+2z}{(z-1)(z-2)(z-3)}$ (iii) $\dfrac{z}{(z-1)(z-2)(z-3)}$

(iv) $\dfrac{z}{z^2+11z+30}$ (v) $\dfrac{5z}{(2z-1)(z-3)}$ (vi) $\dfrac{z^2}{(z+2)(z^2+4)}$

(vii) $\dfrac{4z^{-1}}{(1-z^{-1})^2}$ (viii) $F(z) = \dfrac{9z^3}{(z-2)(3z-1)^3}$ (ix) $\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$.

(*M.T.U. 2014*)

2. Using convolution theorem find the Z^{-1} of

(i) $\dfrac{z^2}{(z-4)(z-3)}$ (ii) $\dfrac{8z^2}{(2z-1)(4z-1)}$ (iii) $\dfrac{z^2}{(z-a)^2}$.

3. Solve the following difference equations:

(i) $u(k+2) - 5u(k+1) + 6u(k) = 6k$ if $u(0) = u(1) = 0$

(ii) $y_{k+2} - 3y_{k+1} - 4y_k = 0$; $y_0 = 3$, $y_1 = -2$

(iii) $y_{k+2} + y_{k+1} - 2y_k = 0$; $y_0 = 4$, $y_1 = 0$

(iv) $y_{k+2} - 2y_{k+1} + y_k = 3k + 5$, $y(0) = 0$, $y(1) = 1$ (*M.T.U. 2014*)

by using Z-transform.

4. Solve by Z-transform:

(i) $y_{k+2} - 4y_{k+1} + 3y_k = 5^k$ (*G.B.T.U. 2011*) (ii) $y_{k+2} - 5y_{k+1} - 6y_k = 2^k$ (iii) $y_{k+2} - 6y_{k+1} + 9y_k = 3^k$.

(iv) $y_{k+2} - 4y_k = k - 1$ (v) $y_{k+2} - 6y_{k+1} + 8y_k = 2^k + 6k$

(vi) $y_{k+1} - 2y_{k-1} = 0$, $k \geq 1$, $y(0) = 1$ (*U.P.T.U. 2015*)

5. Solve the following difference equations using Z-transform:

(i) $y_{k+2} - 2y_{k+1} + y_k = k$; $y_0 = y_1 = 0$ (ii) $y_{k+2} - 4y_k = 0$; $y_0 = 0$, $y_1 = 2$ (*U.P.T.U. 2008*)

(iii) $y_{k+2} - 2y_{k+1} + y_k = 2^k$; $y_0 = 2$, $y_1 = 1$. (iv) $y_k + 3y_{k-1} + 2y_{k-2} = \delta(k) + 2\delta(k-1)$

6. Find $Z^{-1}\left[\dfrac{9z^3}{(3z-1)^2\,(z-2)}\right]$ (*U.P.T.U. 2015*)

Answers

1. (i) $2k+1$ (ii) $\dfrac{3}{2} - 4(2)^k + \dfrac{5}{2}(3)^k$ (iii) $\dfrac{1}{2} - (2)^k + \dfrac{1}{2}(3)^k$ (iv) $(-5)^k - (-6)^k$

(v) $3^k - \left(\dfrac{1}{2}\right)^k$ (vi) $\dfrac{(-2)^{k+1}}{8} + \dfrac{(2i)^{k+1}}{8i\,(1+i)} - \dfrac{(2i)^{k+1}}{2i\,(1-i)}$ (vii) $2k(k-1)\,U(k)$

(viii) $f(k) = \dfrac{36}{125}(2)^k - \left(\dfrac{25k^2+85k+72}{250}\right)\left(\dfrac{1}{3}\right)^k$

2. (i) $4^{k+1} - 3^{k+1}$ (ii) $\left(\dfrac{1}{2}\right)^{k-1} - \left(\dfrac{1}{2}\right)^{2k}$ (iii) $(k+1)\,a^k\,U(k)$

3. (*i*) $u(k) = \frac{1}{12}(6)^k - \frac{1}{3}(3)^k + \frac{1}{4}(2)^k$ (*ii*) $y_k = \frac{1}{5}(4)^k + \frac{14}{5}(-1)^k$

(*iii*) $y_k = \frac{8}{3} + \frac{4}{3}(-2)^k$ (*iv*) $y_k = \frac{k(k-1)(k+3)}{2} + k$

4. (*i*) $y_k = c_1 + c_2(3)^k + \frac{5^k}{8}$ (*ii*) $y_k = c_1(-1)^k + c_2(6)^k - \frac{2^k}{12}$

(*iii*) $y_k = (c_1 + c_2 k)(3)^k + \frac{1}{2}k(k-1)(3)^{k-2}$ (*iv*) $y_k = c_1(2)^k + c_2(-2)^k - \frac{k}{3} + \frac{1}{9}$

(*v*) $y_k = c_1(4)^k + \left(c_2 - \frac{k}{4}\right)(2)^k + 2k - \frac{8}{3}$ (*vi*) $y_k = \frac{1}{2}\left\{(\sqrt{2})^k + (-\sqrt{2})^k\right\}$

5. (*i*) $y_k = \frac{k-1}{4}\{1-(-1)^k\}$ (*ii*) $y_k = 2^{k-1} + (-2)^{k-1}$ (*iii*) $y_k = 1 - 2k + 2^k$

(*iv*) $\{(-1)^k u(k)\}$

6. $\frac{36}{25}(2)^k - \left(\frac{5k+11}{25}\right)\left(\frac{1}{3}\right)^k$.

TEST YOUR KNOWLEDGE

1. Find Fourier cosine and sine transform of $f(x) = \begin{cases} k, & 0 < x < a \\ 0, & x > a \end{cases}$ where k is a constant.
2. Find Fourier cosine transform of $e^{-ax}\cos ax$.
3. Find $f(x)$ if its Fourier cosine transform is $\frac{1}{2\pi}\left(a - \frac{p}{2}\right)$ if $p < 2a$ and 0 if $p \geq 2a$.
4. Solve the integral equation $\int_0^\infty f(x)\sin \alpha x \, dx = \begin{cases} 1-\alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$
5. Solve: $\int_0^\infty f(x)\cos \alpha x \, dx = \begin{cases} 1-\alpha, & 0 \leq \alpha < 1 \\ 0, & \alpha > 1 \end{cases}$
6. Define Fourier transform of a function $f(x)$. (*U.P.T.U. 2014*)
7. Define Fourier sine and cosine transform.
8. Write any two properties of Fourier transform with proof.
9. Find the complex Fourier transform of dirac-delta function $\delta(x-a)$.
10. State and prove convolution theorem of Fourier transform. (*M.T.U. 2014*)
11. Define Z-transform of a sequence $\{f(k)\}$.
12. Define inverse Z-transform of a function $F(z)$.
13. Define unit step sequence and find its Z-transform.
14. Define unit impulse sequence and find its Z-transform.
15. Find the Z-transform of $\{^nC_k\}$, $0 \leq k \leq n$. (*M.T.U. 2014*)
16. State and prove change of scale property of Z-transform.
17. Find the Z-transform of $\{k^3\}$, $k \geq 0$.
18. State and prove convolution theorem for Z-transform.

19. Find the Z-transform of $\left\{\dfrac{k^2+3k+2}{2}\right\}$, $k \geq 0$.

20. Explain the inverse integral method of finding inverse Z-transform of a function F(Z).

21. Define a difference equation. Define its order and degree. How is Z-transform useful in finding the solution of a difference equation?

22. Find the inverse Z-transform of $\dfrac{4z^{-1}}{(1-z^{-1})^2}$.

23. Solve by Z-transform: $y_{k+2} + y_{k+1} - 2y_k = 0$, $y_0 = 4$, $y_1 = 0$.

24. Solve: $y_{k+2} - 5y_{k+1} - 6y_k = 2^k$ by Z-transform.

25. Find the Z-transform of $\{a^k\}$, $k \geq 0$. (*U.P.T.U. 2014*)

Answers

1. $\dfrac{k \sin ap}{p}, \dfrac{k(1-\cos ap)}{p}$

2. $\dfrac{a(p^2+2a^2)}{p^4+4a^4}$

3. $\dfrac{2}{\pi^2} \cdot \dfrac{\sin^2 ax}{x^2}$

4. $\dfrac{2}{\pi}\left(\dfrac{x-\sin x}{x^2}\right)$

5. $\dfrac{2(1-\cos x)}{\pi x^2}$

9. e^{iap}

13. $\dfrac{z}{z-1}$

14. 1

15. $\left(1+\dfrac{1}{z}\right)^n$

17. $\dfrac{z^3+4z^2+z}{(z-1)^4}$

19. $\dfrac{1}{2}\left[\dfrac{z(z+1)}{(z-1)^3} + \dfrac{3z}{(z-1)^2} + \dfrac{2z}{z-1}\right]$

22. $2k(k-1)\,u(k)$

23. $y_k = \dfrac{8}{3} + \dfrac{4}{3}(-2)^k$

24. $y_k = c_1(-1)^k + c_2(6)^k - \dfrac{(2)^k}{12}$

25. $\dfrac{z}{z-a}$.

UNIT 3

Statistical Techniques

3.1 MOMENTS

Moments are statistical tools, used in statistical investigations. The moments of a distribution are the arithmetic means of the various powers of the deviations of items from some given number.

3.2 MOMENTS ABOUT MEAN (Central Moments)

3.2.1. For an Individual Series

If $x_1, x_2, ..., x_n$ are the values of the variable under consideration, the r^{th} moment μ_r about mean $\bar{x}$ is defined as

$$\mu_r = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^r}{n} \; ; r = 0, 1, 2, ...$$

3.2.2. For a Frequency Distribution

If $x_1, x_2, ..., x_n$ are the values of a variable x with the corresponding frequencies $f_1, f_2, ..., f_n$ respectively then r^{th} moment μ_r about the mean $\bar{x}$ is defined as

$$\mu_r = \frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^r}{\text{N}} \; ; r = 0, 1, 2, ... \quad \text{where} \quad \text{N} = \sum_{i=1}^{n} f_i$$

In particular, $$\mu_0 = \frac{1}{\text{N}} \sum_{i=1}^{n} f_i (x_i - \bar{x})^0 = \frac{1}{\text{N}} \sum_{i=1}^{n} f_i = \frac{\text{N}}{\text{N}} = 1$$

$\therefore$ For any distribution, $\boxed{\mu_0 = 1}$

For $r = 1$,

$$\mu_1 = \frac{1}{\text{N}} \sum_{i=1}^{n} f_i (x_i - \bar{x}) = \frac{1}{\text{N}} \sum_{i=1}^{n} f_i x_i - \bar{x} \left(\frac{1}{\text{N}} \sum_{i=1}^{n} f_i \right) = \bar{x} - \bar{x} = 0$$

$\therefore$ For any distribution, $\boxed{\mu_1 = 0}$

For $r = 2$,

$$\mu_2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2 = (\text{S.D.})^2 = \text{Variance}$$

$\therefore$ For any distribution, μ_2 coincides with the variance of the distribution.

Similarly, $$\mu_3 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^3,\ \mu_4 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^4$$

and so on.

Note. In case of a frequency distribution with class intervals, the values of x are the mid-points of the intervals.

EXAMPLES

Example 1. *Find the first four moments for the following individual series:*

x	*3*	*6*	*8*	*10*	*18*

Sol. **Calculation of Moments**

S. No.	*x*	$x - \bar{x}$	$(x - \bar{x})^2$	$(x - \bar{x})^3$	$(x - \bar{x})^4$
1	3	– 6	36	– 216	1296
2	6	– 3	9	– 27	81
3	8	– 1	1	– 1	1
4	10	1	1	1	1
5	18	9	81	729	6561
$n = 5$	$\Sigma x = 45$	$\Sigma(x - \bar{x}) = 0$	$\Sigma(x - \bar{x})^2 = 128$	$\Sigma(x - \bar{x})^3 = 486$	$\Sigma(x - \bar{x})^4 = 7940$

Now, $$\bar{x} = \frac{\Sigma x}{n} = \frac{45}{5} = 9$$

$\therefore$ $$\mu_1 = \frac{\Sigma(x-\bar{x})}{n} = \frac{0}{5} = 0, \qquad \mu_2 = \frac{\Sigma(x-\bar{x})^2}{n} = \frac{128}{5} = 25.6$$

$$\mu_3 = \frac{\Sigma(x-\bar{x})^3}{n} = \frac{486}{5} = 97.2, \qquad \mu_4 = \frac{\Sigma(x-\bar{x})^4}{n} = \frac{7940}{5} = 1588.$$

Example 2. *Calculate μ_1, μ_2, μ_3, μ_4 for the following frequency distribution:*

Marks	*5–15*	*15–25*	*25–35*	*35–45*	*45–55*	*55–65*
No. of students	*10*	*20*	*25*	*20*	*15*	*10*

Sol. **Calculation of Moments**

Marks	*No. of students (f)*	*Mid-point (x)*	*fx*	$x - \bar{x}$	$f(x - \bar{x})$	$f(x - \bar{x})^2$	$f(x - \bar{x})^3$	$f(x - \bar{x})^4$
5–15	10	10	100	– 24	– 240	5760	– 138240	3317760
15–25	20	20	400	– 14	– 280	3920	– 54880	768320
25–35	25	30	750	– 4	– 100	400	– 1600	6400
35–45	20	40	800	6	120	720	4320	25920
45–55	15	50	750	16	240	3840	61440	983040
55–65	10	60	600	26	260	6760	175760	4569760
	N = 100		Σfx = 3400		$\Sigma f(x - \bar{x})$ = 0	$\Sigma f(x - \bar{x})^2$ = 21400	$\Sigma f(x - \bar{x})^3$ = 46800	$\Sigma f(x - \bar{x})^4$ = 9671200

Now, $\quad \bar{x} = \dfrac{\Sigma fx}{N} = \dfrac{3400}{100} = 34$

$\therefore \quad \mu_1 = \dfrac{\Sigma f(x-\bar{x})}{N} = \dfrac{0}{100} = 0, \qquad \mu_2 = \dfrac{\Sigma f(x-\bar{x})^2}{N} = \dfrac{21400}{100} = 214$

$\mu_3 = \dfrac{\Sigma f(x-\bar{x})^3}{N} = \dfrac{46800}{100} = 468, \qquad \mu_4 = \dfrac{\Sigma f(x-\bar{x})^4}{N} = \dfrac{9671200}{100} = 96712.$

3.3 SHEPPARD'S CORRECTIONS FOR MOMENTS

While computing moments for frequency distribution with class intervals, we take variable x as the mid-point of class-intervals which means that we have assumed the frequencies concentrated at the mid-points of class-intervals.

The above assumption is true when the distribution is symmetrical and the no. of class-intervals is not greater than $\dfrac{1}{20}^{\text{th}}$ of the range, otherwise the computation of moments will have certain error called **grouping error.**

This error is corrected by the following formulae given by **W.F. Sheppard.**

$$\mu_2 \text{ (corrected)} = \mu_2 - \frac{h^2}{12}$$

$$\mu_4 \text{ (corrected)} = \mu_4 - \frac{1}{2}h^2\mu_2 + \frac{7}{240}h^4$$

where h is the width of the class-interval while μ_1 and μ_3 require no correction.

These formulae are known as **Sheppard's corrections.**

Example 3. *Find the corrected values of the following moments using Sheppard's correction. The width of classes in the distribution is 10:*

$\mu_2 = 214, \qquad \mu_3 = 468, \qquad \mu_4 = 96712.$

Sol. We have $\quad \mu_2 = 214, \qquad \mu_3 = 468, \qquad \mu_4 = 96712, \qquad h = 10.$

Now, $\quad \mu_2 \text{ (corrected)} = \mu_2 - \dfrac{h^2}{12} = 214 - \dfrac{(10)^2}{12} = 214 - 8.333 = 205.667.$

$\mu_3 \text{ (corrected)} = \mu_3 = 468$

$$\mu_4 \text{ (corrected)} = \mu_4 - \frac{1}{2}h^2\mu_2 + \frac{7}{240}h^4 = 96712 - \frac{(10)^2}{2}(214) + \frac{7}{240}(10)^4$$
$$= 96712 - 10700 - 291.667 = 86303.667.$$

3.4 MOMENTS ABOUT AN ARBITRARY NUMBER (Raw Moments)

If $x_1, x_2, x_3, ..., x_n$ are the values of a variable x with the corresponding frequencies $f_1, f_2, f_3, ..., f_n$ respectively then r^{th} moment μ_r' about the number $x = A$ is defined as

$$\mu_r' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)^r \;;\; r = 0, 1, 2, ... \quad \text{where,} \quad N = \sum_{i=1}^{n} f_i$$

For $r = 0$, $$\mu_0' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)^0 = 1$$

For $r = 1$, $$\mu_1' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A) = \frac{1}{N}\sum_{i=1}^{n} f_i x_i - \frac{A}{N}\sum_{i=1}^{n} f_i = \bar{x} - A$$

For $r = 2$, $$\mu_2' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)^2$$

For $r = 3$, $$\mu_3' = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)^3 \text{ and so on.}$$

In calculation work, if we find that there is some common factor h (> 1) in values of $x - A$, we can ease our calculation work by defining $u = \frac{x - A}{h}$. In that case, we have

$$\mu_r' = \frac{1}{N}\left(\sum_{i=1}^{n} f_i u_i^r\right) h^r \;;\; r = 0, 1, 2, ...$$

Note. For an individual series,

1. $\mu_r' = \frac{1}{n}\sum_{i=1}^{n}(x_i - A)^r \;;\; r = 0, 1, 2, ...$

2. $\mu_r' = \frac{1}{N}\left(\sum_{i=1}^{n} u_i^r\right) h^r \;;\; r = 0, 1, 2, ...$ | for $u = \frac{x - A}{h}$

3.5 MOMENTS ABOUT THE ORIGIN

If $x_1, x_2, ..., x_n$ be the values of a variable x with corresponding frequencies $f_1, f_2, ..., f_n$ respectively then r^{th} moment about the origin ν_r is defined as

$$\nu_r = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^r \;;\; r = 0, 1, 2, ... \quad \text{where,} \quad N = \sum_{i=1}^{n} f_i$$

For $r = 0$, $\quad \nu_0 = \dfrac{1}{N}\sum_{i=1}^{n} f_i x_i^{\,0} = \dfrac{N}{N} = 1$

For $r = 1$, $\quad \nu_1 = \dfrac{1}{N}\sum_{i=1}^{n} f_i x_i = \bar{x}$

For $r = 2$, $\quad \nu_2 = \dfrac{1}{N}\sum_{i=1}^{n} f_i x_i^{\,2}$ and so on.

3.6 RELATION BETWEEN μ_r AND μ_r'

We know that,

$$\mu_r = \frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^r}{N} = \frac{1}{N}\sum_{i=1}^{n} f_i\,[(x_i - A) - (\bar{x} - A)]^r$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i\,[(x_i - A) - \mu'_1]^r \qquad | \because \ \mu_1' = \bar{x} - A$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i\,[(x_i - A)^r - {}^r c_1 (x_i - A)^{r-1}\mu_1' + {}^r c_2 (x_i - A)^{r-2}\mu_1'^2 - \ldots\ldots + (-1)^r \mu_1'^r]$$

| Using binomial theorem

$$\Rightarrow \qquad \mu_r = \mu_r' - {}^r c_1\, \mu'_{r-1}\mu_1' + {}^r c_2\, \mu'_{r-2}\mu_1'^2 - \ldots\ldots + (-1)^r \mu_1'^r$$

Putting $r = 2, 3, 4$, we get

$$\mu_2 = \mu_2' - 2\mu_1'^2 + \mu_1'^2 = \mu_2' - \mu_1'^2 \qquad | \because \ \mu_0' = 1$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 3\mu_1'^3 - \mu_1'^3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

Hence, we have the following relations:

$$\boxed{\mu_1 = 0}$$

$$\boxed{\mu_2 = \mu_2' - \mu_1'^2}$$

$$\boxed{\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}$$

and

$$\boxed{\mu_4 = \mu_4' - 4\mu'_3\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}$$

3.7 RELATION BETWEEN ν_r AND μ_r

We know that,

$$\nu_r = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^{\,r}\ ;\ r = 0, 1, 2, \ldots$$

$$= \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - A + A)^r$$

$$= \frac{1}{N} \sum_{i=1}^{n} f_i [(x_i - A)^r + {}^rc_1 (x_i - A)^{r-1} . A + \ldots + A^r]$$

$$= \mu_r' + {}^rc_1 \mu_{r-1} A + \ldots + A^r$$

If we take, $A = \bar{x}$ (for μ_r) then

$$\nu_r = \mu_r + {}^rc_1 \mu_{r-1} \bar{x} + {}^rc_2 \mu_{r-2} \bar{x}^2 + \ldots + \bar{x}^r \quad \ldots(1)$$

Putting, $r = 1, 2, 3, 4$ in (1), we get

$$\nu_1 = \mu_1 + \mu_0 \bar{x} = \bar{x} \quad | \because \mu_1 = 0, \mu_0 = 1$$

$$\nu_2 = \mu_2 + {}^2c_1 \mu_1 \bar{x} + {}^2c_2 \mu_0 \bar{x}^2 = \mu_2 + \bar{x}^2$$

$$\nu_3 = \mu_3 + {}^3c_1 \mu_2 \bar{x} + {}^3c_2 \mu_1 \bar{x}^2 + {}^3c_3 \mu_0 \bar{x}^3 = \mu_3 + 3\mu_2 \bar{x} + \bar{x}^3$$

$$\nu_4 = \mu_4 + {}^4c_1 \mu_3 \bar{x} + {}^4c_2 \mu_2 \bar{x}^2 + {}^4c_3 \mu_1 \bar{x}^3 + {}^4c_4 \mu_0 \bar{x}^4$$

$$= \mu_4 + 4\mu_3 \bar{x} + 6\mu_2 \bar{x}^2 + \bar{x}^4$$

Hence we have the following relations:

$$\boxed{\nu_1 = \bar{x}} \qquad \boxed{\nu_2 = \mu_2 + \bar{x}^2}$$

$$\boxed{\nu_3 = \mu_3 + 3\mu_2 \bar{x} + \bar{x}^3} \quad \text{and} \quad \boxed{\nu_4 = \mu_4 + 4\mu_3 \bar{x} + 6\mu_2 \bar{x}^2 + \bar{x}^4.}$$

3.8 KARL PEARSON'S β AND γ COEFFICIENTS

Karl Pearson defined the following four coefficients based upon the first four moments of a frequency distribution about its mean:

$$\left.\begin{aligned} \beta_1 &= \frac{\mu_3^2}{\mu_2^3} \\ \beta_2 &= \frac{\mu_4}{\mu_2^2} \end{aligned}\right\} \quad (\beta\text{-coefficients})$$

$$\left.\begin{aligned} \gamma_1 &= + \sqrt{\beta_1} \\ \gamma_2 &= \beta_2 - 3 \end{aligned}\right\} \quad (\gamma\text{-coefficients})$$

The practical use of these coefficients is to measure the skewness and kurtosis of a frequency distribution. These coefficients are pure numbers independent of units of measurement.

EXAMPLES

Example 1. *The first three moments of a distribution, about the value '2' of the variable are 1, 16 and – 40. Show that the mean is 3, variance is 15 and $\mu_3 = -86$.*

Sol. We have $A = 2$, $\mu_1' = 1$, $\mu_2' = 16$, and $\mu_3' = -40$

We know that $\mu_1' = \bar{x} - A \Rightarrow \bar{x} = \mu_1' + A = 1 + 2 = 3$

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2 = 16 - (1)^2 = 15$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -40 - 3(16)(1) + 2(1)^3 = -40 - 48 + 2 = -86.$$

Example 2. *The first four moments of a distribution, about the value '35' are – 1.8, 240, – 1020 and 144000. Find the values of* $\mu_1, \mu_2, \mu_3, \mu_4$.

Sol. $\mu_1 = 0.$

$\mu_2 = \mu_2' - \mu_1'^2 = 240 - (-1.8)^2 = 236.76$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -1020 - 3(240)(-1.8) + 2(-1.8)^3 = 264.36$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 144000 - 4(-1020)(-1.8) + 6(240)(-1.8)^2 - 3(-1.8)^4 = 141290.11.$

Example 3. *Calculate the variance and third central moment from the following data:*

x_i	0	1	2	3	4	5	6	7	8
f_i	1	9	26	59	72	52	29	7	1

(U.P.T.U. 2006)

Sol. **Calculation of Moments**

x	f	$u = \frac{x-A}{h}$ $A = 4, h = 1$	fu	fu^2	fu^3
0	1	– 4	– 4	16	– 64
1	9	– 3	– 27	81	– 243
2	26	– 2	– 52	104	– 208
3	59	– 1	– 59	59	– 59
4	72	0	0	0	0
5	52	1	52	52	52
6	29	2	58	116	232
7	7	3	21	63	189
8	1	4	4	16	64
	$N = \Sigma f = 256$		$\Sigma fu = -7$	$\Sigma fu^2 = 507$	$\Sigma fu^3 = -37$

Now, moments about the point $x = A = 4$ are

$$\mu_1' = \left(\frac{\Sigma fu}{N}\right)h = \frac{-7}{256} = -0.02734$$

$$\mu_2' = \left(\frac{\Sigma fu^2}{N}\right)h^2 = \frac{507}{256} = 1.9805$$

$$\mu_3' = \left(\frac{\Sigma fu^3}{N}\right)h^3 = \frac{-37}{256} = -0.1445$$

Moments about mean

$\mu_1 = 0$

$\mu_2 = \mu_2' - \mu_1'^2 = 1.9805 - (-.02734)^2 = 1.97975$

$\therefore$ Variance = 1.97975

Also, $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= (-0.1445) - 3\,(1.9805)\,(-.02734) + 2\,(-.02734)^3$
$= 0.0178997$

$\therefore$ Third central moment = 0.0178997.

Example 4. *The first three moments of a distribution about the value 2 of the variable are 1, 16 and – 40 respectively. Find the values of the first three moments about the origin.*

Sol. We have $A = 2, \quad \mu_1' = 1, \quad \mu_2' = 16, \; \mu_3' = -40$

$$\therefore \quad \nu_1 = \bar{x} = A + \mu_1' = 2 + 1 = 3$$
$$\nu_2 = \mu_2 + \bar{x}^2 = 15 + (3)^2 = 24$$
$$\nu_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3 = -86 + 3(15)(3) + (3)^3 = 76.$$

Example 5. *The first four moments of a distribution about x = 2 are 1, 2.5, 5.5 and 16. Calculate the first four moments about the mean and about origin.*

Sol. We have $A = 2, \quad \mu_1' = 1, \quad \mu_2' = 2.5, \quad \mu_3' = 5.5, \quad \mu_4' = 16.$

Moments about mean

$$\mu_1 = 0$$
$$\mu_2 = \mu_2' - (\mu_1')^2 = 2.5 - (1)^2 = 1.5$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = 5.5 - 3(2.5)(1) + 2(1)^3 = 0$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 16 - 4(5.5)(1) + 6(2.5)(1)^2 - 3(1)^4 = 6.$$

Moments about origin

$$\nu_1 = \bar{x} = A + \mu_1', \qquad \nu_2 = \mu_2 + \bar{x}^2$$
$$\nu_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3, \qquad \nu_4 = \mu_4 + 4\mu_3\bar{x} + 6\mu_2\bar{x}^2 + \bar{x}^4$$
$$\therefore \quad \nu_1 = \bar{x} = 2 + 1 = 3, \qquad \nu_2 = 1.5 + (3)^2 = 10.5$$
$$\nu_3 = 0 + 3(1.5)(3) + (3)^3 = 40.5, \qquad \nu_4 = 6 + 4(0)(3) + 6(1.5)(3)^2 + (3)^4 = 168.$$

Example 6. *For a distribution, the mean is 10, variance is 16, γ_1 is 1, and β_2 is 4. Find the first four moments about the origin.*

Sol. $\bar{x} = 10, \quad \mu_2 = 16, \quad \gamma_1 = 1, \quad \beta_2 = 4$ | given

Now, $\gamma_1 = 1$

$\Rightarrow \quad \beta_1 = 1$ | $\because \; \gamma_1 = \sqrt{\beta_1}$

$$\Rightarrow \quad \frac{\mu_3^2}{\mu_2^3} = 1 \quad \Rightarrow \quad \mu_3{}^2 = \mu_2{}^3 = (16)^3 = (64)^2$$

$\Rightarrow \quad \mu_3 = 64$

and $\quad \beta_2 = 4$

$$\Rightarrow \quad \frac{\mu_4}{\mu_2^2} = 4 \quad \Rightarrow \quad \mu_4 = 4(16)^2 = 1024$$ | $\because \; \mu_2 = 16$

Moments about the origin

$$\nu_1 = \bar{x} = 10$$
$$\nu_2 = \mu_2 + \bar{x}^2 = 16 + 100 = 116$$
$$\nu_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3 = 64 + 480 + 1000 = 1544$$
$$\nu_4 = \mu_4 + 4\mu_3\bar{x} + 6\mu_2\bar{x}^2 + \bar{x}^4 = 1024 + 4(64)(10) + 6(16)(100) + (10)^4$$
$$= 22184$$

Example 7. *In a certain distribution, the first four moments about the point x = 4 are – 1.5, 17, – 30 and 308. Find the moments about mean and about origin. Also, Calculate β_1 and β_2.*
(U.P.T.U. 2014)

Sol. We have, $A = 4, \mu'_1 = -1.5, \mu'_2 = 17, \mu'_3 = -30, \mu'_4 = 308$

Moments about mean

$$\mu_1 = 0$$
$$\mu_2 = \mu'_2 - \mu_1'^2 = 17 - (-1.5)^2 = 14.75$$
$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3 = -30 - 3(17)(-1.5) + 2(-1.5)^3 = 39.75$$
$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu_1'^2 - 3\mu_1'^4$$
$$= 308 - 4(-30)(-1.5) + 6(17)(-1.5)^2 - 3(-1.5)^4 = 342.3125$$

Moments about origin

$$\nu_1 = \bar{x} = \mu'_1 + A = -1.5 + 4 = 2.5$$
$$\nu_2 = \mu_2 + \bar{x}^2 = 14.75 + (2.5)^2 = 21$$
$$\nu_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3 = 166$$
$$\nu_4 = \mu_4 + 4\mu_3\bar{x} + 6\mu_2\bar{x}^2 + \bar{x}^4 = 1332$$

Calculation of β_1 and β_2

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0.492377 \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} = 1.573398$$

Example 8. *The first four moments of a distribution about the value '4' of the variable are – 1.5, 17, – 30 and 108. Find the moments about mean, about origin ; β_1 and β_2. Also find the moments about the point $x = 2$.* (U.P.T.U. 2007)

Sol. We have $A = 4, \mu'_1 = -1.5, \mu'_2 = 17, \mu'_3 = -30, \mu'_4 = 108$

Moments about mean

$$\mu_1 = 0$$
$$\mu_2 = \mu'_2 - \mu_1'^2 = 14.75$$
$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3 = 39.75$$
$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu_1'^2 - 3\mu_1'^4 = 142.3125$$

Also, $\bar{x} = \mu'_1 + A = -1.5 + 4 = 2.5$

Moments about origin

$$\nu_1 = \bar{x} = 2.5$$
$$\nu_2 = \mu_2 + \bar{x}^2 = 14.75 + (2.5)^2 = 21$$
$$\nu_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3 = 166$$
$$\nu_4 = \mu_4 + 4\mu_3\bar{x} + 6\mu_2\bar{x}^2 + \bar{x}^4 = 1132$$

Calculation of β_1 and β_2

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0.492377 \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} = 0.654122$$

Moments about the point x = 2

$$\mu'_1 = \bar{x} - A = 2.5 - 2 = 0.5$$
$$\mu'_2 = \mu_2 + \mu_1'^2 = 14.75 + (.5)^2 = 15$$
$$\mu'_3 = \mu_3 + 3\mu'_2\mu'_1 - 2\mu_1'^3 = 39.75 + 3(15)(.5) - 2(.5)^3 = 62$$
$$\mu'_4 = \mu_4 + 4\mu'_3\mu'_1 - 6\mu'_2\mu_1'^2 + 3\mu_1'^4 = 244$$

ASSIGNMENT

1. (*i*) Calculate first four moments about mean, for the following individual series:

5, 5, 5, 5, 5, 5.

(*ii*) Find the first four moments about the mean of the following series:

1, 3, 7, 9, 10.

(*iii*) Calculate μ_1, μ_2, μ_3, μ_4 for the series : 4, 7, 10, 13, 16, 19, 22.

2. (*i*) Find the first four moments for the following frequency distribution:

x	1	2	3	4	5	6	7	8	9
f	1	2	3	4	5	4	3	2	1

(*ii*) Calculate the first four moments of the following distribution about the mean and hence find β_1 and β_2.

x	0	1	2	3	4	5	6	7
f	1	8	28	56	70	56	28	8

(*iii*) The number of flowers on Sunflower plants are given below:

No. of flowers	3	6	12	16	25
No. of plants	1	2	3	4	5

Calculate the first four moments about mean. [*M.T.U. (B. Pharma) 2011*]

3. (*i*) Find the first four moments about mean for the following frequency distribution :

Marks	0–10	10–20	20–30	30–40	40–50
No. of students	5	10	40	20	25

(*ii*) Calculate the first four moments about the mean for the following:

Classes	5–15	15–25	25–35	35–45	45–55
f	14	22	36	18	10

(*iii*) Calculate the first four moments about the mean for the following data:

Class-interval	0–10	10–20	20–30	30–40	40–50
f	10	20	40	20	10

(*M.T.U. 2014*)

4. Calculate the first four moments about $x = 15$ and hence find the moments about the mean of the following distribution :

x	10	11	12	13	14	15	16	17	18	19	20	21
f	9	36	75	105	116	107	88	66	45	30	18	5

5. (*i*) The first three moments of a distribution about the value 4 of the variable are 1.5, 17 and – 30. Find the moments about mean.

(*ii*) The first four moments of a distribution about $x = 4$ are 1, 4, 10 and 45. Show that the mean is 5, the variance is 3, μ_3 is 0 and μ_4 is 26.

(*iii*) The first four moments of a distribution about the value 5 of the variable are 2, 20, 40 and 50. Calculate mean, variance, μ_3 and μ_4. (*G.B.T.U. 2011*)

6. If the first four moments of a distribution about the value 5 are equal to – 4, 22, – 117 and 560. Determine the corresponding moments :

(*i*) about the mean, and (*ii*) about zero.

7. Compute first four moments of the data 3, 5, 7, 9 about the mean. Also, compute the first four moments about the point 4.

8. In a frequency distribution, the mean is 1.5, variance is 0.64, β_2 is 2.5 and γ_1 is 0.3. Find μ_3 and μ_4 and also the first four moments about the origin.

9. The first four moments of a distribution about the value '0' are – 0.20, 1.76, – 2.36 and 10.88. Find the moments about the mean. (*U.P.T.U. 2009*)

Answers

1. (*i*) $\mu_1 = 0, \mu_2 = 0, \mu_3 = 0, \mu_4 = 0$ (*ii*) $\mu_1 = 0, \mu_2 = 12, \mu_3 = -12, \mu_4 = 208.8$
(*iii*) $\mu_1 = 0, \mu_2 = 36, \mu_3 = 0, \mu_4 = 2268$

2. (*i*) $\mu_1 = 0, \mu_2 = 4, \mu_3 = 0, \mu_4 = 37.6$ (*ii*) $\mu_1 = 0, \mu_2 = 2, \mu_3 = 0, \mu_4 = 11, \beta_1 = 0, \beta_2 = 2.75$
(*iii*) $\mu_1 = 0, \mu_2 = 54.8, \mu_3 = -49.6, \mu_4 = 5475.6$

3. (*i*) $\mu_1 = 0, \mu_2 = 125, \mu_3 = -300, \mu_4 = 37625$ (*ii*) $\mu_1 = 0, \mu_2 = 134.56, \mu_3 = 126.14, \mu_4 = 41840.82$
(*iii*) $\mu_1 = 0, \mu_2 = 120, \mu_3 = 0, \mu_4 = 36000$

4. $\mu_1 = 0, \mu_2 = 5.5, \mu_3 = 4.4, \mu_4 = 77.8$

5. (*i*) $\mu_1 = 0, \mu_2 = 14.75, \mu_3 = -99.75$ (*iii*) 7, 16, – 64, 162

6. (*i*) 0, 6, 19, 32 (*ii*) 1, 7, 38, 145

7. $\mu_1 = 0, \mu_2 = 5, \mu_3 = 0, \mu_4 = 41; \mu_1' = 2, \mu_2' = 9, \mu_3' = 38.25, \mu_4' = 177$

8. $\mu_3 = 0.1536, \mu_4 = 1.024, \nu_1 = 1.5, \nu_2 = 2.89, \nu_3 = 6.4086, \nu_4 = 15.6481$

9. $\mu_1 = 0, \mu_2 = 1.72, \mu_3 = -1.32, \mu_4 = 9.4096.$

3.9 MOMENT GENERATING FUNCTION

[*U.P.T.U. 2014 ; G.B.T.U. 2012, M.T.U. 2013*]

For certain theoretical developments, an indirect method for computing moments is used. The method depends on the finding of the moment generating function.

3.9.1. In Case of a Continuous Variable *x*, it is defined as

$$M(t) = \int_a^b e^{tx} f(x)\, dx \qquad ...(1)$$

where integral is a function of parameter t only. The limits a, b can be $-\infty$ and ∞ respectively. It is possible to associate a moment generating function with the distribution only when all the moments of the distribution are finite.

Let us see how $M(t)$ generates moments. For this, let us assume that $f(x)$ is a distribution function for which the integral given by (1) exists.

Then e^{tx} may be expanded in a power series and the integration may be performed term by term. It follows that

$$M(t) = \int_a^b \left(1 + tx + \frac{t^2}{2!}x^2 + \ldots\right) f(x)\, dx$$

$$= \int_a^b f(x)\, dx + t\int_a^b x f(x)\, dx + \ldots$$

$$= \nu_0 + \nu_1 t + \nu_2 \cdot \frac{t^2}{2!} + \ldots \qquad \ldots(2)$$

Obviously, the coefficient of $\frac{t^r}{r!}$ in (2) is the r^{th} moment about the origin.

Also, $$\left| \frac{d^r}{dt^r} \mathrm{M}(t) \right|_{t=0} = \left| \frac{\nu_r}{r!} r! + \nu_{r+1} t + \ldots \right|_{t=0} = \nu_r \qquad \ldots(3)$$

Thus, ν_r about origin = r^{th} derivative of M(t) with $t = 0$.

Although the moment generating function (m.g.f.) has been defined for the variable x only, the definition can be generalized so that it holds for a variable z where z is a function of x. *e.g.,* if $z = x - m$ (m is mean), the r^{th} moment about z will give r^{th} moment of x about the mean m.

Moment generating function for z will clearly be given as

$$\mathrm{M}_z(t) = \int_a^b e^{tz} f(x)\, dx$$

$$\mathrm{M}_{x-m}(t) = \int_a^b e^{t(x-m)} f(x)\, dx = e^{-mt} \int_a^b e^{tx} f(x) dx = e^{-mt}\, \mathrm{M}_x(t).$$

3.9.2. In Case of Discrete Distribution of the Variable x

We know that, for a variable x,

$$\nu_r = \Sigma\, x^r \,.\, \mathrm{P}$$

where P is the probability that the variable takes on the value x.

If z is any function of x, we get r^{th} moment for z by the relation

$$\nu_r = \Sigma z^r \mathrm{P}$$

and the moment generating function is given by

$$\mathrm{M}_z(t) = \Sigma e^{tz}\, \mathrm{P} \qquad \ldots(1)$$

To verify that this function generates moments, we will expand e^{tz} and then sum term by term,

$$\therefore \qquad \mathrm{M}_z(t) = \sum \left(1 + tz + \frac{t^2}{2!} z^2 + \ldots\right) \mathrm{P} = \Sigma\mathrm{P} + t\Sigma z\mathrm{P} + \frac{t^2}{2!} \Sigma z^2 \,.\, \mathrm{P} + \ldots$$

$$= \nu_0 + t\nu_1 + \frac{t^2}{2!} \nu_2 + \ldots$$

In this case, we can also show that $\nu_r = \left| \frac{d^r}{dt^r} \mathrm{M}_z(t) \right|_{t=0}$

M(t) is clearly the expected value of e^{tx} and hence can be written as E(e^{tx}) which gives the moment generating function incase of discrete as well as continuous variables.

Expectation of any function $\phi(x)$ is given by

$$\mathrm{E}\{\phi(x)\} = \sum_i \phi(x_i)\, f(x_i) \qquad | \text{ for discrete distribution}$$

or, $$\mathrm{E}\{\phi(x)\} = \int_{-\infty}^{\infty} \phi(x)\, f(x)\, dx \qquad | \text{ for continuous distribution}$$

Eqn. (1) can also be rewritten as

$$M_{x-a}(t) = E\,[e^{t(x-a)}] = \sum_i e^{t(x_i - a)}\, P_i = e^{-at} \sum_i e^{tx_i}\, P_i = e^{-at}\, M_0(t)$$

Therefore the moment generating function about the point 'a' is equal to e^{-at} times the moment generating function about the origin.

Note. m.g.f. is not always defined since $E\{|\, e^{tx} \,|\}$ does not always exist for all values of t.

e.g., if $f(x) = \dfrac{6}{\pi^2 x^2}$, $x = 1, 2, 3, \ldots$ then m.g.f. does not exist.

m.g.f. always exists for $t = 0$ since $M_{x=0}(0) = 1$.

3.9.3. Properties of Moment Generating Function (*M.T.U. 2013*)

(1) *The moment generating function of the sum of two independent chance variables is the product of their respective moment generating functions.*

Symbolically, $M_{x+y}(t) = M_x(t) \times M_y(t)$ *provided that x and y are independent random variables.*

Proof. Let x and y be two independent random variables so that $x + y$ is also a random variable.

The m.g.f. of the sum $x + y$ w.r.t. origin is

$$M_{x+y}(t) = E\{e^{t(x+y)}\} = E(e^{tx} \,.\, e^{ty}) = E(e^{tx}) \,.\, E(e^{ty})$$

Since x and y are independent variables and so are e^{tx} and e^{ty}.

$$\therefore \qquad M_{x+y}(t) = M_x(t) \,.\, M_y(t)$$

Hence the theorem.

(2) *Effect of change of origin and scale on m.g.f.*

$$\boxed{M_u(t) = e^{-at/h}\, M_x(t/h)} \qquad \text{where } u = \frac{x-a}{h}$$

Proof. Let u be a new random variable given by $u = \dfrac{x-a}{h}$ so that $x = a + hu$

then by definition, the effect of linear transformation on m.g.f. is governed by

$$M_x(t) = E(e^{tx}) = E[e^{t(a+hu)}] = E(e^{at} \,.\, e^{thu})$$

$$= e^{at}\, E(e^{thu}) = e^{at}\, M_u(th)$$

Also,

$$M_u(t) = E(e^{tu})$$

$$= E\left[e^{t\left(\frac{x-a}{h}\right)}\right] = e^{-\frac{at}{h}}\, M_x\left(\frac{t}{h}\right)$$

(3) $M_{cx}(t) = M_x(ct)$, *c being a constant.*

Proof. By definition,

$$\text{LHS} = M_{cx}(t) = E(e^{tcx}) = M_x(ct) = \text{RHS}$$

Hence the result.

EXAMPLES

Example 1. *Find the moment generating function of the exponential distribution*

$$f(x) = \frac{1}{c}\, e^{-x/c}\ ;\ 0 \le x \le \infty,\ c > 0 \qquad \text{(M.T.U. 2014)}$$

Hence find its mean and standard deviation.

Sol. Moment generating function about the origin is given by

$$M_x(t) = \int_0^\infty e^{tx} \cdot \frac{1}{c} e^{-x/c}\, dx$$

$$= \frac{1}{c}\int_0^\infty e^{\left(t-\frac{1}{c}\right)x}\, dx = \frac{1}{c}\left[\frac{e^{\left(t-\frac{1}{c}\right)x}}{\left(t-\frac{1}{c}\right)}\right]_0^\infty$$

$$= (1 - ct)^{-1} = 1 + ct + c^2t^2 + c^3t^3 + \ldots$$

$\therefore$
$$\nu_1 = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = (c + 2c^2t + 3c^3t^2 + \ldots)_{t=0} = c$$

and
$$\nu_2 = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = 2c^2$$

Now, mean $\quad \bar{x} = \nu_1 = c$

Variance $\quad \mu_2 = \nu_2 - \bar{x}^2 = \nu_2 - \nu_1^2 = 2c^2 - c^2 = c^2$

$\therefore$ Standard deviation $= \sqrt{\mu_2} = c.$

Example 2. *Obtain the moment generating function of the random variable x having probability distribution*

$$f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 \le x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

[M.T.U. 2012; G.B.T.U. (C.O.) 2011; G.B.T.U. 2013]

Also determine mean ν_1, ν_2 *and variance* μ_2.

Sol. $M_x(t) = E(e^{tx})$

$$= \int_0^1 x \,.\, e^{tx}\, dx + \int_1^2 (2-x)\, e^{tx}\, dx + \int_2^\infty 0 \,.\, e^{tx}\, dx$$

$$= \left(\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2}\right)_0^1 + \left(\frac{2e^{tx}}{t} - \frac{xe^{tx}}{t} + \frac{e^{tx}}{t^2}\right)_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \left[\left(\frac{2e^{2t}}{t} - \frac{2e^{2t}}{t} + \frac{e^{2t}}{t^2}\right) - \left(\frac{2e^t}{t} - \frac{e^t}{t} + \frac{e^t}{t^2}\right)\right] = \frac{e^{2t} - 2e^t + 1}{t^2}$$

$$= \left(\frac{e^t - 1}{t}\right)^2 = \frac{\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right)^2}{t^2} = 1 + t + t^2 + \ldots$$

$$\text{Mean} = \nu_1 = \left[\frac{d}{dt}\mathrm{M}_x(t)\right]_{t=0} = 1$$

Similarly, $\nu_2 = 2$, $\mu_2 = \nu_2 - \bar{x}^2 = \nu_2 - \nu_1^2 = 2 - (1)^2 = 1$ = Variance.

Example 3. *Find the moment generating function of the random variable whose moments are* $\nu_r = (r + 1)!\, 2^r$.

Sol.
$$\mathrm{M}_x(t) = \mathrm{E}(e^{tx}) = \sum_{x=0}^{\infty} e^{tx}\, \mathrm{P}(\mathrm{X} = x)$$
$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\nu_r = \sum_{r=0}^{\infty} \frac{t^r}{r!}(r+1)!\,.\,2^r = \sum_{r=0}^{\infty}(r+1)\,(2t)^r$$
$$= 1 + 2.2t + 3\,.\,(2t)^2 + \ldots = (1 - 2t)^{-2}.$$

Example 4. *Find the moment generating function of the probability distribution function*
$$f(z) = e^{-z}\,(1 + e^{-z})^{-2}, \quad -\infty < z < \infty.$$

Sol.
$$\mathrm{M}_z(t) = \mathrm{E}(e^{tz})$$
$$= \int_{-\infty}^{\infty} e^{tz}\,.\,e^{-z}(1+e^{-z})^{-2}\,dz$$
$$= \int_1^{\infty} u^2\,(u-1)^{-t}\,du \quad \text{where } 1 + e^{-z} = u \;\Rightarrow\; -e^{-z}\,dz = du$$
$$= \int_0^1 v^{-t}\,(1-v)^t\,dv \quad \text{where } v = 1 - \frac{1}{u} \;\Rightarrow\; dv = \frac{1}{u^2}\,du$$
$$= \beta(1 - t, 1 + t)\,;\; 1 - t > 0$$
$$= \pi t \operatorname{cosec} \pi t,\; t < 1.$$

Example 5. *Find the moment generating function of the negative exponential function*
$$f(x) = \lambda e^{-\lambda x}\,;\; x, \lambda > 0.$$

Sol.
$$\mathrm{M}_x(t) = \lambda\int_0^{\infty} e^{tx}\,.\,e^{-\lambda x}\,dx = \lambda\int_0^{\infty} e^{(t-\lambda)x}\,dx = \lambda\int_0^{\infty} e^{-(\lambda - t)x}\,dx$$
$$= \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1} = \sum_{r=0}^{\infty}\left(\frac{t}{\lambda}\right)^r\,;\; \lambda > t$$

Example 6. *Find the moment generating function of the discrete binomial distribution given by*
$$P(x) = {}^nc_x\, p^x\, q^{n-x} \quad (\text{where } q = 1 - p)$$
Also find the first and second moments about the mean. (U.P.T.U. 2008)

Sol. Moment generating function about the origin is given by
$$\mathrm{M}_x(t) = \Sigma\, e^{tx}\,.\,{}^nc_x\,.\,p^x q^{n-x}$$
$$= \Sigma\, {}^nc_x\,(pe^t)^x\, q^{n-x} = (q + pe^t)^n$$
$$\nu_1 = \left[\frac{d}{dt}\mathrm{M}_x(t)\right]_{t=0} = [n(q + pe^t)^{n-1}\,.\,pe^t]_{t=0} = np \qquad |\text{Since } q + p = 1$$
$$\nu_2 = \left[\frac{d^2}{dt^2}\mathrm{M}_x(t)\right]_{t=0}$$
$$= [np\{e^t\,.\,(n-1)\,(q + pe^t)^{n-2}\,pe^t + (q + pe^t)^{n-1}\,.\,e^t\}]_{t=0}$$

$$= [np\,(q + pe^t)^{n-2} \,.\, e^t\, \{(n-1)pe^t + (q + pe^t)\}]_{t=0}$$
$$= [np(q + pe^t)^{n-2} \,.\, e^t(q + npe^t)]_{t=0}$$
$$= np(q + np) \qquad | \because \quad q + p = 1$$
$$= npq + n^2 p^2$$

Hence first and second moments about the mean are given by

$$\mu_1 = 0$$

Since $\quad \bar{x} = \nu_1 = np$

$\therefore \quad \mu_2 = \nu_2 - \bar{x}^2 = \nu_2 - \nu_1^2 = npq + n^2p^2 - n^2p^2 = npq$

Hence, $\quad$ mean $= np$, S.D. $= \sqrt{\mu_2} = \sqrt{npq}$.

Example 7. *Find the moment generating function of the discrete Poisson distribution given by* $P(x) = e^{-\lambda} \cdot \dfrac{\lambda^x}{x\,!}$. *Also find the first and second moments about the mean.*

(M.T.U. 2013)

Sol. Moment generating function about the origin is given by

$$M_x(t) = \Sigma e^{tx} \,.\, e^{-\lambda} \,.\, \frac{\lambda^x}{x\,!} = e^{-\lambda}\, \Sigma \frac{(\lambda e^t)^x}{x\,!} = e^{-\lambda} \,.\, e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

$$\nu_1 = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = [e^{\lambda(e^t-1)}\, \lambda e^t]_{t=0} = \lambda$$

$$\nu_2 = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = [\lambda\, \{e^t \,.\, e^{\lambda(e^t-1)} \,.\, \lambda e^t + e^{\lambda(e^t-1)} e^t\}]_{t=0}$$

$$= [\lambda e^{\lambda(e^t-1)}\, e^t\, (\lambda\, e^t + 1)]_{t=0} = \lambda(\lambda + 1)$$

Hence first and second moments about the mean are given by

$$\mu_1 = 0$$

Since $\quad \nu_1 = \bar{x} = \lambda$

$\therefore \quad \mu_2 = \nu_2 - \bar{x}^2 = \nu_2 - \nu_1{}^2 = \lambda\,(\lambda + 1) - \lambda^2 = \lambda$

Example 8. *Find the moment generating function of the continuous normal distribution given by* $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$; $-\infty < x < \infty$.

Sol. Moment generating function about the origin is defined as

$$M_x(t) = E\,(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \,.\, e^{-t\sigma z}\, dz \qquad \text{where } z = \frac{x-\mu}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{\left(\mu t + \frac{1}{2}t^2\sigma^2\right)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - t\sigma)^2}\, dz$$

$$= e^{\mu t + \frac{1}{2}t^2\sigma^2} \cdot 1 = e^{\mu t + \frac{1}{2}t^2\sigma^2}. \qquad \left| \because \int_0^\infty e^{-z^2}\, dz = \frac{\sqrt{\pi}}{2} \right.$$

Example 9. *The random variable X assuming only non-negative values has a Gamma probability distribution if its probability distribution is given by*

$$f(x) = \begin{cases} \dfrac{\alpha^\beta}{\Gamma\beta} x^{\beta-1} e^{-\alpha x} & ; x > 0, \alpha > 0, \beta > 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the moment generating function of Gamma probability distribution.

Sol. $\quad M_x(t) = E(e^{tx})$

$$= \int_0^\infty e^{tx} \cdot \frac{\alpha^\beta}{\Gamma\beta} \cdot x^{\beta-1} e^{-\alpha x}\, dx = \frac{\alpha^\beta}{\Gamma\beta} \int_0^\infty x^{\beta-1} e^{-x(\alpha-t)}\, dx$$

$$= \frac{\alpha^\beta}{(\alpha-t)^\beta\, \Gamma\beta} \int_0^\infty y^{\beta-1} e^{-\beta}\, dy \quad | \text{ where } y = x(\alpha - t) \text{ so that } dy = (\alpha - t)\, dx$$

$$= \frac{1}{\left(1 - \frac{t}{\alpha}\right)^\beta} \cdot \frac{1}{\Gamma\beta} \Gamma\beta = \left(1 - \frac{t}{\alpha}\right)^{-\beta} ; \; |\, t\, | < \alpha.$$

Example 10. *Let the random variable X assume the value 'n' with the probability law $p(X = n) = pq^{n-1}$, $n = 1, 2, 3, \ldots$. Find the moment generating function and hence mean and variance.* (G.B.T.U. 2010)

Sol. The given distribution is a discrete distribution.

$$M_n(t) = \Sigma\, e^{tn}\, pq^{n-1} = \frac{p}{q} \Sigma (e^t q)^n = \frac{p}{q}(1 - e^t q)^{-1} = \frac{p}{q(1 - qe^t)}$$

which is the moment generating function.

$$\nu_1 = \left[\frac{d}{dt} M_n(t)\right]_{t=0} = \frac{p}{q}\left[\frac{q e^t}{(1 - q e^t)^2}\right]_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\nu_2 = \left[\frac{d^2}{dt^2} M_n(t)\right]_{t=0} = p\left[\frac{d}{dt}\left\{\frac{e^t}{(1 - qe^t)^2}\right\}\right]_{t=0}$$

$$= p\left[\frac{(1 - qe^t)^2 \cdot e^t - e^t \cdot 2(1 - qe^t)(-qe^t)}{(1 - qe^t)^4}\right]_{t=0}$$

$$= p\left[\frac{(1-q)^2 + 2q(1-q)}{(1-q)^4}\right] = \frac{1}{p} + \frac{2q}{p^2}$$

$$\text{Mean} = \bar{x} = \nu_1 = \frac{1}{p}$$

$$\text{Variance} = \mu_2 = \nu_2 - \bar{x}^2 = \frac{1}{p} + \frac{2q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

ASSIGNMENT

1. Define moment generating function. Find the moment generating function of a random variable X whose probability function is given by:

$$P(X = x) = p(1-p)^x,\ x = 0, 1, 2, \ldots, \infty$$ *(G.B.T.U. 2012)*

2. Define moment generating function and two properties of moment generating function with proof. *(M.T.U. 2013)*

3. The probability density function of the random variable X is $f(x) = \frac{1}{2\theta} \exp\left(-\frac{|x-\theta|}{\theta}\right), -\infty < x < \infty.$ Find moment generating function of X. Hence find the mean E(X) and variance V(X). *(M.T.U. 2013)*

$$\left[\textbf{Hint: } M_x(t) = \frac{1}{2\theta}\int_{-\infty}^{\theta} \exp\left(-\frac{\theta - x}{\theta}\right) e^{tx}\, dx + \frac{1}{2\theta}\int_{\theta}^{\infty} \exp\left(-\frac{x-\theta}{\theta}\right) e^{tx}\, dx\right]$$

4. Show that the moment generating function of random variable X having the p.d.f.

$$f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases} \quad \text{is } M_X(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

5. Find the moment generating function for triangular distribution defined by

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \end{cases}$$ *(M.T.U. 2013)*

6. If $P(X = x) = \frac{1}{2^x}$, $x = 1, 2, 3, \ldots$, find the moment generating function of x. Hence obtain the variance. *(U.P.T.U. 2014)*

Answers

1. $\dfrac{p}{1 - e^t(1-p)}$

3. $\dfrac{e^{\theta t}}{1 - \theta^2 t^2}$ or $1 + \theta t + \dfrac{3\theta^2 t^2}{2!} + \ldots$; $E(X) = \theta$, $V(x) = 2\theta^2$

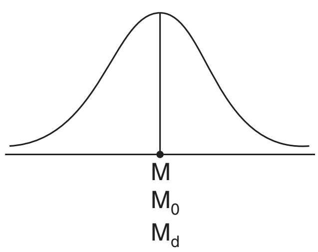
5. $M_x(t) = 1 + t + \dfrac{1}{18} t^2 + \ldots$

6. $\dfrac{e^t}{2 - e^t}$; $\dfrac{1}{2}$.

3.10 SKEWNESS

For a symmetrical distribution, the frequencies are symmetrically distributed about the mean *i.e.,* variates equidistant from the mean have equal frequencies. Also, the mean, mode and median coincide and median lies half-way between the two quartiles.

Thus, $\quad M = M_0 = M_d$ and $Q_3 - M = M - Q_1$.

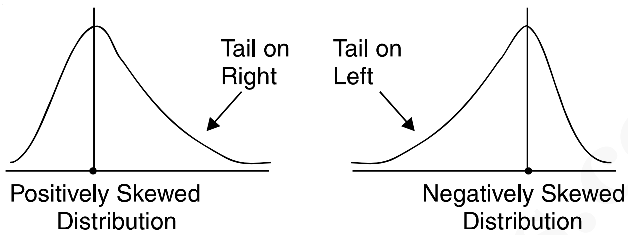


Symmetrical distribution

3.11 MEANING OF SKEWNESS (*U.P.T.U. 2015*)

If the curve of the distribution is not symmetrical, it may admit of tail on either side of the distribution. **Skewness means lack of symmetry or lopsidedness in a frequency distribution.**

The object of measuring skewness is to estimate the extent to which a distribution is distorted from a perfectly symmetrical distribution. Skewness indicates whether the curve is turned more to one side than to other *i.e.,* whether the curve has a longer tail on one side. Skewness can be positive as well as negative.



Skewness is positive if the longer tail of the distribution lies towards the right and negative if it lies towards the left.

3.12 TESTS OF SKEWNESS

1. If A.M. = Mode = Median, then there is no skewness in the distribution. In other words, the curve of the frequency distribution would be symmetrical, bell-shaped.
2. If A.M. is less than (greater than), the value of mode, the tail would be on left (right) side, *i.e.,* the distribution is negatively (positively) skewed.
3. If sum of frequencies of values less than mode is equal to the sum of frequencies of values greater than mode, then there would be no skewness.
4. If quartiles are equidistant from median, then there would be no skewness.

3.13 METHODS OF MEASURING SKEWNESS (*U.P.T.U. 2007*)

Relative measures of skewness are called the **coefficient of skewness**. They are independent of the units of measurement and as such, they are pure numbers.

Following are the methods of measuring skewness:

1. Karl Pearson's Method
2. Bowley's Method
3. Kelly's Method
4. Method of Moments.

Here, we will discuss Karl Pearson's method and the method of moments only.

3.13.1. Karl Pearson's Method

This method is based on the fact that in a symmetrical distribution, the value of A.M. is equal to that of mode. As we have already noted that the distribution is positively skewed if A.M. > Mode and negatively skewed if A.M. < Mode. The Karl Pearson's coefficient of skewness is given by:

$$\textbf{Karl Pearson's coefficient of skewness} = \frac{\textbf{A.M.} - \textbf{Mode}}{\textbf{S.D.}}$$

We have already studied the methods of calculating A.M., Mode and S.D. of frequency distributions. If mode is ill-defined in some frequency distribution, then the value of empirical mode is used in the formula.

Empirical mode = 3 Median – 2 A.M.

$$\therefore \quad \text{Coeff. of skewness} = \frac{\text{A.M.} - \text{Mode}}{\text{S.D.}}$$

$$= \frac{\text{A.M.} - (3\text{ Median} - 2\text{ A.M.})}{\text{S.D.}} = \frac{3\text{ A.M.} - 3\text{ Median}}{\text{S.D.}}$$

$$\therefore \quad \textbf{Karl Pearson's coefficient of skewness} = \frac{\mathbf{3(A.M. - Median)}}{\mathbf{S.D.}}$$

The coefficient of skewness as calculated by using this method gives magnitude as well as direction of skewness, present in the distribution. Practically, its value lies between – 1 and 1. For a symmetrical distribution, its value comes out to be zero.

The Karl Pearson's coefficient of skewness is generally denoted by 'SK_P'.

(*i*) If $SK_P = 0$ $\Leftrightarrow \dfrac{\text{Mean} - \text{Mode}}{\text{S.D.}} = 0$

$\Leftrightarrow$ Mean = Mode

$\Leftrightarrow$ Distribution is symmetrical.

Thus a distribution is a symmetrical distribution iff $SK_P = 0$.

(*ii*) If $SK_P > 0$ $\Leftrightarrow \dfrac{\text{Mean} - \text{Mode}}{\text{S.D.}} > 0$

$\Leftrightarrow$ Mean – Mode > 0

$\Leftrightarrow$ Mean > Mode

$\Leftrightarrow$ Distribution is positively skewed.

Thus a distribution is a positively skewed distribution iff $SK_P > 0$.

(*iii*) If $SK_P < 0$ $\Leftrightarrow \dfrac{\text{Mean} - \text{Mode}}{\text{S.D.}} < 0$

$\Leftrightarrow$ Mean – Mode < 0

$\Leftrightarrow$ Mean < Mode

$\Leftrightarrow$ Distribution is negatively skewed.

Thus a distribution is a negatively skewed distribution iff $SK_P < 0$.

EXAMPLES

Example 1. *Karl Pearson's coefficient of skewness of a distribution is 0.32, its standard deviation is 6.5 and mean is 29.6. Find the mode of the distribution.*

Sol. We have $SK_P = 0.32$, S.D. = 6.5, $\bar{x} = 29.6$.

Now $$SK_P = \frac{\bar{x} - \text{Mode}}{\text{S.D.}}$$

$$\therefore \quad 0.32 = \frac{29.6 - \text{Mode}}{6.5}$$

$$\Rightarrow \quad 29.6 - \text{Mode} = 0.32 \times 6.5 = 2.08$$

$$\Rightarrow \quad \text{Mode} = 29.6 - 2.08 = 27.52.$$

Example 2. *For a moderately skewed data, the arithmetic mean is 100, the variance is 35 and Karl Pearson's coefficient of skewness is 0.2. Find its mode and median.*

Sol. We have $\bar{x}$ = 100, Variance = 35, SK_P = 0.2.

Now $$SK_P = \frac{\bar{x} - \text{Mode}}{\sigma}$$

$\therefore$ $$0.2 = \frac{100 - \text{Mode}}{\sqrt{35}} \qquad (\because \quad \text{S.D.} = \sqrt{\text{variance}})$$

$\Rightarrow$ $$100 - \text{Mode} = 0.2 \times 5.92 = 1.184$$

$\Rightarrow$ $$\text{Mode} = 100 - 1.184 = 98.816.$$

Also, $$\text{Mode} = 3\text{ Median} - 2\bar{x} \quad \Rightarrow \quad 98.816 = 3 \text{ Median} - 2(100).$$

$\therefore$ $$3 \text{ Median} = 98.816 + 200 = 298.816$$

$\therefore$ $$\text{Median} = \frac{298.816}{3} = 99.61.$$

Example 3. *In a certain distribution, the following results were obtained :*

A.M. = 45, Median = 48, Coefficient of skewness = – 0.4. The person who gave you this data, failed to give the value of S.D. You are required to estimate it with the help of available data.

Sol. We have

coeff. of skewness = – 0.4, A.M. = 45, median = 48.

Now, coeff. of skewness $$= \frac{3(\bar{x} - \text{Median})}{\text{S.D.}}$$

$\Rightarrow$ $$-\frac{4}{10} = \frac{3(45-48)}{\text{S.D.}} = \frac{-9}{\text{S.D.}}$$

$\Rightarrow$ $$4 \text{ S.D.} = 90$$

$$\text{S.D.} = \frac{90}{4} = 22.5.$$

Example 4. *The sum of 20 observations is 300 and sum of their squares is 5000. The median is 15. Find the Karl Pearson's coefficient of skewness.*

Sol. Let 'x' be the variable under consideration.

We have n = 20, Σx = 300, Σx^2 = 5000, median = 15.

Now, $$\bar{x} = \frac{\Sigma x}{n} = \frac{300}{20} = 15$$

$$\text{S.D.} = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2} = \sqrt{\frac{5000}{20} - (15)^2} = \sqrt{250 - 225} = \sqrt{25} = 5$$

Now, Karl Pearson's coeff. of skewness

$$= \frac{3(\bar{x} - \text{Median})}{\text{S.D.}} = \frac{3(15-15)}{5} = \frac{0}{5} = 0\,.$$

Example 5. *Find the coefficient of skewness by Karl Pearson's method for the following data:*

Value	*6*	*12*	*18*	*24*	*30*	*36*	*42*
Frequency	*4*	*7*	*9*	*18*	*15*	*10*	*3*

Sol. **Calculation of $\overline{\mathbf{x}}$, S.D.**

Value *x*	*f*	*d = x – A* *A = 24*	*u = d/h* *h = 6*	*fu*	*fu*^{2}
6	4	– 18	– 3	– 12	36
12	7	– 12	– 2	– 14	28
18	9	– 6	– 1	– 9	9
24	18	0	0	0	0
30	15	6	1	15	15
36	10	12	2	20	40
42	3	18	3	9	27
	N = 66			$\Sigma fu = 9$	$\Sigma fu^2 = 155$

$$\textbf{A.M. } \bar{x} = A + \left(\frac{\Sigma fu}{N}\right)h = 24 + \left(\frac{9}{66}\right)6 = 24.82$$

$$\textbf{S.D.} = \sqrt{\frac{\Sigma fu^2}{N} - \left(\frac{\Sigma fu}{N}\right)^2} \times h = 6 \times \sqrt{\frac{155}{66} - \left(\frac{9}{66}\right)^2} = 8.94$$

Mode. **Grouping Table**

x	*I* *f*	*II*	*III*	*IV*	*V*	*VI*
6	4					
12	7	11				
18	9		16	20		
24	**18**	**27**			**34**	
30	15		**33**	**43**		**42**
36	10	25			28	
42	3		13			

Analysis Table

Column	*24*	*18*	*30*	*36*	*12*
I	1				
II	1	1			
III	1		1		
IV	1		1	1	
V	1	1			
VI	1	1	1		1
Total	6	3	3	1	1

∴ Mode = 24

$$\therefore \quad SK_P = \frac{\bar{x} - \text{Mode}}{\text{S.D.}} = \frac{24.82 - 24}{8.94} = \frac{0.82}{8.94} = 0.092.$$

Example 6. *Calculate Karl Pearson's coefficient of skewness for the following data:*

Income (in ₹)	*500—600*	*600—700*	*700—800*	*800—900*	*900—1000*	*1000—1100*
No. of employees	*8*	*12*	*4*	*2*	*1*	*1*

Sol. **Calculation of $\bar{x}$, Mode, S.D.**

Income (in ₹)	*No. of employees f*	*Mid-points of classes x*	*d = x – A A = 750*	*u = d/h h = 100*	*fu*	*fu²*
500—600	8	550	– 200	– 2	– 16	32
600—700	12	650	– 100	– 1	– 12	12
700—800	4	750	0	0	0	0
800—900	2	850	100	1	2	2
900—1000	1	950	200	2	2	4
1000—1100	1	1050	300	3	3	9
	N = 28				$\Sigma fu = -21$	$\Sigma fu^2 = 59$

A.M. $\bar{x} = \text{A} + \left(\frac{\Sigma fu}{\text{N}}\right)h = 750 + \left(-\frac{21}{28}\right)(100) = 750 - 75 = ₹\ 675$

Mode. By inspection, modal class is 600–700

$$\therefore \quad \text{Mode} = l + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)h$$

Here $l = 600, \Delta_1 = 12 - 8 = 4, \Delta_2 = 12 - 4 = 8, h = 100$

$$\therefore \quad \text{Mode} = 600 + \left(\frac{4}{4+8}\right)(100) = 600 + 33.33 = ₹\ 633.33$$

$$\textbf{S.D.} = \left[\sqrt{\frac{\Sigma fu^2}{\text{N}} - \left(\frac{\Sigma fu}{\text{N}}\right)^2}\right] \times h = \left[\sqrt{\frac{59}{28} - \left(-\frac{21}{28}\right)^2}\right] \times 100$$

$$= \sqrt{2.1071 - 0.5625} \times 100 = \sqrt{1.5446} \times 100 = 1.2428 \times 100 = ₹\ 124.28$$

Now, Karl Pearson's coeff. of skewness $= \frac{\bar{x} - \text{Mode}}{\text{S.D.}} = \frac{675 - 633.33}{124.28} = 0.34.$

ASSIGNMENT

1. A frequency distribution gives the following results:

Coeff. of variation = 5

Karl Pearson's Coeff. of Skewness = 0.5

S.D. = 2

Find A.M. and Mode of the distribution.

2. Find Pearson's coeff. of skewness from the following frequency distribution:

Height (in inches)	60–62	63–65	66–68	69–71	72–74
Frequency	5	18	42	27	8

3. From the following data, calculate the coefficient of skewness based on mean, median and S.D.

Variable	100–110	110–120	120–130	130–140	140–150	150–160	160–170	170–180
Frequency	4	16	36	52	64	40	32	11

4. From the following data, find out the Karl Pearson's coefficient of skewness:

Measurement	10	11	12	13	14	15
Frequency	2	4	10	8	5	1

5. Calculate Karl Pearson's coefficient of skewness for the following frequency distribution:

Marks more than	0	10	20	30	40	50	60	70
No. of students	100	90	75	50	25	15	5	0

6. For the following frequency distribution, calculate the value of Karl Pearson's coeff. of skewness:

Temp. (°C)	– 40 to – 30	– 30 to – 20	– 20 to – 10	– 10 to 0	0 to 10	10 to 20	20 to 30
No. of days	10	28	30	42	65	180	10

7. From the following data, calculate Karl Pearson's coefficient of skewness:

Marks (above)	0	10	20	30	40	50	60	70	80
No. of students	150	140	100	80	80	70	30	14	0

8. Find out the mean wage and coefficient of skewness from the following data:

35 men gets at the rate of ₹ 4.5 per man
40 men gets at the rate of ₹ 5.5 per man
48 men gets at the rate of ₹ 6.5 per man
100 men gets at the rate of ₹ 7.5 per man
125 men gets at the rate of ₹ 8.5 per man
87 men gets at the rate of ₹ 9.5 per man
43 men gets at the rate of ₹ 10.5 per man
22 men gets at the rate of ₹ 11.5 per man.

9. Calculate Karl Pearson's coefficient of skewness:

Wages (in ₹)	70–80	80–90	90–100	100–110	110–120	120–130	130–140	140–150
No. of workers	12	18	35	42	50	45	20	8

10. Find the mean, mode, S.D. and Karl Pearson's coeff. of skewness for the following:

Yrs. under	10	20	30	40	50	60
No. of persons	15	32	51	78	97	109

11. Compute the coeff. of skewness from the following figures : 25, 15, 23, 40, 27, 25, 23, 25, 20.

12. In a discrete series of 20 terms, the sum of the terms is 200, the sum of the squares of the terms is 5000 and the median is 15. Find Karl Pearson's coefficient of skewness.

13. Calculate the coefficient of skewness based on mean, median and standard deviation from the following data:

C.I.	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80
f	3	6	11	24	28	16	9	3

[*M.T.U. (MBA) 2012*]

14. Which of the following two series is symmetrical?

Series α: Mean = 22, Median = 24, S.D. = 10

Series β: Mean = 22, Median = 25, S.D. = 12

15. Following table gives the data relating to marks obtained by students who appeared for B. Tech. III Semester examination in Mathematics III at a centre. Calculate Karl Pearson's coefficient of skewness from the said data:

Marks	0 to 10	10 to 20	20 to 30	30 to 40	40 to 50	50 to 60	60 to 70	70 to 80
No. of students	10	40	20	0	10	40	16	14

[**Hint.** Since max. frequency corresponds to two classes very far from each other so mode is ill-defined and $SK_P = \dfrac{3(\text{Mean} - \text{Median})}{\text{S.D.}}$].

Answers

1. Mean = 40, Mode = 39 **2.** 0.0356 **3.** – 0.0087 **4.** 0.3604

5. – 0.0627 **6.** – 0.6617 **7.** – 0.7539

8. Mean wage = ₹ 8.07, Coeff. of skewness = – 0.2422 **9.** – 0.3314

10. Mean = 29.95, Mode = 35, S.D. = 15.49, Coeff. of skewness = – 0.32 **11.** – 0.03

12. – 1.22 **13.** – 0.08608 **14.** Series α **15.** 0.754.

3.13.2. Method of Moments

In this method, second and third central moments of the distribution are used. This measure of skewness is called the **Moment coefficient of skewness** and is given by:

$$\boxed{\textbf{Moment coefficient of skewness} = \frac{\mu_3}{\sqrt{\mu_2^3}}.}$$ [*G.B.T.U. (C.O.) 2009, 2011*]

For a symmetrical distribution, its value would come out to be zero. The coefficient of skewness as calculated by this method gives the magnitude as well as direction of the skewness present in the distribution.

In Statistics, we define $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$.

$\therefore$ Moment coefficient of skewness can also be written as $= \dfrac{\mu_3}{\sqrt{\mu_2^3}} = \pm\sqrt{\beta_1}$.

The sign with $\sqrt{\beta_1}$ is to be taken as that of μ_3. The moment coefficient of skewness is also denoted by γ_1. The moment coefficient of skewness is generally denoted by 'SK_M'.

EXAMPLES

Example 1. *The first three central moments of a distribution are 0, 15, – 31. Find the moment coefficient of skewness.*

Sol. We have $\mu_1 = 0, \mu_2 = 15$ and $\mu_3 = -31$

Moment coefficient of skewness $= \dfrac{\mu_3}{\sqrt{\mu_2{}^3}} = \dfrac{-31}{\sqrt{(15)^3}} = -\dfrac{31}{58.09} = -0.53.$

Example 2. *The first four moments of a distribution about the value 5 of the variable are 2, 20, 40 and 50. Calculate the moment coefficient of skewness.*

Sol. We have $A = 5, \mu_1' = 2, \mu_2' = 20, \mu_3' = 40$ and $\mu_4' = 50$

Now $\mu_2 = \mu_2' - (\mu_1')^2 = 20 - (2)^2 = 16$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3 = 40 - 3(2)(20) + 2(2)^3$$
$$= 40 - 120 + 16 = -64$$

Moment coefficient of skewness $= \dfrac{\mu_3}{\sqrt{\mu_2{}^3}} = \dfrac{-64}{\sqrt{(16)^3}} = \dfrac{-64}{64} = -1.$

Example 3. *Calculate the moment coefficient of skewness for the following distribution :*

Classes	*2.5–7.5*	*7.5–12.5*	*12.5–17.5*	*17.5–22.5*	*22.5–27.5*	*27.5–32.5*	*32.5–37.5*
Frequency	*8*	*15*	*20*	*32*	*23*	*17*	*5*

Sol. **Calculation of Moment Coefficient of Skewness**

Classes	*f*	*Mid-pts.* *x*	$d = x - A$ $A = 20$	$u = d/h$ $h = 5$	*fu*	fu^2	fu^3
2.5–7.5	8	5	– 15	– 3	– 24	72	– 216
7.5–12.5	15	10	– 10	– 2	– 30	60	– 120
12.5–17.5	20	15	– 5	– 1	– 20	20	– 20
17.5–22.5	32	20	0	0	0	0	0
22.5–27.5	23	25	5	1	23	23	23
27.5–32.5	17	30	10	2	34	68	136
32.5–37.5	5	35	15	3	15	45	135
	N = 120				$\Sigma fu = -2$	$\Sigma fu^2 = 288$	$\Sigma fu^3 = -62$

Now, $\mu_1' = \left(\dfrac{\Sigma fu}{N}\right)h = \left(\dfrac{-2}{120}\right) 5 = -0.083$

$$\mu_2' = \left(\frac{\Sigma fu^2}{N}\right)h^2 = \left(\frac{288}{120}\right) 5^2 = 60$$

$$\mu_3' = \left(\frac{\Sigma fu^3}{N}\right)h^3 = \left(\frac{-62}{162}\right) 5^3 = -64.583$$

Now, $\mu_2 = \mu_2' - \mu_1'^2 = 60 - (-0.083)^2 = 59.993$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3 = -64.583 - 3(-0.083)(60) + 2(-0.083)^3$$
$$= -49.644.$$

$\therefore$ Moment coefficient of skewness $= \dfrac{\mu_3}{\sqrt{\mu_2^3}} = \dfrac{-49.644}{\sqrt{(59.993)^3}} = -0.1068.$

ASSIGNMENT

1. The first-three central moments of a distribution are 0, 2.5, 0.7. Find the value of the moment coefficient of skewness. (*U.P.T.U. 2015*)

2. In a certain distribution, the first four moments about the point 4 are – 1.5, 17, – 30 and 308. Calculate the moment coefficient of skewness. (*U.P.T.U. 2014*)

3. The first three moments of a frequency distribution about origin ‘5’ are – 0.55, 4.46 and – 0.43. Find the moment coefficient of skewness.

4. Calculate the moment coefficient of skewness for the following data:

Marks	0–10	10–20	20–30	30–40	40–50	50–60	60–70
No. of students	8	12	20	30	15	10	5

5. Calculate the moment coefficient of skewness from the following data:

x	0	1	2	3	4	5	6	7	8
f	1	8	28	56	70	56	28	8	1

6. For the following frequency distribution, find the first four moments about the mean. Also find the value of β_1. Is it a symmetrical distribution?

x	2	3	4	5	6
f	1	3	7	3	1

7. Compute the coefficient of skewness from the following data: [*G.B.T.U. (C.O.) 2009*]

x	6	7	8	9	10	11	12
f	3	6	9	13	8	5	4

8. In two frequency distributions, the second moments about mean are 36 and 49 respectively while third moments about mean are 43.2 and 85.75. Compare the skewness in the two frequency distributions. (*G.B.T.U. 2012*)

9. The first three moments about the origin are given by $\nu_1 = \frac{n+1}{2}$, $\nu_2 = \frac{(n+1)(2n+1)}{6}$, and $\nu_3 = \frac{n(n+1)^2}{4}$. Examine the skewness of the data.

10. (*i*) Define skewness of a distribution. (*U.P.T.U. 2015*)
(*ii*) Define the coefficients of skewness. (*M.T.U. 2012, 2013*)

Answers

1. 0.17708 **2.** 0.7017 **3.** 0.7781 **4.** 0.0726

5. 0 **6.** 0, 0.933, 0, 2.533, Yes **7.** 0.0903.

8. γ_1 (for I distribution) = 0.2, γ_2 (for II distribution) = 0.25
Second distribution is more positively skewed than the first.

9. data is symmetrical.

3.14 KURTOSIS [*U.P.T.U. (C.O.) 2008; U.P.T.U. 2006, 2015*]

Given two frequency distributions which have the same variability as measured by the standard deviation, they may be relatively more or less flat topped than the normal curve. A frequency curve may be symmetrical but it may not be equally flat topped with the normal curve. The relative flatness of the top is called **kurtosis** and is measured by β_2. Kurtosis refers to the bulginess of the curve of a frequency distribution.

Curves which are neither flat nor sharply peaked are called normal curves or **mesokurtic curves.**

Curves which are flatter than the normal curve are called **platykurtic curves.**

Curves which are more sharply peaked than the normal curve are called **leptokurtic curves.**

3.15 MEASURE OF KURTOSIS [*G.B.T.U. (C.O.) 2009, 2011; U.P.T.U. 2007*]

The measure of kurtosis is denoted by β_2 and is defined as

$$\boxed{\beta_2 = \frac{\mu_4}{\mu_2^2}}$$

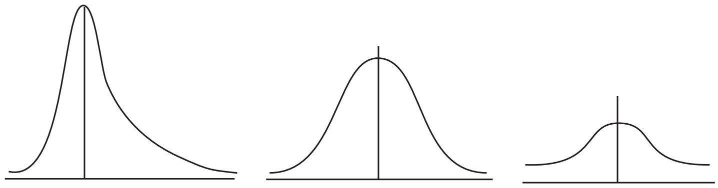
where μ_2 and μ_4 are respectively the second and fourth moments about mean of the distribution.

If $\boldsymbol{\beta_2 > 3}$, the distribution is **leptokurtic.** If $\boldsymbol{\beta_2 = 3}$, the distribution is ***mesokurtic***. If $\boldsymbol{\beta_2 < 3}$, the distribution is **platykurtic.** The kurtosis of a distribution is also measured by using Greek letter 'γ_2' which is defined as $\gamma_2 = \beta_2 - 3$.

$\therefore \quad \gamma_2 > 0 \Rightarrow \beta_2 - 3 > 0 \Rightarrow \beta_2 > 3 \Rightarrow$ the distribution is **leptokurtic.**

Similarly, if $\gamma_2 = 0$, then $\beta_2 = 3 \Rightarrow$ The distribution is **mesokurtic.**

$\gamma_2 < 0 \Rightarrow \beta_2 < 3 \Rightarrow$ the distribution is **platykurtic.**



3.15.1. Steps for Computing β_2

I. *If the value of μ_2 and μ_4 are given, then find β_2 by using the formula:* $\beta_2 = \dfrac{\mu_4}{{\mu_2}^2}$.

II. *If raw moments μ_1', μ_2', μ_3' and μ_4' are given, then calculate:*
$\mu_2 = \mu_2' - \mu_1'^2$ *and* $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

Now, find $\beta_2 = \dfrac{\mu_4}{\mu_2^2}$.

III. *If moments are not given, then first find μ_2 and μ_4 by using the given data and then use the formula:* $\beta_2 = \dfrac{\mu_4}{\mu_2^2}$.

IV. *The given distribution is leptokurtic, mesokurtic and platykurtic according as $\beta_2 > 3$, $\beta_2 = 3$ and $\beta_2 < 3$ respectively.*

EXAMPLES

Example 1. *The first four moments about mean of a frequency distribution are 0, 100, – 7 and 35000. Discuss the kurtosis of the distribution.*

Sol. We have, $\mu_1 = 0,\ \mu_2 = 100,\ \mu_3 = -7$ and $\mu_4 = 35000$

Now,
$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{35000}{(100)^2} = 3.5 > 3$$

$\therefore$ The distribution is **leptokurtic.**

Example 2. *The first four moments of a distribution about the value '4' of the variable are – 1.5, 17,– 30 and 108. State whether the distribution is leptokurtic or platykurtic.*

(U.P.T.U. 2007, 2014)

Sol. We have, $\mu_1' = -1.5,\ \mu_2' = 17,\ \mu_3' = -30,\ \mu_4' = 108$

Moments about mean:

$$\mu_2 = \mu_2' - \mu_1'^2 = 17 - (-1.5)^2 = 14.75$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$
$$= 108 - 4(-30)(-1.5) + 6(17)(-1.5)^2 - 3(-1.5)^4 = 142.3125$$

Kurtosis:
$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{142.3125}{(14.75)^2} = 0.6541$$

Since $\beta_2 < 3$, the distribution is **platykurtic.**

Example 3. *The first four moments of a distribution about x = 4 are 1, 4, 10 and 45. Obtain the various characteristics of the distribution on the basis of the given information. Comment upon the nature of the distribution.*

Sol. We have $A = 4, \mu_1' = 1, \mu_2' = 4, \mu_3' = 10$ and $\mu_4' = 45$

Moments about mean:

$$\mu_1 = 0 \text{ (always)}$$
$$\mu_2 = \mu_2' - \mu_1'^2 = 4 - (1)^2 = 3$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 10 - 3(4)(1) + 2(1)^3 = 0$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$
$$= 45 - 4(10)(1) + 6(4)(1)^2 - 3(1)^4 = 26$$

Skewness: Moment coefficient of skewness, $\gamma_1 = \dfrac{\mu_3}{\sqrt{\mu_2^3}} = \dfrac{0}{\sqrt{(3)^3}} = 0$.

$\therefore$ The distribution is **symmetrical.**

Kurtosis: $\beta_2 = \dfrac{\mu_4}{\mu_2{}^2} = \dfrac{26}{(3)^2} = 2.89 < 3$ $\therefore$ The distribution is **platykurtic.**

Example 4. *The standard deviation of a symmetric distribution is 5. What must be the value of the fourth moment about the mean in order that the distribution be*

(i) leptokurtic (ii) mesokurtic (iii) platykurtic?

Sol. We have, $\sigma = 5 \Rightarrow \sigma^2 = 25 \Rightarrow \mu_2 = 25$ | $\because \mu_2 = \sigma^2$

Now, $\beta_2 = \dfrac{\mu_4}{\mu_2{}^2} = \dfrac{\mu_4}{625}$

Thus, the distribution will be

(*i*) Leptokurtic if $\beta_2 > 3 \Rightarrow \dfrac{\mu_4}{625} > 3 \Rightarrow \mu_4 > 1875$

(*ii*) Mesokurtic if $\beta_2 = 3 \Rightarrow \dfrac{\mu_4}{625} = 3 \Rightarrow \mu_4 = 1875$

(*iii*) Platykurtic if $\beta_2 < 3 \Rightarrow \dfrac{\mu_4}{625} < 3 \Rightarrow \mu_4 < 1875$.

Example 5. *The first four moments about the working mean 28.5 of a distribution are 0.294, 7.144, 42.409 and 454.98. Calculate the moments about the mean. Also evaluate β_1, β_2 and comment upon the skewness and kurtosis of the distribution.* (U.P.T.U. 2006)

Sol. We have, $\mu_1' = 0.294, \mu_2' = 7.144, \mu_3' = 42.409, \mu_4' = 454.98$

Moments about mean

$$\mu_1 = 0$$
$$\mu_2 = \mu_2' - \mu_1'^2 = 7.144 - (.294)^2 = 7.0576$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$
$$= 42.409 - 3(7.144)(.294) + 2(.294)^3 = 36.1588$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$
$$= 454.98 - 4(42.409)(.294) + 6(7.144)(.294)^2 - 3(.294)^4$$
$$= 408.7896$$

Calculation of β_1 and β_2

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 3.7193 \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} = 8.2070$$

Skewness

Since β_1 is positive, $\gamma_1 = 1.9285$ | μ_3 is positive

$\therefore$ The distribution is **positively skewed.**

Kurtosis

Since $\beta_2 = 8.2070 > 3$

$\therefore$ The distribution is **leptokurtic.**

Example 6. *The first four moments of a distribution about the value '0' are – 0.20, 1.76, – 2.36 and 10.88. Find the moments about the mean and measure the kurtosis.*

(U.P.T.U. 2009)

Sol. We have, $\mu'_1 = -0.20, \mu'_2 = 1.76, \mu'_3 = -2.36, \mu'_4 = 10.88$

Moments about the mean:

$$\mu_1 = 0$$

$$\mu_2 = \mu'_2 - \mu_1'^2 = 1.76 - (-0.20)^2 = 1.72$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$$
$$= -2.36 - 3\,(1.76)\,(-0.20) + 2\,(-0.20)^3 = -1.32$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu_1'^2 - 3\mu_1'^4$$
$$= 10.88 - 4\,(-2.36)\,(-0.20) + 6\,(1.76)\,(-0.20)^2 - 3\,(-0.20)^4$$
$$= 9.4096$$

Kurtosis:

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3.180638$$

Since, $\beta_2 > 3$ hence the distribution is **leptokurtic.**

Example 7. *The following table represents the height of a batch of 100 students. Calculate kurtosis.*

Height (in cm)	*59*	*61*	*63*	*65*	*67*	*69*	*71*	*73*	*75*
No. of students	*0*	*2*	*6*	*20*	*40*	*20*	*8*	*2*	*2*

[U.P.T.U. (C.O.) 2008]

Sol. To calculate β_2, we will have to first find the values of μ_2 and μ_4.

Moments about 67

$$\mu_1' = \left(\frac{\Sigma fu}{N}\right)h = \left(\frac{12}{100}\right)(2) = 0.24$$

$$\mu_2' = \left(\frac{\Sigma fu^2}{N}\right)h^2 = \left(\frac{164}{100}\right)(4) = 6.56$$

Height (cm) x	*No. of students f*	$u = \frac{x - 67}{2}$	*fu*	fu^2	fu^3	fu^4
59	0	– 4	0	0	0	0
61	2	– 3	– 6	18	– 54	162
63	6	– 2	– 12	24	– 48	96
65	20	– 1	– 20	20	– 20	20
67	40	0	0	0	0	0
69	20	1	20	20	20	20
71	8	2	16	32	64	128
73	2	3	6	18	54	162
75	2	4	8	32	128	512
	$N = \Sigma f = 100$		$\Sigma fu = 12$	$\Sigma fu^2 = 164$	$\Sigma fu^3 = 144$	$\Sigma fu^4 = 1100$

$$\mu_3' = \left(\frac{\Sigma fu^3}{N}\right) h^3 = \frac{144}{100} \times 8 = 11.52$$

$$\mu_4' = \left(\frac{\Sigma fu^4}{N}\right) h^4 = \frac{1100}{100} \times 16 = 176$$

Moments about mean

$$\mu_2 = \mu_2' - \mu_1'^2 = 6.56 - (.24)^2 = 6.5024$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 176 - 4\,(11.52)\,(0.24) + 6\,(6.56)\,(0.24)^2 - 3\,(0.24)^4 = 167.19798$$

Kurtosis

Measure of kurtosis $\beta_2 = \frac{\mu_4}{\mu_2^2} = 3.9544 > 3$

Hence, the distribution is **leptokurtic.**

ASSIGNMENT

1. The first four moments about mean of a frequency distribution are 0, 60, – 50 and 8020 respectively. Discuss the kurtosis of the distribution.

2. The μ_2 and μ_4 for a distribution are found to be 2 and 12 respectively. Discuss the kurtosis of the distribution.

3. (*i*) The first four central moments of a distribution are 0, 2.5, 0.7 and 18.75. Test the kurtosis of the distribution. (*M.T.U. 2013*)

(*ii*) Define skewness and kurtosis of a distribution. The first four moments of a distribution are 0, 2.5, 0.7 and 18.71. Find the coefficient of skewness and kurtosis. (*U.P.T.U. 2015*)

4. The standard deviation of symmetric distribution is 4. What must be the value of μ_4 so that the distribution may be mesokurtic?

5. (*i*) If the first four moments about the value '5' of the variable are – 4, 22, – 117 and 560, find the value of β_2 and discuss the kurtosis.

(*ii*) The first four moments of a distribution about the value 5 of the variable are 2, 20, 40 and 50. Calculate the moments about the mean and comment upon the skewness and kurtosis of the distribution. (*G.B.T.U. 2011*)

6. (*i*) Calculate the value of β_2 for the following distribution:

Class	2.5–7.5	7.5–12.5	12.5–17.5	17.5–22.5	22.5–27.5	27.5–32.5	32.5–37.5
Frequency	8	15	20	32	23	17	5

(*ii*) Compute the value of β_2 for the following distribution. Is the distribution platykurtic?

Class	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Frequency	1	20	69	108	78	22	2

7. (*i*) Calculate μ_1, μ_2, μ_3 , μ_4 for the frequency distribution of heights of 100 students given in the following table and hence find coefficient of skewness and kurtosis.

Height (cm.) Class interval	144.5 – 149.5	149.5 – 154.5	154.5 – 159.5	159.5 – 164.5	164.5 – 169.5	169.5 – 174.5	174.5 – 179.5
Frequency	2	4	13	31	32	15	3

(*G.B.T.U. 2011*)

(*ii*) Find all four central moments and discuss skewness and kurtosis for the frequency distribution given in the following table:

Range of Expenditure (in ₹ 100 per month)	2–4	4–6	6–8	8–10	10–12
No. of families	38	292	389	212	69

[*G.B.T.U. 2013; M.T.U. 2012*]

8. (*i*) Find the measures of skewness and kurtosis on the basis of moments for the following distribution:

x	1	3	5	7	9
f	1	4	6	4	1

[*G.B.T.U.* (*C.O.*) *2011*]

(*ii*) Find the measure of skewness and kurtosis on the basis of moments for the following distribution:

Marks	5–15	15–25	25–35	35–45	45–55
No. of Students	1	3	5	7	4

(*M.T.U. (MBA) 2011*)

9. Calculate β_1 and β_2 from the following data:

Profit (in lakhs of ₹)	10–20	20–30	30–40	40–50	50–60
No. of companies	18	20	30	22	10

Indicate the nature of frequency curve.

10. Prove that the frequency distribution curve of the following frequency distribution is leptokurtic.

Class	10–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55
Frequency	1	4	8	19	35	20	7	5	1

11. Calculate the first four moments about the mean of the following distribution:

x	2	2.5	3	3.5	4	4.5	5
f	5	38	65	92	70	40	10

Also find the measures of skewness and kurtosis. (*M.T.U. 2012*)

12. Calculate the first four moments about the mean for the following frequency distribution and hence find the coefficient of skewness and kurtosis and comment upon the nature of the distribution.

Class-interval	5–10	10–15	15–20	20–25	25–30	30–35	35–40
Frequency	6	8	17	21	15	11	2

(*G.B.T.U. 2013*)

13. Define the coefficients of kurtosis. [*M.T.U. 2014; G.B.T.U. (C.O.) 2009, 2011; U.P.T.U. 2007*]

14. (*i*) What do you mean by kurtosis? Explain in brief. [*U.P.T.U. (C.O.) 2008*]

(*ii*) Define kurtosis of a distribution. (*U.P.T.U. 2006*)

Answers

1. $\beta_2 = 2.2278$, Platykurtic **2.** $\beta_2 = 3$, Mesokurtic

3. (*i*) $\beta_2 = 3$, Mesokurtic (*ii*) 0.17708, 2.9936 **4.** $\mu_4 = 768$

5. (*i*) $\beta_2 = 0.8889$, Platykurtic

(*ii*) $\mu_1 = 0$, $\mu_2 = 16$, $\mu_3 = -64$, $\mu_4 = 162$

$\gamma_1 = -1$, $\beta_2 = 0.6328$; Negatively skewed and platykurtic

6. (*i*) $\beta_2 = 2.3216$, Platykurtic (*ii*) $\beta_2 = 2.7240$, Yes

7. (*i*) $\mu_1 = 0$, $\mu_2 = 36.66$, $\mu_3 = -85.104$, $\mu_4 = 4373.3832$, $\gamma_1 = -0.3834$, $\beta_2 = 3.2541$

(*ii*) $\mu_1 = 0$, $\mu_2 = 37267.04$, $\mu_3 = 1746530.688$, $\mu_4 = 3567851989$

$y_1 = 0.24275$, $\beta_2 = 2.5689$, positively skewed and platykurtic.

8. (*i*) $\gamma_1 = 0$, $\beta_2 = 2.5$

(*ii*) $\mu_1 = 0$, $\mu_2 = 125$, $\mu_3 = -600$, $\mu_4 = 37625$, $\gamma_1 = -0.4293$, $\beta_2 = 2.408$, negatively skewed and platykurtic.

9. $\beta_1 = 0.0001$, $\beta_2 = 2.047$, Platykurtic.

11. $\mu_1 = 0$, $\mu_2 = 0.45328125$, $\mu_3 = 0.009890625$, $\mu_4 = 0.502111743$, $\gamma_1 = 0.0324$, $\beta_2 = 2.44379$, positively skewed and platykurtic.

12. $\mu_1 = 0$, $\mu_2 = 56$, $\mu_3 = -176.5625$, $\mu_4 = 7502.9375$, $\gamma_1 = -0.4213$, $\beta_2 = 2.3925$; negatively skewed and platykurtic.

3.16 CURVE FITTING

Let there be two variables x and y which give us a set of n pairs of numerical values (x_1, y_1), (x_2, y_2).......(x_n, y_n). In order to have an approximate idea about the relationship of these two variables, we plot these n paired points on a graph thus, we get a diagram showing the simultaneous variation in values of both the variables called *scatter or dot diagram.* From scatter diagram, we get only an approximate non-mathematical relation between two variables. Curve fitting means an exact relationship between two variables by algebraic equations, infact this relationship is the equation of the curve. Therefore, curve fitting means to form an equation of the curve from the given data. Curve fitting is considered of immense importance both from the point of view of theoretical and practical statistics.

Theoretically, it is useful in the study of correlation and regression. Practically, it enables us to represent the relationship between two variables by simple algebraic expressions *e.g.*, polynomials, exponential or logarithmic functions.

It is also used to estimate the values of one variable corresponding to the specified values of the other variable.

The constants occurring in the equation of approximate curve can be found by following methods:

(*i*) Graphical method (*ii*) Method of group averages

(*iii*) Method of least squares (*iv*) Method of moments.

Out of the above four methods, we will only discuss and study here *method of least squares.*

3.17 METHOD OF LEAST SQUARES

[*U.P.T.U. MCA (C.O.) 2008; U.P.T.U. (C.O.) 2008 ; U.P.T.U. 2008*]

Method of least squares provides a unique set of values to the constants and hence suggests a curve of best fit to the given data.

Suppose we have m-paired observations (x_1, y_1), (x_2, y_2), ..., (x_m, y_m) of two variables x and y. It is required to fit a polynomial of degree n of the type

$$y = a + bx + cx^2 + \ldots\ldots + kx^n \qquad \ldots(1)$$

of these values. We have to determine the constants $a, b, c, \ldots, k$ such that it represents the curve of best fit of that degree.

In case $m = n$, we get in general a unique set of values satisfying the given system of equations.

But if $m > n$ then, we get m equations by putting different values of x and y in equation (1) and we want to find only the values of n constants. Thus, there may be no such solution to satisfy all m equations.

Therefore, we try to find out those values of $a, b, c, \ldots\ldots, k$ which satisfy all the equations as nearly as possible. We apply the method of least squares in such cases.

Putting $x_1, x_2, \ldots, x_m$ for x in (1), we get

$$\begin{aligned} y_1' &= a + bx_1 + cx_1^2 + \ldots\ldots + kx_1^n \\ y_2' &= a + bx_2 + cx_2^2 + \ldots\ldots + kx_2^n \\ &\vdots \\ y_m' &= a + bx_m + cx_m^2 + \ldots\ldots + kx_m^n \end{aligned}$$

where $y_1', y_2', \ldots\ldots, y_m'$ are the expected values of y for $x = x_1, x_2, \ldots\ldots, x_m$ respectively. The values $y_1, y_2, \ldots\ldots, y_m$ are called observed values of y corresponding to $x = x_1, x_2, \ldots\ldots, x_m$ respectively.

The expected values are different from observed values, the difference $y_r - y_r'$ for different values of r are called *residuals*.

Introduce a new quantity U such that

$$U = \Sigma(y_r - y_r')^2 = \Sigma(y_r - a - bx_r - cx_r^2 - ..... - kx_r^n)^2$$

The constants $a, b, c, ..., k$ are chosen in such a way that the sum of the squares of residuals is minimum.

Now the condition for U to be maximum or minimum is $\frac{\partial U}{\partial a} = 0 = \frac{\partial U}{\partial b} = \frac{\partial U}{\partial c} = ... = \frac{\partial U}{\partial k}$.

On simplifying these relations, we get

$$\begin{aligned}
\Sigma y &= ma + b\Sigma x + ... + k\Sigma x^n \\
\Sigma xy &= a\Sigma x + b\Sigma x^2 + ... + k\,\Sigma x^{n+1} \\
\Sigma x^2 y &= a\Sigma x^2 + b\Sigma x^3 + ... + k\,\Sigma x^{n+2} \\
&\vdots \qquad\qquad \vdots \\
\Sigma x^n y &= a\Sigma x^n + b\Sigma x^{n+1} + ... + k\,\Sigma x^{2n}
\end{aligned}$$

These are known as *Normal equations* and can be solved as simultaneous equations to give the values of the constants $a, b, c, ..., k$. These equations are $(n + 1)$ in number.

If we calculate the second order partial derivatives and these values are put, they give a positive value of the function, so U is minimum.

This method does not help us to choose the degree of the curve to be fitted but helps us in finding the values of the constants when the form of the curve has already been chosen.

3.18 FITTING A STRAIGHT LINE

Let (x_i, y_i), $i = 1, 2, ..., n$ be n sets of observations of related data and

$$y = a + bx \qquad ...(1)$$

be the straight line to be fitted. The residual at $x = x_i$ is

$$E_i = y_i - f(x_i) = y_i - a - bx_i$$

Introduce a new quantity U such that

$$U = \sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

By the principle of Least squares, U is minimum

$$\therefore \qquad \frac{\partial U}{\partial a} = 0 \quad \text{and} \quad \frac{\partial U}{\partial b} = 0$$

$$\therefore \qquad 2\sum_{i=1}^{n} (y_i - a - bx_i)(-1) = 0 \quad \text{or} \quad \boxed{\Sigma y = na + b\Sigma x} \qquad ...(2)$$

and

$$2\sum_{i=1}^{n} (y_i - a - bx_i)(-x_i) = 0 \quad \text{or} \quad \boxed{\Sigma xy = a\Sigma x + b\Sigma x^2} \qquad ...(3)$$

Since x_i, y_i are known, equations (2) and (3) result two equations in a and b. Solving these, the best values for a and b can be known and hence equation (1).

Note. In case of change of origin,

if n is odd then, $u = \frac{x - (\text{middle term})}{\text{interval }(h)}$

but if n is even then, $u = \frac{x - (\text{mean of two middle terms})}{\frac{1}{2}(\text{interval})}$.

EXAMPLES

Example 1. *By the method of least squares, find the straight line that best fits the following data:*

x:	1	2	3	4	5
y:	14	27	40	55	68.

(U.P.T.U. 2008)

Sol. Let the straight line of best fit be $y = a + bx$ ...(1)

Normal equations are $\Sigma y = ma + b\Sigma x$ ...(2)

and $\Sigma xy = a\Sigma x + b\Sigma x^2$ ...(3)

Here $m = 5$

Table is as below:

x	y	xy	x^2
1	14	14	1
2	27	54	4
3	40	120	9
4	55	220	16
5	68	340	25
$\Sigma x = 15$	$\Sigma y = 204$	$\Sigma xy = 748$	$\Sigma x^2 = 55$

Substituting in (2) and (3), we get

$$204 = 5a + 15b$$
$$748 = 15a + 55b$$

Solving, we get $a = 0$, $b = 13.6$

Hence required straight line is $y = 13.6x$

Example 2. *Fit a straight line to the following data by least square method:*

x:	0	1	2	3	4
y:	1	1.8	3.3	4.5	6.3.

(*U.K.T.U. 2011*)

Sol. Let the straight line obtained from the given data be $y = a + bx$ then the normal equations are

$$\Sigma y = ma + b\,\Sigma x \quad ...(1)$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad ...(2)$$

Here, $m = 5$

x	y	xy	x^2
0	1	0	0
1	1.8	1.8	1
2	3.3	6.6	4
3	4.5	13.5	9
4	6.3	25.2	16
$\Sigma x = 10$	$\Sigma y = 16.9$	$\Sigma xy = 47.1$	$\Sigma x^2 = 30$

From (1) and (2), $16.9 = 5a + 10b$

and $47.1 = 10a + 30b$

Solving, we get $a = 0.72, b = 1.33$

$\therefore$ Required line is $y = 0.72 + 1.33\,x$.

Example 3. *Show that the best fitting linear function for the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ may be expressed in the form*

$$\begin{vmatrix} x & y & 1 \\ \Sigma x_i & \Sigma y_i & n \\ \Sigma x_i^2 & \Sigma x_i y_i & \Sigma x_i \end{vmatrix} = 0 \qquad (i = 1, 2, \ldots, n)$$

Show that the line passes through the mean point $(\bar{x}, \bar{y})$.

Sol. Let the best fitting linear function be $y = a + bx$ ...(1)

Then the normal equations are

$$\Sigma y_i = na + b\Sigma x_i \qquad \ldots(2)$$

and

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 \qquad \ldots(3)$$

Equations (1), (2), (3) may be rewritten as

$$bx - y + a = 0$$

$$b\Sigma x_i - \Sigma y_i + na = 0$$

and

$$b\Sigma x_i^2 - \Sigma x_i y_i + a\Sigma x_i = 0$$

Eliminating a and b between these equations

$$\begin{vmatrix} x & y & 1 \\ \Sigma x_i & \Sigma y_i & n \\ \Sigma x_i^2 & \Sigma x_i y_i & \Sigma x_i \end{vmatrix} = 0 \qquad \ldots(4)$$

which is the required best fitting linear function for the mean point $(\bar{x}, \bar{y})$,

$$\bar{x} = \frac{1}{n}\Sigma x_i, \quad \bar{y} = \frac{1}{n}\Sigma y_i\,.$$

Clearly, the line (4) passes through point $(\bar{x}, \bar{y})$ as two rows of determinant being equal make it zero.

ASSIGNMENT

1. Fit a straight line to the following data regarding x as the independent variable:

(*i*)

x	1	2	3	4	5	6
y	1200	900	600	200	110	50

(*ii*)

x	71	68	73	69	67	65	66	67
y	69	72	70	70	68	67	68	64

(*iii*)

x	0	5	10	15	20	25
y	12	15	17	22	24	30

2. (*i*) Find the best values of a and b so that $y = a + bx$ fits the given data:

x	0	1	2	3	4
y	1.0	2.9	4.8	6.7	8.6

(*ii*) Fit a straight line of the form $y = a_0 + a_1x$ to the data: [*U.P.T.U. (C.O.) 2008*]

x	1	2	3	4	6	8
y	2.4	3.1	3.5	4.2	5.0	6.0

3. Fit a straight line approximate to the data:

x	1	2	3	4
y	3	7	13	21

4. A simply supported beam carries a concentrated load P(*lb*) at its mid-point. Corresponding to various values of P, the maximum deflection Y (*in*) is measured. The data are given below. Find a law of the type $Y = a + bP$

P	100	120	140	160	180	200
Y	0.45	0.55	0.60	0.70	0.80	0.85

5. What straight line best fits the following data in the least square sense?

x	1	2	3	4
y	0	1	1	2

[*G.B.T.U. (MCA) 2010*]

6. The weight of a calf taken at weekly intervals are given below. Fit a straight line using method of least squares and calculate the average rate of growth per week.

Age	1	2	3	4	5	6	7	8	9	10
Weight	52.5	58.7	65	70.2	75.4	81.1	87.2	95.5	102.2	108.4

7. Find the least square line for the data points

(– 1, 10), (0, 9), (1, 7), (2, 5), (3, 4), (4, 3), (5, 0) and (6, –1).

8. Find the least square line $y = a + bx$ for the data:

x_i	– 2	– 1	0	1	2
y_i	1	2	3	3	4

9. If P is the pull required to lift a load W by means of a pulley block, find a linear law of the form $P = mW + c$ connecting P and W, using the data:

P	12	15	21	25
W	50	70	100	120

where P and W are taken in kg-wt. (*U.P.T.U. 2007*)

10. (*i*) Using the method of least squares, fit a straight line to the following data:

x	1	2	3	4	5
y	2	4	6	8	10

(*ii*) Using the method of least squares, fit a straight line from the following data:

x	0	2	4	5	6
y	5.012	10	15	21	30

(*U.P.T.U. 2009*)

(*iii*) Find the least square line that fits the following data, assuming that x-values are free from error: [*U.P.T.U. MCA (SUM) 2008*]

x	1	2	3	4	5	6
y	5.04	8.12	10.64	13.18	16.20	20.04

Answers

1. (*i*) $y = 1361.97 - 243.42\,x$ (*ii*) $y = 39.5454 + 0.4242\,x$ (*iii*) $y = 11.285 + 0.7\,x$

2. (*i*) $y = 1 + 1.9x$ (*ii*) $y = 2.0253 + 0.502\,x$ **3.** $y = -4 + 6x$

4. $Y = 0.004P + 0.048$ **5.** $y = -0.5 + 0.6x$ **6.** $y = 45.74 + 6.16x$, 6.16

7. $y = -1.6071429x + 8.6428571$ **8.** $y = 2.6 + (0.7)\,x$ **9.** $P = 2.2759 + 0.1879\,W$

10. (*i*) $y = 2x$ (*ii*) $y = 3.07734 + 3.86031\,x$ (*iii*) $y = 2.0253 + 2.908\,x$.

3.19 FITTING OF AN EXPONENTIAL CURVE $y = ae^{bx}$

Taking logarithm on both sides, we get

$$\log_{10} y = \log_{10} a + bx \log_{10} e$$

i.e.,

$$\boxed{Y = A + BX} \qquad \ldots(1)$$

where $Y = \log_{10} y$, $A = \log_{10} a$, $B = b \log_{10} e$ and $X = x$

The normal equations for (1) are

$$\Sigma Y = nA + B\Sigma X \quad \text{and} \quad \Sigma XY = A\Sigma X + B\Sigma X^2$$

Solving these, we get A and B.

Then $a = \text{antilog } A$ and $b = \dfrac{B}{\log_{10} e}$.

3.20 FITTING OF THE CURVE $y = ax^b$

Taking logarithm on both sides, we get

$$\log_{10} y = \log_{10} a + b \log_{10} x$$

i.e.,

$$\boxed{Y = A + BX} \qquad \ldots(1)$$

where $Y = \log_{10} y$, $A = \log_{10} a$, $B = b$ and $X = \log_{10} x$.

The normal equations to (1) are

$$\Sigma Y = nA + B\Sigma X$$

and $$\Sigma XY = A\Sigma X + B\Sigma X^2$$

which results A and B on solving and a = antilog A, b = B.

3.21 FITTING OF THE CURVE y = ab^x

Taking logarithm on both sides, we get

$$\log y = \log a + x \log b$$

$$\Rightarrow \qquad \boxed{Y = A + BX} \qquad \ldots(1)$$

where $Y = \log y$, $A = \log a$, $B = \log b$, $X = x$.

This is a linear equation in Y and X.

For estimating A and B, normal equations are

$$\Sigma Y = nA + B\,\Sigma X$$

and $$\Sigma XY = A\,\Sigma X + B\,\Sigma X^2$$

where n is the number of pairs of values of x and y.

Ultimately, a = antilog (A) and b = antilog (B).

3.22 FITTING OF THE CURVE pv^γ = k

$$pv^{\gamma} = k \quad \Rightarrow \quad v = k^{1/\gamma}\, p^{-1/\gamma}$$

Taking logarithm on both sides, we get

$$\log v = \frac{1}{\gamma}\log k - \frac{1}{\gamma}\log p$$

$$\Rightarrow \qquad \boxed{Y = A + BX}$$

where $Y = \log v$, $A = \dfrac{1}{\gamma}\log k$, $B = -\dfrac{1}{\gamma}$ and $X = \log p$

γ and k are determined by above equations. Normal equations are obtained as that of the straight line.

3.23 FITTING OF THE CURVE OF TYPE xy = b + ax

$$xy = b + ax \qquad \Rightarrow \quad y = \frac{b}{x} + a$$

$$\Rightarrow \qquad Y = bX + a, \qquad \text{where } X = \frac{1}{x}.$$

Normal equations are $\Sigma Y = na + b\Sigma X$ and $\Sigma XY = a\Sigma X + b\Sigma X^2$.

3.24 FITTING OF THE CURVE $y = ax^2 + \frac{b}{x}$

Let the n points be $(x_1, y_1), (x_2, y_2), ... , (x_n, y_n)$

Error of estimate for i^{th} point (x_i, y_i) is

$$E_i = \left(y_i - ax_i^2 - \frac{b}{x_i}\right)$$

By principle of Least squares, the values of a and b are such that

$$U = \sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n} \left(y_i - ax_i^2 - \frac{b}{x_i}\right)^2 \text{ is minimum.}$$

Normal equations are given by

$$\frac{\partial U}{\partial a} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} x_i^2 y_i = a \sum_{i=1}^{n} x_i^4 + b \sum_{i=1}^{n} x_i$$

and
$$\frac{\partial U}{\partial b} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} \frac{y_i}{x_i} = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} \frac{1}{x_i^2}$$

or Dropping the suffix i, normal equations are

$$\boxed{\Sigma x^2 y = a\,\Sigma x^4 + b\Sigma x} \quad \text{and} \quad \boxed{\sum \frac{y}{x} = a\,\Sigma x + b \sum \frac{1}{x^2}.}$$

3.25 FITTING OF THE CURVE $y = ax + bx^2$ *(U.P.T.U. 2014)*

Error of estimate for i^{th} point (x_i, y_i) is $E_i = (y_i - ax_i - bx_i^2)$

By principle of Least squares, the values of a and b are such that

$$U = \sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n} (y_i - ax_i - bx_i^2)^2 \text{ is minimum.}$$

Normal equations are given by

$$\frac{\partial U}{\partial a} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} x_i y_i = a \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i^3$$

and
$$\frac{\partial U}{\partial b} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} x_i^2 y_i = a \sum_{i=1}^{n} x_i^3 + b \sum_{i=1}^{n} x_i^4$$

or Dropping the suffix i, normal equations are

$$\boxed{\Sigma xy = a\,\Sigma x^2 + b\Sigma x^3} \quad \text{and} \quad \boxed{\Sigma x^2 y = a\,\Sigma x^3 + b\Sigma x^4.}$$

3.26 FITTING OF THE CURVE $y = ax + \frac{b}{x}$

Error of estimate for i^{th} point (x_i, y_i) is

$$E_i = y_i - ax_i - \frac{b}{x_i}$$

By principle of Least squares, the values of a and b are such that

$$U = \sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n} \left(y_i - ax_i - \frac{b}{x_i} \right)^2 \text{ is minimum.}$$

Normal equations are given by

$$\frac{\partial U}{\partial a} = 0$$

$$\Rightarrow \quad 2\sum_{i=1}^{n} \left(y_i - ax_i - \frac{b}{x_i} \right)(-x_i) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} x_i y_i = a \sum_{i=1}^{n} x_i^2 + nb \qquad \ldots(1)$$

and

$$\frac{\partial U}{\partial b} = 0$$

$$\Rightarrow \quad 2\sum_{i=1}^{n} \left(y_i - ax_i - \frac{b}{x_i} \right)\left(-\frac{1}{x_i} \right) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} \frac{y_i}{x_i} = na + b \sum_{i=1}^{n} \frac{1}{x_i^2} \qquad \ldots(2)$$

Dropping the suffix i, normal equations are

$$\boxed{\Sigma xy = a\Sigma x^2 + nb} \text{ and } \boxed{\sum \frac{y}{x} = na + b \sum \frac{1}{x^2}}$$

where n is the no. of pairs of values of x and y.

3.27 FITTING OF THE CURVE $y = a + \frac{b}{x} + \frac{c}{x^2}$

Normal equations are

$$\Sigma y = ma + b \sum \frac{1}{x} + c \sum \frac{1}{x^2}$$

$$\sum \frac{y}{x} = a \sum \frac{1}{x} + b \sum \frac{1}{x^2} + c \sum \frac{1}{x^3}$$

$$\sum \frac{y}{x^2} = a \sum \frac{1}{x^2} + b \sum \frac{1}{x^3} + c \sum \frac{1}{x^4}$$

where m is number of pairs of values of x and y.

3.28 FITTING OF THE CURVE $y = \frac{c_0}{x} + c_1 \sqrt{x}$

Error of estimate for i^{th} point (x_i, y_i) is

$$E_i = y_i - \frac{c_0}{x_i} - c_1 \sqrt{x_i}$$

By principle of Least squares, the values of a and b are such that

$$U = \sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n} \left(y_i - \frac{c_0}{x_i} - c_i\sqrt{x_i}\right)^2 \text{ is minimum.}$$

Normal equations are given by

$$\frac{\partial U}{\partial c_0} = 0 \quad \text{and} \quad \frac{\partial U}{\partial c_1} = 0$$

Now, $$\frac{\partial U}{\partial c_0} = 0$$

$$\Rightarrow \qquad 2\sum_{i=1}^{n} \left(y_i - \frac{c_0}{x_i} - c_1\sqrt{x_i}\right)\left(-\frac{1}{x_i}\right) = 0$$

$$\Rightarrow \qquad \sum_{i=1}^{n} \frac{y_i}{x_i} = c_0 \sum_{i=1}^{n} \frac{1}{x_i^2} + c_1 \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}} \qquad ...(1)$$

Also, $$\frac{\partial U}{\partial c_1} = 0$$

$$\Rightarrow \qquad 2\sum_{i=1}^{n} \left(y_i - \frac{c_0}{x_i} - c_1\sqrt{x_i}\right)(-\sqrt{x_i}) = 0$$

$$\Rightarrow \qquad \sum_{i=1}^{n} y_i \sqrt{x_i} = c_0 \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}} + c_1 \sum_{i=1}^{n} x_i \qquad ...(2)$$

Dropping suffix i, normal equations (1) and (2) become

$$\sum \frac{y}{x} = c_0 \sum \frac{1}{x^2} + c_1 \sum \frac{1}{\sqrt{x}}$$

and $$\sum y\sqrt{x} = c_0 \sum \frac{1}{\sqrt{x}} + c_1 \sum x.$$

3.29 FITTING OF THE CURVE $2^x = ax^2 + bx + c$

Normal equations are

$$\Sigma\, 2^x x^2 = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$$

$$\Sigma\, 2^x x = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$

and $$\Sigma\, 2^x = a\Sigma x^2 + b\Sigma x + mc$$

where, m is number of points (x_i, y_i)

3.30 FITTING OF THE CURVE $y = ae^{-3x} + be^{-2x}$

Normal equations are

$$\Sigma y e^{-3x} = a\, \Sigma e^{-6x} + b\, \Sigma e^{-5x}$$

and $$\Sigma y e^{-2x} = a\, \Sigma e^{-5x} + b\, \Sigma e^{-4x}$$

EXAMPLES

Example 1. *Find the curve of best fit of the type $y = ae^{bx}$ to the following data by the method of Least squares:*

x:	*1*	*5*	*7*	*9*	*12*
y:	*10*	*15*	*12*	*15*	*21.*

Sol. The curve to be fitted is $y = ae^{bx}$

or $\quad Y = A + BX$, where, $Y = \log_{10} y$, $A = \log_{10} a$, $X = x$ and $B = b \log_{10} e$

$\therefore$ The normal equations are $\Sigma Y = 5A + B\Sigma X$

and $\quad \Sigma XY = A\Sigma X + B\Sigma X^2$

$X = x$	y	$Y = \log_{10} y$	X^2	XY
1	10	1.0000	1	1
5	15	1.1761	25	5.8805
7	12	1.0792	49	7.5544
9	15	1.1761	81	10.5849
12	21	1.3222	144	15.8664
$\Sigma X = 34$		$\Sigma Y = 5.7536$	$\Sigma X^2 = 300$	$\Sigma XY = 40.8862$

Substituting the above values in the normal equations, we get

$$5.7536 = 5A + 34B$$

and $$40.8862 = 34A + 300B$$

On solving, $\quad A = 0.9766$; $B = 0.02561$

$$\therefore \quad a = \text{antilog}_{10} A = 9.4754 \text{ ; } b = \frac{B}{\log_{10} e} = 0.059$$

Hence the required curve is $\quad y = 9.4754e^{0.059x}$.

Example 2. *Determine the constants a and b by the method of least squares such that $y = ae^{bx}$ fits the following data:*

x	*2*	*4*	*6*	*8*	*10*
y	*4.077*	*11.084*	*30.128*	*81.897*	*222.62*

Sol. $\quad y = ae^{bx}$

Taking log on both sides

$$\log y = \log a + bx \log e$$

or $$Y = A + BX,$$

where, $\quad Y = \log y$, $A = \log a$, $B = b \log_{10} e$, $X = x$.

Normal equations are

$$\Sigma Y = mA + B\Sigma X \quad ...(1)$$

and $$\Sigma XY = A\Sigma X + B\Sigma X^2. \quad ...(2)$$

Here, $m = 5$.

Table is as follows:

x	y	X	Y	XY	X^2
2	4.077	2	.61034	1.22068	4
4	11.084	4	1.04469	4.17876	16
6	30.128	6	1.47897	8.87382	36
8	81.897	8	1.91326	15.30608	64
10	222.62	10	2.347564	23.47564	100
		$\Sigma X = 30$	$\Sigma Y = 7.394824$	$\Sigma XY = 53.05498$	$\Sigma X^2 = 220$

Substituting these values in equations (1) and (2), we get

$$7.394824 = 5A + 30B$$

and

$$53.05498 = 30A + 220B.$$

Solving, we get $A = 0.1760594$ and $B = 0.2171509$

$\therefore$ $a = \text{antilog}\,(A) = \text{antilog}\,(0.1760594) = 1.49989$

and

$$b = \frac{B}{\log_{10} e} = \frac{0.2171509}{0.4342945} = 0.50001$$

Hence the required equation is

$$y = 1.49989\, e^{0.50001x}.$$

Example 3. *Obtain a relation of the form $y = ab^x$ for the following data by the method of least squares:* [G.B.T.U. MCA (SUM) 2010]

x	2	3	4	5	6
y	8.3	15.4	33.1	65.2	127.4

Sol. The curve to be fitted is $y = ab^x$

or $Y = A + Bx,$

where, $A = \log_{10} a$, $B = \log_{10} b$ and $Y = \log_{10} y$.

$\therefore$ The normal equations are $\Sigma Y = 5A + B\Sigma x$

and $\Sigma xY = A\Sigma x + B\Sigma x^2$.

x	y	$Y = \log_{10} y$	x^2	xY
2	8.3	0.9191	4	1.8382
3	15.4	1.1872	9	3.5616
4	33.1	1.5198	16	6.0792
5	65.2	1.8142	25	9.0710
6	127.4	2.1052	36	12.6312
$\Sigma x = 20$		$\Sigma Y = 7.5455$	$\Sigma x^2 = 90$	$\Sigma xY = 33.1812$

Substituting the above values, we get

$$7.5455 = 5A + 20B \quad \text{and} \quad 33.1812 = 20A + 90B.$$

On solving $A = 0.31$ and $B = 0.3$

$\therefore$ $a = \text{antilog } A = 2.04$ and $b = \text{antilog } B = 1.995.$

Hence the required curve is $y = 2.04(1.995)^x$.

Example 4. *Obtain the least squares fit of the form $f(t) = a\,e^{-3t} + be^{-2t}$ for the data:* (U.P.T.U. 2008)

t:	0.1	0.2	0.3	0.4
$f(t)$:	0.76	0.58	0.44	0.35

Sol. Normal equations to the curve $f(t) = a\,e^{-3t} + be^{-2t}$ are:

$$\Sigma f(t)\, e^{-3t} = a\,\Sigma\, e^{-6t} + b\,\Sigma\, e^{-5t} \qquad \ldots(1)$$

$$\Sigma f(t)\, e^{-2t} = a\,\Sigma\, e^{-5t} + b\,\Sigma\, e^{-4t} \qquad \ldots(2)$$

See art. 3.30

Table of values is

t	$f(t)$	e^{-4t}	e^{-5t}	e^{-6t}	$f(t)\, e^{-2t}$	$f(t)\, e^{-3t}$
0.1	0.76	0.6703	0.6065	0.5488	0.6222	0.5630
0.2	0.58	0.4493	0.3679	0.3012	0.3888	0.3183
0.3	0.44	0.3012	0.2231	0.1653	0.2415	0.1789
0.4	0.35	0.2019	0.1353	0.0907	0.1573	0.1054
Total		Σe^{-4t} = 1.6227	Σe^{-5t} = 1.3328	Σe^{-6t} = 1.106	$\Sigma f(t)\, e^{-2t}$ = 1.4098	$\Sigma f(t)\, e^{-3t}$ = 1.1656

Substituting values in (1) and (2), we get

$$1.106\, a + 1.3328\, b = 1.1656$$

$$1.3328\, a + 1.6227\, b = 1.4098$$

On solving, we get $a = 0.6778,\quad b = 0.3121$.

Hence the least squares fit is $f(t) = 0.6778\, e^{-3t} + 0.3121 e^{-2t}$.

Example 5. *By the method of least squares, find the curve $y = ax + bx^2$ that best fits the following data:* (U.P.T.U. 2014)

x	1	2	3	4	5
y	1.8	5.1	8.9	14.1	19.8

Sol. Normal equations are

$$\Sigma xy = a\Sigma x^2 + b\Sigma x^3 \qquad \ldots(1)$$

and

$$\Sigma x^2 y = a\Sigma x^3 + b\Sigma x^4 \qquad \ldots(2)$$

Let us form a table as below:

x	y	x^2	x^3	x^4	xy	x^2y
1	1.8	1	1	1	1.8	1.8
2	5.1	4	8	16	10.2	20.4
3	8.9	9	27	81	26.7	80.1
4	14.1	16	64	256	56.4	225.6
5	19.8	25	125	625	99	495
Total		$\Sigma x^2 = 55$	$\Sigma x^3 = 225$	$\Sigma x^4 = 979$	$\Sigma xy = 194.1$	$\Sigma x^2 y = 822.9$

Substituting these values in equations (1) and (2), we get

$$194.1 = 55\,a + 225\,b$$

and
$$822.9 = 225\,a + 979\,b$$

$$\Rightarrow \qquad a = \frac{83.85}{55} \simeq 1.52 \quad \text{and} \quad b = \frac{317.4}{664} \simeq .49$$

Hence required parabolic curve is $y = 1.52\,x + 0.49\,x^2$.

Example 6. *Fit the curve $pv^\gamma = k$ to the following data:* [U.P.T.U. MCA (C.O.) 2007]

p (kg/cm²)	*0.5*	*1*	*1.5*	*2*	*2.5*	*3*
v (litres)	*1620*	*1000*	*750*	*620*	*520*	*460*

Sol. $$pv^\gamma = k$$

$$\Rightarrow \qquad v = \left(\frac{k}{p}\right)^{1/\gamma} = k^{1/\gamma}\,p^{-1/\gamma}$$

Taking log, $$\log v = \frac{1}{\gamma}\log k - \frac{1}{\gamma}\log p$$

which is of the form $$\text{Y} = \text{A} + \text{BX}$$

where $\text{Y} = \log v$, $\text{X} = \log p$, $\text{A} = \frac{1}{\gamma}\log k$ and $\text{B} = -\frac{1}{\gamma}$

p	v	X	Y	XY	X^2
0.5	1620	– 0.30103	3.20952	– 0.96616	0.09062
1	1000	0	3	0	0
1.5	750	0.17609	2.87506	0.50627	0.03101
2	620	0.30103	2.79239	0.84059	0.09062
2.5	520	0.39794	2.716	1.08080	0.15836
3	460	0.47712	2.66276	1.27046	0.22764
Total		$\Sigma X = 1.05115$	$\Sigma Y = 17.25573$	$\Sigma XY = 2.73196$	$\Sigma X^2 = 0.59825$

Here, $m = 6$

Normal equations are

$$17.25573 = 6\text{A} + 1.05115\text{ B}$$

and
$$2.73196 = 1.05115\text{ A} + 0.59825\text{ B}$$

Solving these, we get

$$\text{A} = 2.99911 \quad \text{and} \quad \text{B} = -\,0.70298$$

$$\therefore \qquad \gamma = -\frac{1}{\text{B}} = \frac{1}{0.70298} = 1.42252$$

Again, $$\log k = \gamma\text{A} = 4.26629$$

$$\therefore \qquad k = \text{antilog}\,(4.26629) = 18462.48$$

Hence, required curve is

$$pv^{1.42252} = 18462.48.$$

Example 7. *The pressure of the gas corresponding to various volumes V is measured, given by the following data:*

V (cm^3):	*50*	*60*	*70*	*90*	*100*
P (kg cm^{-2}):	*64.7*	*51.3*	*40.5*	*25.9*	*78*

Fit the data to the equation $PV^{\gamma} = C$.

Sol. $$PV^{\gamma} = C$$

$$\Rightarrow \quad P = CV^{-\gamma}$$

Taking log on both sides, we get

$$\log P = \log C - \gamma \log V$$

$$\Rightarrow \quad Y = A + BX$$

where, $Y = \log P$, $A = \log C$, $B = -\gamma$, $X = \log V$

Normal equations are

$$\Sigma Y = mA + B\Sigma X \qquad \ldots(1)$$

and $$\Sigma XY = A\Sigma X + B\Sigma X^2 \qquad \ldots(2)$$

Here $m = 5$

The table is as below:

V	*P*	*X = log V*	*Y = log P*	*XY*	X^2
50	64.7	1.69897	1.81090	3.07666	2.88650
60	51.3	1.77815	1.71012	3.04085	3.16182
70	40.5	1.84510	1.60746	2.96592	3.40439
90	25.9	1.95424	1.41330	2.76193	3.81905
100	78	2	1.89209	3.78418	4
		$\Sigma X = 9.27646$	$\Sigma Y = 8.43387$	$\Sigma XY = 15.62954$	$\Sigma X^2 = 17.27176$

From Normal equations, we have

$$8.43387 = 5A + 9.27646\, B$$

and $$15.62954 = 9.27646\, A + 17.27176\, B$$

Solving these, we get

$$A = 2.22476, \quad B = -\,0.28997$$

$$\therefore \quad \gamma = -\,B = 0.28997$$

$$C = \text{antilog } (A) = \text{antilog } (2.22476) = 167.78765$$

Hence, the required equation of curve is

$$PV^{0.28997} = 167.78765.$$

Example 8. (*i*) *Given the following experimental values:*

x:	*0*	*1*	*2*	*3*
y:	*2*	*4*	*10*	*15*

Fit by the method of Least squares a parabola of the type $y = a + bx^2$.

(*ii*) *Find the Least squares fit of the form* $y = a_0 + a_1x^2$ *to the following data:*

x:	*– 1*	*0*	*1*	*2*
y:	*2*	*5*	*3*	*0*

(U.P.T.U. 2008)

Sol. (*i*) Error of estimate for i^{th} point (x_i, y_i) is $E_i = (y_i - a - bx_i^2)$

By method of Least squares, the values of a, b are chosen such that

$$U = \sum_{i=1}^{4} E_i^2 = \sum_{i=1}^{4} (y_i - a - bx_i^2)^2 \text{ is minimum.}$$

Normal equation are given by

$$\frac{\partial U}{\partial a} = 0 \Rightarrow \Sigma y = ma + b\Sigma x^2 \quad \text{...(1)}$$

and $$\frac{\partial U}{\partial b} = 0 \Rightarrow \Sigma x^2 y = a\Sigma x^2 + b\Sigma x^4 \quad \text{...(2)}$$

x	y	x^2	x^2y	x^4
0	2	0	0	0
1	4	1	4	1
2	10	4	40	16
3	15	9	135	81
Total	$\Sigma y = 31$	$\Sigma x^2 = 14$	$\Sigma x^2y = 179$	$\Sigma x^4 = 98$

Here $m = 4$

From (1) and (2), $31 = 4a + 14b$ and $179 = 14a + 98b$

Solving for a and b, we get $a = 2.71$, $b = 1.44$

Hence the required curve is $y = 2.71 + 1.44\, x^2$.

(*ii*) Normal equations are

$$\Sigma y = ma_0 + a_1 \Sigma x^2 \quad \text{...(1)}$$

and $$\Sigma x^2 y = a_0 \Sigma x^2 + a_1 \Sigma x^4 \quad \text{...(2)}$$

The table is as follows:

x	y	x^2	x^2y	x^4
– 1	2	1	2	1
0	5	0	0	0
1	3	1	3	1
2	0	4	0	16
	$\Sigma y = 10$	$\Sigma x^2 = 6$	$\Sigma x^2y = 5$	$\Sigma x^4 = 18$

Here, $m = 4$

From (1) and (2), $10 = 4a_0 + 6a_1$

$5 = 6a_0 + 18a_1$

$\Rightarrow$ $a_0 = 4.1667,\ a_1 = -1.1111$

Hence, the required curve is $y = 4.1667 - 1.1111\,x^2$

Example 9. *Use the method of Least squares to fit the curve:* $y = \dfrac{c_0}{x} + c_1\sqrt{x}$ *to the following table of values:* [G.B.T.U. (MCA) 2007, 2011]

x:	*0.1*	*0.2*	*0.4*	*0.5*	*1*	*2*
y:	*21*	*11*	*7*	*6*	*5*	*6.*

Sol. As derived in art. 3.28, normal equations to the curve $y = \dfrac{c_0}{x} + c_1\sqrt{x}$ are

$$\sum \frac{y}{x} = c_0 \sum \frac{1}{x^2} + c_1 \sum \frac{1}{\sqrt{x}} \qquad ...(1)$$

and

$$\sum y\sqrt{x} = c_0 \sum \frac{1}{\sqrt{x}} + c_1 \sum x \qquad ...(2)$$

Table is as below:

x	y	y/x	$y\sqrt{x}$	$\dfrac{1}{\sqrt{x}}$	$\dfrac{1}{x^2}$
0.1	21	210	6.64078	3.16228	100
0.2	11	55	4.91935	2.23607	25
0.4	7	17.5	4.42719	1.58114	6.25
0.5	6	12	4.24264	1.41421	4
1	5	5	5	1	1
2	6	3	8.48528	0.70711	0.25
$\Sigma x = 4.2$		$\Sigma(y/x) = 302.5$	$\Sigma y\sqrt{x} = 33.71524$	$\sum \dfrac{1}{\sqrt{x}} = 10.10081$	$\sum \dfrac{1}{x^2} = 136.5$

From equations (1) and (2), we have

$$302.5 = 136.5\,c_0 + 10.10081\,c_1$$

and

$$33.71524 = 10.10081\,c_0 + 4.2\,c_1$$

Solving these, we get

$$c_0 = 1.97327 \quad \text{and} \quad c_1 = 3.28182$$

Hence the required equation of curve is

$$y = \frac{1.97327}{x} + 3.28182\sqrt{x}.$$

ASSIGNMENT

1. (*i*) Using the method of least squares, fit the non-linear curve of the form $y = ae^{bx}$ to the following data:

x	0	2	4
y	5.012	10	31.62

(*ii*) The voltage V across a capacitor at time t seconds is given by the following table. Use the principle of least squares to fit a curve of the form $V = ae^{kt}$ to the data:

t	0	2	4	6	8
V	150	63	28	12	5.6

(*iii*) For the data given below, find the equation to the best fitting exponential curve of the form $y = ae^{bx}$.

x	1	2	3	4	5	6
y	1.6	4.5	13.8	40.2	125	300

2. (*i*) State some important curve-fitting procedures. (*U.P.T.U. 2008*)

(*ii*) Derive the least square equations for fitting a curve of the type $y = ax + \frac{b}{x}$ to a set of n points (x_i, y_i) ; $i = 1, 2, \ldots, n$.

3. (*i*) Fit a curve $y = ax^b$ to the following data:

x	1	2	3	4	5	6
y	2.98	4.26	5.21	6.1	6.8	7.5

(*ii*) Fit a least square geometric curve $y = ax^b$ to the following data:

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

(*iii*) Fit a curve of the form $y = ax^b$ to the data given below:

x	1	2	3	4	5
y	7.1	27.8	62.1	110	161

4. (*i*) Fit a curve of the form $y = ab^x$ in least square sence to the data given below:

x	2	3	4	5	6
y	144	172.8	207.4	248.8	298.5

(*ii*) Fit an exponential curve of the form $y = ab^x$ to the following data:

x	1	2	3	4	5	6	7	8
y	1	1.2	1.8	2.5	3.6	4.7	6.6	9.1

5. The pressure and volume of a gas are related by the equation $pv^a = b$ where a and b are constants. Fit this equation to the following set of data:

p (kg/cm²)	0.5	1	1.5	2	2.5	3
v (litres)	1.62	1	0.75	0.62	0.52	0.46

6. (*i*) Determine the constants of the curve $y = ax + bx^2$ for the following data:

x	0	1	2	3	4
y	2.1	2.4	2.6	2.7	3.4

(*ii*) Using method of least squares, derive the normal equations to fit the curve $y = ax^2 + bx$. Hence fit this curve to the following data: (*G.B.T.U. 2011*)

x	1	2	3	4	5	6	7	8
y	1	1.2	1.8	2.5	3.6	4.7	6.6	9.1

7. Fit a curve of the type $xy = ax + b$ to the following data:

x	1	3	5	7	9	10
y	36	29	28	26	24	15

8. Fit a relation $y = ax + \frac{b}{x}$ which satisfies the following data, using method of least squares:

x	1	2	3	4	5	6	7	8
y	5.4	6.2	8.2	10.3	12.6	14.8	17.2	19.5

(*G.B.T.U. 2010*)

9. Derive the least square equations for fitting a curve of the type $y = ax^2 + \frac{b}{x}$ to a set of n points.

Hence fit a curve of this type to the data:

x	1	2	3	4
y	– 1.51	0.99	3.88	7.66

10. Derive the least squares approximations of the type $ax^2 + bx + c$ to the function 2^x at the points $x_i = 0, 1, 2, 3, 4$.

11. A person runs the same race track for 5 consecutive days and is timed as follows:

Day (x)	1	2	3	4	5
Time (y)	15.3	15.1	15	14.5	14

Make a least square fit to the above data using a function $a + \frac{b}{x} + \frac{c}{x^2}$.

12. Use the method of least squares to fit the curve $y = c_0 x + \dfrac{c_1}{\sqrt{x}}$ for the following data:

x	0.2	0.3	0.5	1	2
y	16	14	11	6	3

13. Experiments with a periodic process gave the following data:

$t°$	0	50	100	150	200	250	300	350
y	0.754	1.762	2.041	1.412	0.303	– 0.484	– 0.38	0.520

Estimate the parameters a and b in the model $y = b + a \sin t$ using the least squares approximation.

14. A physicist wants to approximate the following data:

x	0	0.5	1	2
$f(x)$	0	0.57	1.46	5.05

using a function $a\,e^{bx} + c$. He believes that $b \approx 1$. Compute the values of a and c that give the best least squares approximation assuming that $b = 1$.

15. Determine the normal equations if the cubic polynomial $y = a_0 + a_1x + a_2x^2 + a_3x^3$ is fitted to the data (x_i, y_i); $0 \le i \le m$.

16. Estimate y at $x = 5$ by fitting a least squares curve of the form $y = \dfrac{b}{x\,(x - a)}$ to the following data:

x	3.6	4.8	6	7.2	8.4	9.6	10.8
y	0.83	0.31	0.17	0.10	0.07	0.05	0.04

$\left(\textbf{Hint:}\text{ Rewrite the equation as } \dfrac{1}{y} = -\dfrac{a}{b}x + \dfrac{1}{b}x^2\right)$

Answers

1. (*i*) $y = 4.642\,e^{0.46x}$ (*ii*) $V = 146.3\,e^{-0.4118t}$ (*iii*) $y = 0.5580\,e^{1.0631\,x}$

3. (*i*) $y = 2.978\,x^{0.5143}$ (*ii*) $y = 0.5012\,x^{1.9977}$ (*iii*) $y = 7.173x^{1.952}$

4. (*i*) $y = 99.86\,(1.2)^x$ (*ii*) $y = 0.6823\,(1.384)^x$ **5.** $pv^{1.42} = 0.99$

6. (*i*) $a = 1.97$, $b = -0.298$ (*ii*) $y = 0.107798\,x^2 + 0.217125\,x$ **7.** $xy = 16.18x + 40.78$

8. $y = 2.39188\,x + \dfrac{2.98195}{x}$ **9.** $y = 0.509x^2 - \dfrac{2.04}{x}$

10. $y = 1.143x^2 - 0.971x + 1.286$ **11.** $y = 13.0065 + \dfrac{6.7512}{x} - \dfrac{4.4738}{x^2}$

12. $y = -1.1836\,x + \dfrac{7.5961}{\sqrt{x}}$ **13.** $a = 1.312810$, $b = 0.752575$

14. $a = 0.784976$, $b = -0.733298$

15. $\Sigma y = m\, a_0 + a_1\, \Sigma x + a_2\, \Sigma x^2 + a_3\, \Sigma x^3$

$\Sigma xy = a_0\, \Sigma x + a_1\, \Sigma x^2 + a_2\, \Sigma x^3 + a_3\, \Sigma x^4$

$\Sigma x^2 y = a_0\, \Sigma x^2 + a_1\, \Sigma x^3 + a_2\, \Sigma x^4 + a_3\, \Sigma x^5$

$\Sigma x^3 y = a_0\, \Sigma x^3 + a_1\, \Sigma x^4 + a_2\, \Sigma x^5 + a_3\, \Sigma x^6.$

16. $y = \dfrac{3.774}{x\,(x-2)}$; $y\,(5) = 0.2516$

3.31 CORRELATION

In a bivariate distribution, if the change in one variable affects a change in the other variable, the variables are said to be *correlated.*

If the two variables deviate in the same direction *i.e.,* if the increase (or decrease) in one results in a corresponding increase (or decrease) in the other, correlation is said to be *direct or positive.*

e.g., the correlation between income and expenditure is positive.

If the two variables deviate in opposite direction *i.e.,* if the increase (or decrease) in one results in a corresponding decrease (or increase) in the other, correlation is said to be *inverse or negative.*

e.g., the correlation between volume and the pressure of a perfect gas or the correlation between price and demand is negative.

Correlation is said to be *perfect* if the deviation in one variable is followed by a corresponding **proportional deviation** in the other.

3.32 REASONS RESPONSIBLE FOR THE EXISTENCE OF CORRELATION

1. Due to mere chance. The correlation between variables may be due to mere chance. Consider the data regarding six students selected at random from a college.

Students:	A	B	C	D	E	F
% of marks obtained in: *previous exam.*	43%	47%	60%	80%	55%	40%
Height (in inches):	60	62	65	70	64	59

Here the variables are moving in the same direction and a high degree of correlation is expected between the variables. We cannot expect this degree of correlation to hold good for any other sample drawn from the concerned population. In this case, the correlation has occurred just due to chance.

2. Due to the effect of some common cause. The correlation between variables may be due to the effect of some common cause. For example, positive correlation between the number of girls seeking admission in colleges A and B of a city may be due to the effect of increasing interest of girls towards higher education.

3. Due to the presence of cause-effect relationship between variables. For example, a high degree correlation between 'temperature' and 'sale of coffee' is due to the fact that people like taking coffee in winter season.

4. Due to the presence of interdependent relationship between the variables. For example, the presence of correlation between amount spent on entertainment of family and total expenditure of family is due to fact that both variables affect each other.

3.33 SCATTER OR DOT DIAGRAMS

It is the simplest method of the diagrammatic representation of bivariate data. Let (x_i, y_i), $i = 1, 2, 3, \ldots\ldots, n$ be a bivariate distribution. Let the values of the variables x and y be plotted along the x-axis and y-axis on a suitable scale. Then corresponding to every ordered pair, there corresponds a point or dot in the xy-plane. The diagram of dots so obtained is called a *dot or scatter diagram.*

If the dots are very close to each other and the number of observations is not very large, a fairly good correlation is expected. If the dots are widely scattered, a poor correlation is expected.

3.34 KARL PEARSON'S CO-EFFICIENT OF CORRELATION (OR PRODUCT MOMENT CORRELATION CO-EFFICIENT)

[*U.P.T.U. (C.O.) 2009; U.P.T.U. 2006, 2007, 2015*]

Correlation co-efficient between two variables x and y, usually denoted by $r(x, y)$ or r_{xy} is a numerical measure of linear relationship between them and is defined as

$$r_{xy} = \frac{\Sigma(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\Sigma(x_i - \overline{x})^2 \, \Sigma(y_i - \overline{y})^2}} = \frac{\frac{1}{n}\Sigma(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\frac{1}{n}\Sigma(x_i - \overline{x})^2 \cdot \frac{1}{n}\Sigma(y_i - \overline{y})^2}} = \frac{\frac{1}{n}\Sigma(x_i - \overline{x})(y_i - \overline{y})}{\sigma_x \sigma_y}.$$

$$\therefore \qquad \boxed{r_{xy} = \frac{\Sigma(x - \overline{x})(y - \overline{y})}{n\sigma_x\sigma_y}}$$

Alternate form of r(x, y):

$$\boxed{r(x, y) = \frac{n\,\Sigma xy - \Sigma x \Sigma y}{\sqrt{n\,\Sigma x^2 - (\Sigma x)^2}\,\sqrt{n\,\Sigma y^2 - (\Sigma y)^2}}}$$

Here n is the no. of pairs of values of x and y.

Note. Correlation co-efficient is independent of change of origin and scale.

Let us define two new variables u and v as

$$u = \frac{x - a}{h}, \; v = \frac{y - b}{k} \text{ where } a, b, h, k \text{ are constants, then } r_{xy} = r_{uv}.$$

Then, $$\boxed{r(u, v) = \frac{n\,\Sigma uv - \Sigma u \Sigma v}{\sqrt{n\,\Sigma u^2 - (\Sigma u)^2}\,\sqrt{n\,\Sigma v^2 - (\Sigma v)^2}}}.$$

EXAMPLES

Example 1. *Find the coefficient of correlation between the values of x and y:*

[U.P.T.U. (C.O.) 2008]

x	1	3	5	7	8	10
y	8	12	15	17	18	20

Sol. Here, $n = 6$. The table is as follows:

x	y	x^2	y^2	xy
1	8	1	64	8
3	12	9	144	36
5	15	25	225	75
7	17	49	289	119
8	18	64	324	144
10	20	100	400	200
$\Sigma x = 34$	$\Sigma y = 90$	$\Sigma x^2 = 248$	$\Sigma y^2 = 1446$	$\Sigma xy = 582$

Karl Pearson's coefficient of correlation is given by

$$r(x, y) = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{\sqrt{n\,\Sigma x^2 - (\Sigma x)^2}\,\sqrt{n\,\Sigma y^2 - (\Sigma y)^2}}$$

$$= \frac{(6 \times 582) - (34 \times 90)}{\sqrt{(6 \times 248) - (34)^2}\,\sqrt{(6 \times 1446) - (90)^2}} = 0.9879$$

Example 2. *The following data regarding the heights (y) and weights (x) of 100 college students are given:*

$\Sigma x = 15000$, $\Sigma x^2 = 2272500$, $\Sigma y = 6800$, $\Sigma y^2 = 463025$ and $\Sigma xy = 1022250$.

Find the correlation coefficient between height and weight.

Sol. Here, $n = 100$

Correlation co-efficient $r(x, y)$ is given by

$$r = \frac{n\Sigma xy - \Sigma x \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\,\sqrt{n\,\Sigma y^2 - (\Sigma y)^2}}$$

$$= \frac{(100 \times 1022250) - (15000 \times 6800)}{\sqrt{(100 \times 2272500) - (15000)^2}\,\sqrt{(100 \times 463025) - (6800)^2}} = 0.6.$$

Example 3. *Find the co-efficient of correlation for the following table:* (U.K.T.U. 2011)

x:	*10*	*14*	*18*	*22*	*26*	*30*
y:	*18*	*12*	*24*	*6*	*30*	*36*

Sol. Let $u = \dfrac{x - 22}{4}$, $v = \dfrac{y - 24}{6}$

x	y	u	v	u^2	v^2	uv
10	18	– 3	– 1	9	1	3
14	12	– 2	– 2	4	4	4
18	24	– 1	0	1	0	0
22	6	0	– 3	0	9	0
26	30	1	1	1	1	1
30	36	2	2	4	4	4
Total		$\Sigma u = -3$	$\Sigma v = -3$	$\Sigma u^2 = 19$	$\Sigma v^2 = 19$	$\Sigma uv = 12$

Here, $n = 6$, $\bar{u} = \frac{1}{n}\Sigma u = \frac{1}{6}(-3) = -\frac{1}{2}$; $\bar{v} = \frac{1}{n}\Sigma v = \frac{1}{6}(-3) = -\frac{1}{2}$

$$r_{uv} = \frac{n\,\Sigma uv - \Sigma u \Sigma v}{\sqrt{n\,\Sigma u^2 - (\Sigma u)^2}\,\sqrt{n\,\Sigma v^2 - (\Sigma v)^2}}$$

$$= \frac{(6 \times 12) - (-3)(-3)}{\sqrt{(6 \times 19) - (-3)^2}\,\sqrt{(6 \times 19) - (-3)^2}} = \frac{63}{\sqrt{105}\,\sqrt{105}} = 0.6$$

Hence, $r_{xy} = r_{uv} = 0.6$.

Example 4. *Ten students got the following percentage of marks in Principles of Economics and Statistics:*

Roll Nos.:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Marks in Economics:	*78*	*36*	*98*	*25*	*75*	*82*	*90*	*62*	*65*	*39*
Marks in Statistics:	*84*	*51*	*91*	*60*	*68*	*62*	*86*	*58*	*53*	*47*

Calculate the co-efficient of correlation.

Sol. Let the marks in the two subjects be denoted by x and y respectively.

x	y	$u = x - 65$	$v = y - 66$	u^2	v^2	uv
78	84	13	18	169	324	234
36	51	– 29	– 15	841	225	435
98	91	33	25	1089	625	825
25	60	– 40	– 6	1600	36	240
75	68	10	2	100	4	20
82	62	17	– 4	289	16	– 68
90	86	25	20	625	400	500
62	58	– 3	– 8	9	64	24
65	53	0	– 13	0	169	0
39	47	– 26	– 19	676	361	494
Total		$\Sigma u = 0$	$\Sigma v = 0$	$\Sigma u^2 = 5398$	$\Sigma v^2 = 2224$	$\Sigma uv = 2734$

Here, $n = 10$, $\bar{u} = \frac{1}{n}\Sigma u_i = 0$, $\bar{v} = \frac{1}{n}\Sigma v_i = 0$

$$r_{uv} = \frac{n\Sigma uv - \Sigma u \Sigma v}{\sqrt{n\,\Sigma u^2 - (\Sigma u)^2}\,\sqrt{n\,\Sigma v^2 - (\Sigma v)^2}}$$

$$= \frac{(10 \times 2734) - (0 \times 0)}{\sqrt{(10 \times 5398) - (0)^2}\,\sqrt{(10 \times 2224) - (0)^2}} = 0.789$$

Hence, $r_{xy} = r_{uv} = 0.789$.

Example 5. *A computer while calculating correlation co-efficient between two variables X and Y from 25 pairs of observations obtained the following results :*

$n = 25$, $\Sigma X = 125$, $\Sigma X^2 = 650$,

$\Sigma Y = 100$, $\Sigma Y^2 = 460$, $\Sigma XY = 508$.

It was, however, later discovered at the time of checking that he had copied down two pairs as

X	*Y*
6	*14*
8	*6*

while the correct values were

X	*Y*
8	*12*
6	*8*

Obtain the correct value of correlation co-efficient.

$$\left.\begin{aligned}
&\textbf{Sol. } \text{Corrected } \Sigma X = 125 - 6 - 8 + 8 + 6 = 125 \\
&\text{Corrected } \Sigma Y = 100 - 14 - 6 + 12 + 8 = 100 \\
&\text{Corrected } \Sigma X^2 = 650 - 6^2 - 8^2 + 8^2 + 6^2 = 650 \\
&\text{Corrected } \Sigma Y^2 = 460 - 14^2 - 6^2 + 12^2 + 8^2 = 436 \\
&\text{Corrected } \Sigma XY = 508 - 6 \times 14 - 8 \times 6 + 8 \times 12 + 6 \times 8 = 520
\end{aligned}\right\}$$

(Subtract the incorrect values and add the corresponding correct values)

$$\overline{X} = \frac{1}{n}\Sigma X = \frac{1}{25} \times 125 = 5\ ; \quad \overline{Y} = \frac{1}{n}\Sigma Y = \frac{1}{25} \times 100 = 4$$

Corrected $$r_{xy} = \frac{n\,\Sigma XY - \Sigma X \Sigma Y}{\sqrt{n\,\Sigma X^2 - (\Sigma X)^2}\ \sqrt{n\,\Sigma Y^2 - (\Sigma Y)^2}}$$

$$= \frac{(25 \times 520) - (125 \times 100)}{\sqrt{(25 \times 650) - (125)^2}\ \sqrt{(25 \times 436) - (100)^2}} = 0.67.$$

Example 6. *If $z = ax + by$ and r is the correlation co-efficient between x and y, show that*

$$\sigma_z^{\,2} = a^2\sigma_x^{\,2} + b^2\sigma_y^{\,2} + 2abr\ \sigma_x\ \sigma_y.$$

Sol. $$z = ax + by$$

$\Rightarrow$ $$\bar{z} = a\bar{x} + b\bar{y}, \qquad z_i = ax_i + by_i$$

$$z_i - \bar{z} = a(x_i - \bar{x}) + b(y_i - \bar{y})$$

Now, $$\sigma_z^{\,2} = \frac{1}{n}\Sigma(z_i - \bar{z})^2 = \frac{1}{n}\ \Sigma[a(x_i - \bar{x}) + b(y_i - \bar{y})]^2$$

$$= \frac{1}{n}\ \Sigma[a^2(x_i - \bar{x})^2 + b^2(y_i - \bar{y})^2 + 2ab(x_i - \bar{x})(y_i - \bar{y})]$$

$$= a^2 \cdot \frac{1}{n}\Sigma(x_i - \bar{x})^2 + b^2 \cdot \frac{1}{n}\Sigma(y_i - \bar{y})^2 + 2ab \cdot \frac{1}{n}\Sigma(x_i - \bar{x})(y_i - \bar{y})$$

$$= a^2\sigma_x^{\,2} + b^2\sigma_y^{\,2} + 2abr\ \sigma_x\sigma_y \qquad \left|\ \because\ r = \frac{\frac{1}{n}\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sigma_x\sigma_y}\right.$$

Example 7. *Establish the formula:* $\sigma_{x-y}^2 = \sigma_x^{\,2} + \sigma_y^{\,2} - 2r_{xy}\sigma_x\sigma_y$

where r_{xy} is the correlation coefficient between x and y. Using the above formula, calculate the correlation coefficient from the following data relating to the marks of 10 candidates in aptitude test (x) and Achievement rating (y).

Marks

Aptitude (x):	22	53	46	67	43	35	88	11	95	13
Achievement (y):	18	39	31	42	55	64	82	10	96	14

Sol. Let $$z = x - y$$

$\therefore$ $$\bar{z} = \bar{x} - \bar{y}$$

$\therefore$ $$z - \bar{z} = (x - \bar{x}) - (y - \bar{y})$$

or, $$(z - \bar{z})^2 = (x - \bar{x})^2 + (y - \bar{y})^2 - 2(x - \bar{x})(y - \bar{y})$$

Summing up for n terms,

$$\Sigma(z - \bar{z})^2 = \Sigma(x - \bar{x})^2 + \Sigma(y - \bar{y})^2 - 2\Sigma(x - \bar{x})(y - \bar{y})$$

or,
$$\frac{\Sigma(z-\bar{z})^2}{n} = \frac{\Sigma(x-\bar{x})^2}{n} + \frac{\Sigma(y-\bar{y})^2}{n} - \frac{2\Sigma(x-\bar{x})(y-\bar{y})}{n}$$

$\Rightarrow$
$$\sigma_z^{\ 2} = \sigma_x^{\ 2} + \sigma_y^{\ 2} - 2r\sigma_x\sigma_y, \qquad \text{where } r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}$$

$\Rightarrow$
$$\boxed{\sigma_{x-y}^2 = \sigma_x^{\ 2} + \sigma_y^{\ 2} - 2r\sigma_x\sigma_y} \qquad ...(1)$$

Now, $n = 10$

$$\bar{x} = \frac{22+53+46+67+43+35+88+11+95+13}{10} = \frac{473}{10} = 47.3$$

$$\bar{y} = \frac{18+39+31+42+55+64+82+10+96+14}{10} = \frac{451}{10} = 45.1$$

Now we form the table as

x	y	$z = x - y$	$x - \bar{x}$	$y - \bar{y}$	$z - \bar{z}$	$(x - \bar{x})^2$	$(y - \bar{y})^2$	$(z - \bar{z})^2$
22	18	4	– 25.3	– 27.1	1.8	640.09	734.41	3.24
53	39	14	5.7	– 6.1	11.8	32.49	37.21	139.24
46	31	15	– 1.3	– 14.1	12.8	1.69	198.81	163.84
67	42	25	19.7	– 3.1	22.8	38.09	9.61	519.84
43	55	– 12	– 4.3	9.9	– 14.2	18.49	98.01	201.64
35	64	– 29	– 12.3	18.9	– 31.2	151.29	357.21	973.44
88	82	6	40.7	36.9	3.8	1656.49	1361.61	14.44
11	10	1	– 36.3	– 35.1	– 1.2	1317.69	1232.01	1.44
95	96	– 1	47.7	50.9	– 3.2	2275.29	2590.81	10.24
13	14	– 1	– 34.3	– 31.1	– 3.2	1176.49	967.21	10.24
						$\Sigma(x - \bar{x})^2 = 7658.1$	$\Sigma(y - \bar{y})^2 = 7586.9$	$\Sigma(z - \bar{z})^2 = 2037.6$

where,
$$\bar{z} = \frac{\Sigma z}{n} = \frac{22}{10} = 2.2$$

Now,
$$\sigma_x^{\ 2} = \frac{\Sigma(x-\bar{x})^2}{n} = \frac{7658.1}{10} = 765.81, \quad \sigma_y^{\ 2} = \frac{\Sigma(y-\bar{y})^2}{n} = \frac{7586.9}{10} = 758.69$$

$$\sigma_{x-y}^2 = \sigma_z^{\ 2} = \frac{\Sigma(z-\bar{z})^2}{n} = \frac{2037.6}{10} = 203.76.$$

Substituting the values in the formula (1),

$$\sigma_{x-y}^2 = \sigma_x^{\ 2} + \sigma_y^{\ 2} - 2r\,\sigma_x\sigma_y$$

$\Rightarrow$
$$203.76 = 765.81 + 758.69 - 2r\,(27.67)(27.54)$$

$$r = \frac{1524.5 - 203.76}{1524.06} = 0.866.$$

Example 8. *(i) Calculate coefficient of correlation from the following results:*

$n = 10,\ \ \Sigma X = 100,\ \ \Sigma Y = 150,\ \ \Sigma(X - 10)^2 = 180,\ \ \Sigma(Y - 15)^2 = 215,\ \ \Sigma(X - 10)(Y - 15) = 60.$

(ii) Calculate Karl Pearson's coefficient of correlation between X and Y for the following information:

$n = 12,\ \ \Sigma X = 120, \Sigma Y = 130, \Sigma(X - 8)^2 = 150, \Sigma(Y - 10)^2 = 200$ *and* $\Sigma(X - 8)(Y - 10) = 50$

Sol. (*i*) Mean of first series, $\overline{X} = \dfrac{\Sigma X}{n} = \dfrac{100}{10} = 10$

Mean of second series, $\overline{Y} = \dfrac{\Sigma Y}{n} = \dfrac{150}{10} = 15$

Now,
$$r = \frac{\Sigma(X - \overline{X})(Y - \overline{Y})}{\sqrt{\Sigma(X - \overline{X})^2 \,.\, \Sigma(Y - \overline{Y})^2}}$$

$$= \frac{\Sigma(X - 10)(Y - 15)}{\sqrt{\Sigma(X - 10)^2 \,.\, \Sigma(Y - 15)^2}} = \frac{60}{\sqrt{180 \times 215}} = 0.305$$

(*ii*)
$$\Sigma x = \Sigma(X - 8) = \Sigma X - \Sigma 8 = 120 - (8 \times 12) = 24$$
$$\Sigma y = \Sigma(Y - 10) = \Sigma Y - \Sigma 10 = 130 - (10 \times 12) = 10$$
$$\Sigma xy = \Sigma(X - 8)(Y - 10) = 50 \quad \text{(given)}$$
$$\Sigma x^2 = \Sigma(X - 8)^2 = 150$$
$$\Sigma y^2 = \Sigma(Y - 10)^2 = 200$$

Now,
$$r = \frac{n\,\Sigma xy - \Sigma x \Sigma y}{\sqrt{n\,\Sigma x^2 - (\Sigma x)^2}\ \sqrt{n\,\Sigma y^2 - (\Sigma y)^2}} = \frac{(12 \times 50) - (24 \times 10)}{\sqrt{(12 \times 150) - (24)^2}\ \sqrt{(12 \times 200) - (10)^2}}$$

$$= \frac{360}{\sqrt{1224}\ \sqrt{2300}} = 0.2146.$$

3.35 CALCULATION OF CO-EFFICIENT OF CORRELATION FOR A BIVARIATE FREQUENCY DISTRIBUTION

If the bivariate data on x and y is presented on a two way correlation table and f is the frequency of a particular rectangle in the correlation table, then

$$\boxed{r_{xy} = \frac{\Sigma fxy - \dfrac{1}{n}\Sigma fx\, \Sigma fy}{\sqrt{\left[\Sigma fx^2 - \dfrac{1}{n}(\Sigma fx)^2\right]\left[\Sigma fy^2 - \dfrac{1}{n}(\Sigma fy)^2\right]}}}$$

Since change of origin and scale do not affect the co-efficient of correlation.

$\therefore$ $r_{xy} = r_{uv}$ where the new variables u, v are properly chosen.

Example 9. *The following table gives according to age the frequency of marks obtained by 100 students in an intelligence test:*

Marks \ Age (in years)	18	19	20	21	Total
10–20	4	2	2		8
20–30	5	4	6	4	19
30–40	6	8	10	11	35
40–50	4	4	6	8	22
50–60		2	4	4	10
60–70		2	3	1	6
Total	19	22	31	28	100

Calculate the co-efficient of correlation between age and intelligence.

Sol. Let age and intelligence be denoted by x and y respectively.

Mid value	y \ x	18	19	20	21	f	u	fu	fu^2	fuv
15	10–20	4	2	2		8	– 3	– 24	72	30
25	20–30	5	4	6	4	19	– 2	– 38	76	20
35	30–40	6	8	10	11	35	– 1	– 35	35	9
45	40–50	4	4	6	8	22	0	0	0	0
55	50–60		2	4	4	10	1	10	10	2
65	60–70		2	3	1	6	2	12	24	– 2
	f	19	22	31	28	100	Total	**– 75**	**217**	**59**
	v	– 2	– 1	0	1	Total				
	fv	– 38	– 22	0	28	**– 32**				
	fv^2	76	22	0	28	**126**				
	fuv	56	16	0	– 13	**59**				

Let us define two new variables u and v as $u = \dfrac{y-45}{10}$, $v = x - 20$

$$r_{xy} = r_{uv} = \frac{\Sigma fuv - \frac{1}{n}\Sigma fu\, \Sigma fv}{\sqrt{\left[\Sigma fu^2 - \frac{1}{n}(\Sigma fu)^2\right]\left[\Sigma fv^2 - \frac{1}{n}(\Sigma fv)^2\right]}}$$

$$= \frac{59 - \frac{1}{100}(-75)(-32)}{\sqrt{\left[217 - \frac{1}{100}(-75)^2\right]\left[126 - \frac{1}{100}(-32)^2\right]}} = \frac{59-24}{\sqrt{\frac{643}{4} \times \frac{2894}{25}}} = 0.25.$$

3.36 RANK CORRELATION

Sometimes we have to deal with problems in which data cannot be quantitatively measured but qualitative assessment is possible.

Let a group of n individuals be arranged in order of merit or proficiency in possession of two characteristics A and B. The ranks in the two characteristics are, in general, different. For example, if A stands for intelligence and B for beauty, it is not necessary that the most intelligent individual may be the most beautiful and *vica versa.* Thus an individual who is ranked at the top for the characteristic A *may be* ranked at the bottom for the characteristic B. Let (x_i, y_i), $i = 1, 2, \ldots\ldots n$ be the ranks of the n individuals in the group for the characteristics A and B respectively. Pearsonian co-efficient of correlation between the ranks x_i's and y_i's is called the *rank correlation co-efficient* between the characteristics A and B for that group of individuals.

Thus rank correlation co-efficient

$$r = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\Sigma(x_i - \bar{x})^2 \, \Sigma(y_i - \bar{y})^2}} = \frac{\frac{1}{n}\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sigma_x \sigma_y} \qquad \ldots(1)$$

Now x_i's and y_i's are merely the permutations of n numbers from 1 to n. *Assuming that no two individuals are bracketed or tied in either classification i.e., $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$, both x and y take all integral values from 1 to n.*

$$\therefore \qquad \bar{x} = \bar{y} = \frac{1}{n}(1 + 2 + 3 + \ldots\ldots + n) = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}$$

$$\Sigma x_i = 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2} = \Sigma y_i$$

$$\Sigma x_i^2 = 1^2 + 2^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = \Sigma y_i^2$$

If D_i denotes the difference in ranks of the i^{th} individual, then

$$D_i = x_i - y_i = (x_i - \bar{x}) - (y_i - \bar{y}) \qquad [\because \ \bar{x} = \bar{y}]$$

$$\frac{1}{n}\Sigma D_i^2 = \frac{1}{n}\Sigma[(x_i - \bar{x}) - (y_i - \bar{y})]^2$$

$$= \frac{1}{n}\Sigma(x_i - \bar{x})^2 + \frac{1}{n}\Sigma(y_i - \bar{y})^2 - 2 \cdot \frac{1}{n}\Sigma(x_i - \bar{x})(y_i - \bar{y})$$

$$= \sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y \qquad \ldots(2) \quad | \text{ using (1)}$$

But $$\sigma_x^2 = \frac{1}{n}\Sigma x_i^2 - \bar{x}^2 = \frac{1}{n}\Sigma y_i^2 - \bar{y}^2 = \sigma_y^2$$

$\therefore$ From (2), $$\frac{1}{n}\Sigma D_i^2 = 2\sigma_x^2 - 2r\sigma_x^2 = 2(1-r)\,\sigma_x^2 = 2(1-r)\left[\frac{1}{n}\Sigma x_i^2 - \bar{x}^2\right]$$

$$= 2(1-r)\left[\frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}\right]$$

$$= (1-r)(n+1)\left[\frac{4n + 2 - 3n - 3}{6}\right] = \frac{(1-r)(n^2-1)}{6} \quad \text{or} \quad 1 - r = \frac{6\Sigma D_i^2}{n(n^2-1)}$$

Hence, $$\boxed{r = 1 - \left[\frac{6\Sigma D_i^2}{n(n^2-1)}\right]}$$

Note. This is called *Spearman's Formula for* Rank Correlation.

$$\Sigma d_i = \Sigma(x_i - y_i) = \Sigma x_i - \Sigma y_i = 0 \text{ always.}$$

This serves as a check on calculations.

EXAMPLES

Example 1. *Compute the rank correlation coefficient for the following data:*

Person:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Rank in Maths:	*9*	*10*	*6*	*5*	*7*	*2*	*4*	*8*	*1*	*3*
Rank in Physics:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*

Sol. Here the ranks are given and $n = 10$.

Person	R_1	R_2	$D = R_1 - R_2$	D^2
A	9	1	8	64
B	10	2	8	64
C	6	3	3	9
D	5	4	1	1
E	7	5	2	4
F	2	6	– 4	16
G	4	7	– 3	9
H	8	8	0	0
I	1	9	– 8	64
J	3	10	– 7	49
				$\Sigma D^2 = 280$

$$\therefore \qquad r = 1 - \left\{\frac{6\Sigma D^2}{n(n^2-1)}\right\} = 1 - \left\{\frac{6 \times 280}{10(100-1)}\right\} = 1 - 1.697 = -0.697.$$

Example 2. *The marks secured by recruits in the selection test (X) and in the proficiency test (Y) are given below:*

Serial No.:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
X:	*10*	*15*	*12*	*17*	*13*	*16*	*24*	*14*	*22*
Y:	*30*	*42*	*45*	*46*	*33*	*34*	*40*	*35*	*39*

Calculate the rank correlation co-efficient.

Sol. Here the marks are given. Therefore, first of all, write down ranks. In each series, the item with the largest size is ranked 1, next largest 2 and so on. Here $n = 9$.

X	10	15	12	17	13	16	24	14	22	Total
Y	30	42	45	46	33	34	40	35	39	
Ranks in X (x)	9	5	8	3	7	4	1	6	2	
Ranks in Y (y)	9	3	2	1	8	7	4	6	5	
$D = x - y$	0	2	6	2	– 1	– 3	– 3	0	– 3	0
D^2	0	4	36	4	1	9	9	0	9	72

$$\therefore \qquad r = 1 - \left\{\frac{6\Sigma D^2}{n(n^2-1)}\right\} = 1 - \left\{\frac{6 \times 72}{9 \times 80}\right\} = 1 - .6 = 0.4$$

Example 3. *Rank correlation co-efficient of marks obtained by 10 students in Mathematics and English was found to be 0.5. It was later discovered that the difference in ranks in two subjects obtained by one of the students was wrongly taken as 3 instead of 7. Find the correct rank correlation co-efficient.*

Sol. Incorrect $r_k = 0.5$, $n = 10$

$$\therefore \quad \text{Incorrect } \Sigma D^2 = \frac{n(n^2-1)(1-r_k)}{6} = \frac{10 \times 99 \times 0.5}{6} = 82.5$$

Now, correct ΣD^2 = Incorrect $\Sigma D^2 - (3)^2 + (7)^2 = 82.5 - 9 + 49 = 122.5$

$$\therefore \qquad \text{Correct } r_k = 1 - \left\{\frac{6\Sigma D^2}{n(n^2-1)}\right\} = 1 - \left\{\frac{6 \times 122.5}{10(100-1)}\right\} = 0.2575.$$

Example 4. *Ten competitors in a beauty contest were ranked by three judges in the following orders:*

First Judge:	*1*	*6*	*5*	*10*	*3*	*2*	*4*	*9*	*7*	*8*
Second Judge:	*3*	*5*	*8*	*4*	*7*	*10*	*2*	*1*	*6*	*9*
Third Judge:	*6*	*4*	*9*	*8*	*1*	*2*	*3*	*10*	*5*	*7*

Use the method of rank correlation to determine which pair of judges has the nearest approach to common taste in beauty?

Sol. Let R_1, R_2, R_3 be the ranks given by three judges.

Calculation of rank correlation coefficient

Competitor	R_1	R_2	R_3	$D_{12} = R_1 - R_2$	$D_{13} = R_1 - R_3$	$D_{23} = R_2 - R_3$	D_{12}^2	D_{13}^2	D_{23}^2
A	1	3	6	– 2	– 5	– 3	4	25	9
B	6	5	4	1	2	1	1	4	1
C	5	8	9	– 3	– 4	– 1	9	16	1
D	10	4	8	6	2	– 4	36	4	16
E	3	7	1	– 4	2	6	16	4	36
F	2	10	2	– 8	0	8	64	0	64
G	4	2	3	2	1	– 1	4	1	1
H	9	1	10	8	– 1	– 9	64	1	81
I	7	6	5	1	2	1	1	4	1
J	8	9	7	– 1	1	2	1	1	4
$n = 10$							$\Sigma D_{12}^2 = 200$	$\Sigma D_{13}^2 = 60$	$\Sigma D_{23}^2 = 214$

Rank correlation coefficient between first and second judges,

$$r_{k_{12}} = 1 - \left\{\frac{6\Sigma D_{12}^2}{n(n^2-1)}\right\} = 1 - \left\{\frac{6 \times 200}{10\,(99)}\right\} = -\,0.212$$

Rank correlation coefficient between first and third judges,

$$r_{k_{13}} = 1 - \left\{\frac{6\Sigma D_{13}^{\ 2}}{n(n^2-1)}\right\} = 1 - \left\{\frac{6 \times 60}{10 \times 99}\right\} = 0.636$$

Rank correlation coefficient between second and third judges,

$$r_{k_{23}} = 1 - \left\{\frac{6\Sigma D_{23}^{\ 2}}{n(n^2-1)}\right\} = 1 - \left\{\frac{6 \times 214}{10 \times 99}\right\} = -0.297$$

Correlation between first and second judges is negative *i.e.,* their opinions regarding beauty test are opposite to each other. Similarly, opinions of second and third judges are opposite to each other, but the opinions of first and third judges are of similar type as their correlation is positive. It means that their likings and dislikings are very much common.

3.37 TIED RANKS

If any two or more individuals have same rank or the same value in the series of marks, then the above formula fails and requires an adjustment. In such cases, each individual is given an average rank. This common average rank is the average of the ranks which these individuals would have assumed if they were slightly different from each other. Thus, if two individual are ranked equal at the sixth place, they would have assumed the 6th and 7th ranks if they were ranked slightly different. Their common rank = $\frac{6+7}{2}$ = 6.5. If three individuals are ranked equal at fourth place, they would have assumed the 4th, 5th and 6th ranks if they were ranked slightly different. Their common rank = $\frac{4+5+6}{3}$ = 5.

Adjustment. Add $\frac{1}{12}$ $m(m^2 - 1)$ to ΣD^2 where m stands for the number of times an item is repeated.

This adjustment factor is to be added for each repeated item.

Thus
$$r = 1 - \frac{6\left\{\Sigma D^2 + \frac{1}{12} m(m^2-1) + \frac{1}{12} m(m^2-1) + \ldots\right\}}{n(n^2-1)}.$$

Example 5. *Obtain the rank correlation co-efficient for the following data:*

X:	*68*	*64*	*75*	*50*	*64*	*80*	*75*	*40*	*55*	*64*
Y:	*62*	*58*	*68*	*45*	*81*	*60*	*68*	*48*	*50*	*70.*

Sol. Here, marks are given, so write down the ranks.

X	68	64	75	50	64	80	75	40	55	64	Total
Y	62	58	68	45	81	60	68	48	50	70	
Ranks in X (x)	4	6	2.5	9	6	1	2.5	10	8	6	
Ranks in Y (y)	5	7	3.5	10	1	6	3.5	9	8	2	
$D = x - y$	– 1	– 1	– 1	– 1	5	– 5	– 1	1	0	4	0
D^2	1	1	1	1	25	25	1	1	0	16	72

In the X-series, the value 75 occurs twice. Had these values been slightly different, they would have been given the ranks 2 and 3. Therefore, the common rank given to them is $\frac{2+3}{2}$ = 2.5. The value 64 occurs thrice. Had these values been slightly different, they would have been given the ranks 5, 6, and 7. Therefore the common rank given to them is $\frac{5+6+7}{3} = 6$. Similarly, in the Y-series, the value 68 occurs twice. Had these values been slightly different, they would have been given the ranks 3 and 4? Therefore, the common rank given to them is $\frac{3+4}{2} = 3.5$.

Thus, m has the values 2, 3, 2.

$$\therefore \qquad r = 1 - \frac{6\left\{\Sigma D^2 + \frac{1}{12}m_1(m_1{}^2 - 1) + \frac{1}{12}m_2(m_2{}^2 - 1) + \frac{1}{12}m_3(m_3{}^2 - 1)\right\}}{n(n^2 - 1)}$$

$$= 1 - \frac{6\left[72 + \frac{1}{12}\,.\,2(2^2 - 1) + \frac{1}{12}\,.\,3(3^2 - 1) + \frac{1}{12}\,.\,2(2^2 - 1)\right]}{10(10^2 - 1)}$$

$$= 1 - \left\{\frac{6 \times 75}{990}\right\} = \frac{6}{11} = 0.545.$$

ASSIGNMENT

1. Calculate the coefficient of correlation for the following data: *(U.P.T.U. 2006)*

Husband's age (in yrs.) x	23	27	28	28	29	30	31	33	35	36
Wife's age (in yrs.) y	18	20	22	27	21	29	27	29	28	29

2. Calculate the coefficient of correlation for the following data:

Height of father (in inches)	65	66	67	67	68	69	70	72
Height of son (in inches)	67	68	65	68	72	72	69	71

3. Define Karl Pearson's coefficient of correlation. How would you interpret the sign and magnitude of a correlation coefficient?

4. Calculate the coefficient of correlation between the marks obtained by 8 students in Mathematics and Statistics:

Students	A	B	C	D	E	F	G	H
Mathematics	25	30	32	35	37	40	42	45
Statistics	08	10	15	17	20	23	24	25

[*U.P.T.U. (C.O.) 2009*]

5. Find the correlation coefficient between x and y for the following data:

x	60	34	40	50	45	41	22	43
y	75	32	34	40	45	33	12	30

[*U.K.T.U. 2010*]

6. The marks of the same 15 students in two subjects A and B are analysed. The two numbers within the brackets denote the ranks of the same student in A and B respectively:

(1, 10) (2, 7) (3, 2) (4, 6) (5, 4) (6, 8) (7, 3) (8, 1) (9, 11) (10, 15) (11, 9) (12, 5) (13, 14) (14, 12) (15, 13)

Use Spearman's formula to find the rank correlation coefficient.

7. Ten students got the following percentage of marks in Chemistry and Physics:

Students:	1	2	3	4	5	6	7	8	9	10
Marks in Chemistry:	78	36	98	25	75	82	90	62	65	39
Marks in Physics:	84	51	91	60	68	62	86	58	63	47

Calculate the rank correlation co-efficient.

8. A firm not sure of the response to its product in ten different colour shades decides to produce them in those colour shades, if the ranking of these colour shades by two typical consumer judges is highly correlated.

The two judges rank the ten colours in the following order:

Colour :	*Red*	*Green*	*Blue*	*Yellow*	*White*	*Black*	*Pink*	*Purple*	*Orange*	*Ivory*
Ranking by I Judge :	6	4	3	1	2	7	9	8	10	5
Ranking by II Judge :	4	1	6	7	5	8	10	9	3	2

Is there any agreement between the two judges, to allow the introduction of the product by the firm in the market?

9. Two judges in a music competition rank the 12 entries as follows:

x	1	2	3	4	5	6	7	8	9	10	11	12
y	12	9	6	10	3	5	4	7	8	2	11	1

What degree of agreement is there between the judgement of the two judges?

10. Calculate the coefficient of correlation between the following ages of husband (x) and wife (y) by taking 30 and 28 as assumed mean incase of x and y respectively:

x :	24	27	28	28	29	30	32	33	35	35	40
y :	18	20	22	25	22	28	28	30	27	30	32

Answers

1. 0.82 **2.** 0.603 **4.** 0.9804 **5.** 0.9158
6. 0.51 **7.** 0.84 **8.** 0.22, no **9.** – 0.454
10. 0.8926.

3.38 REGRESSION ANALYSIS (*U.P.T.U. 2015*)

The term 'regression' was first used by Sir Francis Galton (1822–1911), a British Biometrician in connection with the height of parents and their offsprings. He found that the offspring of tall or short parents tend to regress to the average height. In other words, though tall fathers do tend to have tall sons yet the average height of tall fathers is more than the average height of their sons and the average height of short fathers is less than the average height of their sons.

The term 'regression' stands for some sort of functional relationship between two or more related variables. The only fundamental difference, if any, between problems of curve-fitting and regression is that in regression, any of the variables may be considered as independent or dependent while in curve-fitting, one variable cannot be dependent.

Regression measures the nature and extent of correlation. Regression is the estimation or prediction of unknown values of one variable from known values of another variable.

3.39 CURVE OF REGRESSION AND REGRESSION EQUATION

If two variates x and y are correlated *i.e.,* there exists an association or relationship between them, then the scatter diagram will be more or less concentrated round a curve. This curve is called the *curve of regression* and the relationship is said to be expressed by means of *curvilinear regression.*

The mathematical equation of the regression curve is called regression equation.

3.40 LINEAR REGRESSION

When the points of the scatter diagram concentrate round a straight line, the regression is called linear and this straight line is known as the line of regression.

Regression will be called non-linear if there exists a relationship other than a straight line between the variables under consideration.

3.41 LINES OF REGRESSION (*U.P.T.U. 2006, 2007*)

A line of regression is the straight line which gives the best fit in the least square sense to the given frequency.

In case of n pairs (x_i, y_i) ; $i = 1, 2, ..., n$ from a bivariate data, we have no reason or justification to assume y as dependent variable and x as independent variable. Either of the two may be estimated for the given values of the other. Thus if we wish to estimate y for given values of x, we shall have the regression equation of the form $y = a + bx$, called the regression line of y on x. If we wish to estimate x for given values of y, we shall have the regression line of the form $x = \text{A} + \text{B}y$, called the regression line of x on y.

Thus it implies, in general, *we always have two lines of regression*.

If the line of regression is so chosen that the sum of squares of deviation parallel to the axis of y is minimised [*See Fig. (a)*], it is called *the line of regression of y on x* and it gives *the best estimate of y for any given value of x.*

If the line of regression is so chosen that the sum of squares of deviations parallel to the axis of x is minimised [*See Fig. (b)*], it is called *the line of regression of x on y* and it gives *the best estimate of x for any given value of y.*

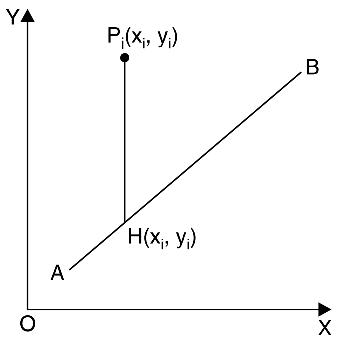


Fig. (a)

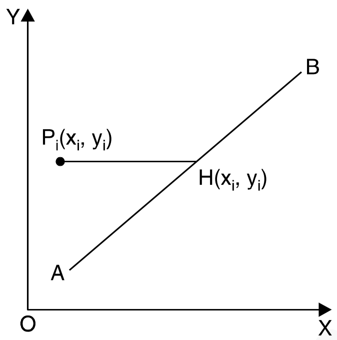


Fig. (b)

The independent variable is called *predictor* or regresser or explanator and the dependent variable is called the *predictant* or regressed or explained variable.

3.42 DERIVATION OF LINES OF REGRESSION

3.42.1. Line of Regression of y on x

To obtain the line of regression of y on x, we shall assume y as dependent variable and x as independent variable. Let $y = a + bx$ be the equation of regression line of y on x.

The residual for i^{th} point is $E_i = y_i - a - bx_i$.

Introduce a new quantity U such that

$$U = \sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2 \qquad \text{...(1)}$$

According to the *principle of Least squares*, the constants a and b are chosen in such a way that the sum of the squares of residuals is minimum.

Now, the condition for U to be maximum or minimum is

$$\frac{\partial U}{\partial a} = 0 \quad \text{and} \quad \frac{\partial U}{\partial b} = 0$$

From (1),
$$\frac{\partial U}{\partial a} = 2\sum_{i=1}^{n} (y_i - a - bx_i)(-1)$$

$$\frac{\partial U}{\partial a} = 0 \text{ gives } \quad 2\sum_{i=1}^{n} (y_i - a - bx_i)(-1) = 0$$

$$\Rightarrow \qquad \boxed{\Sigma y = na + b\,\Sigma x} \qquad \text{..(2)}$$

Also,
$$\frac{\partial U}{\partial b} = 2\sum_{i=1}^{n} (y_i - a - bx_i)(-x_i)$$

$$\frac{\partial U}{\partial b} = 0 \text{ gives } \quad 2\sum_{i=1}^{n} (y_i - a - bx_i)(-x_i) = 0$$

$$\Rightarrow \qquad \boxed{\Sigma xy = a\,\Sigma x + b\,\Sigma x^2} \qquad \text{...(3)}$$

Equations (2) and (3) are called *normal equations*.

Solving (2) and (3) for 'a' and 'b', we get

$$b = \frac{\Sigma xy - \frac{1}{n}\Sigma x\,\Sigma y}{\Sigma x^2 - \frac{1}{n}(\Sigma x)^2} = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma x^2 - (\Sigma x)^2} \quad \text{...(4)}$$

and
$$a = \frac{\Sigma y}{n} - b\frac{\Sigma x}{n} = \bar{y} - b\bar{x} \quad \text{...(5)}$$

Eqn. (5) gives $\bar{y} = a + b\bar{x}$

Hence $y = a + bx$ line passes through point $(\bar{x}, \bar{y})$.

Putting $a = \bar{y} - b\bar{x}$ in equation of line $y = a + bx$, we get

$$\boxed{y - \bar{y} = b(x - \bar{x})} \quad \text{...(6)}$$

Equation (6) is called regression line of y on x. 'b' is called the regression coefficient of y on x and is usually denoted by b_{yx}.

Hence eqn. (6) can be rewritten as

$$\boxed{y - \bar{y} = b_{yx}(x - \bar{x})}$$

where $\bar{x}$ and $\bar{y}$ are mean values while

$$b_{yx} = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma x^2 - (\Sigma x)^2}$$

In equation (3), shifting the origin to $(\bar{x}, \bar{y})$, we get

$$\Sigma(x - \bar{x})(y - \bar{y}) = a\,\Sigma(x - \bar{x}) + b\,\Sigma(x - \bar{x})^2$$

$$\Rightarrow \quad nr\,\sigma_x\sigma_y = a(0) + bn\sigma_x^{\,2}$$

$$\Rightarrow \quad b = r\frac{\sigma_y}{\sigma_x}$$

$\because \; \Sigma(x - \bar{x}) = 0$, $\frac{1}{n}\Sigma(x - \bar{x})^2 = \sigma_x^{\,2}$ and $\frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y} = r$

Hence, regression coefficient b_{yx} can also be defined as

$$\boxed{b_{yx} = r\frac{\sigma_y}{\sigma_x}}$$

where r is the coefficient of correlation, σ_x and σ_y are the standard deviations of x and y series respectively.

3.42.2. Line of Regression of x on y

Proceeding in the same way as 3.42.1, we can derive the regression line of x on y as

$$\boxed{x - \bar{x} = b_{xy}(y - \bar{y})}$$

where b_{xy} is the regression coefficient of x on y and is given by

$$\boxed{b_{xy} = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma y^2 - (\Sigma y)^2}}$$

or
$$\boxed{b_{xy} = r\frac{\sigma_x}{\sigma_y}}$$

where the terms have their usual meanings.

Note. If $r = 0$, the two lines of regression become $y = \bar{y}$ and $x = \bar{x}$ which are two straight lines parallel to x and y axes respectively and passing through their means $\bar{y}$ and $\bar{x}$. They are mutually perpendicular. If $r = \pm 1$, the two lines of regression will coincide.

3.43 USE OF REGRESSION ANALYSIS (*U.P.T.U. 2008*)

(*i*) In the field of Business, this tool of statistical analysis is widely used. Businessmen are interested in predicting future production, consumption, investment, prices, profits and sales etc.

(*ii*) In the field of economic planning and sociological studies, projections of population, birth rates, death rates and other similar variables are of great use.

3.44 COMPARISON OF CORRELATION AND REGRESSION ANALYSIS [*G.B.T.U. M.B.A. (C.O.) 2011*]

Both the correlation and regression analysis helps us in studying the relationship between two variables yet they differ in their approach and objectives.

(*i*) Correlation studies are meant for studying the covariation of the two variables. They tell us whether the variables under study move in the same direction or in reverse directions. The degree of their covariation is also reflected in the correlation co-efficient but the correlation study does not provide the nature of relationship. It does not tell us about the relative movement in the variables and we cannot predict the value of one variable corresponding to the value of other variable. This is possible through regression analysis.

(*ii*) Regression presumes one variable as a cause and the other as its effect. The independent variable is supposed to be affecting the dependent variable and as such we can estimate the values of the dependent variable by projecting the relationship between them. However, correlation between two series is not necessarily a cause-effect relationship.

(*iii*) Coefficient of correlation cannot exceed unity but one of the regression coefficients can have a value higher than unity but the product of two regression coefficients can never exceed unity.

3.45 PROPERTIES OF REGRESSION CO-EFFICIENTS

Property I. *Correlation co-efficient is the geometric mean between the regression co-efficients.*

Proof. The co-efficients of regression are $\dfrac{r\sigma_y}{\sigma_x}$ and $\dfrac{r\sigma_x}{\sigma_y}$.

$$\text{G.M. between them} = \sqrt{\frac{r\sigma_y}{\sigma_x} \times \frac{r\sigma_x}{\sigma_y}} = \sqrt{r^2} = r = \text{co-efficient of correlation.}$$

Property II. *If one of the regression co-efficients is greater than unity, the other must be less than unity.*

Proof. The two regression co-efficients are $b_{yx} = \dfrac{r\sigma_y}{\sigma_x}$ and $b_{xy} = \dfrac{r\sigma_x}{\sigma_y}$.

Let $\quad b_{yx} > 1$, then $\dfrac{1}{b_{yx}} < 1 \quad \ldots(1)$

Since $\quad b_{yx} \cdot b_{xy} = r^2 \le 1 \quad (\because \; -1 \le r \le 1)$

$\therefore \quad b_{xy} \le \dfrac{1}{b_{yx}} < 1. \quad$ | Using (1)

Similarly, if $b_{xy} > 1$, then $b_{yx} < 1$.

Property III. *Arithmetic mean of regression co-efficients is greater than the correlation co-efficient.*

Proof. We have to prove that

$$\frac{b_{yx} + b_{xy}}{2} > r$$

or $$r\frac{\sigma_y}{\sigma_x} + r\frac{\sigma_x}{\sigma_y} > 2r$$

or $$\sigma_x^2 + \sigma_y^2 > 2\sigma_x\sigma_y$$

or $$(\sigma_x - \sigma_y)^2 > 0, \quad \text{which is true.}$$

Property IV. *Regression co-efficients are independent of the origin but not of scale.*

Proof. Let $u = \dfrac{x-a}{h}, v = \dfrac{y-b}{k}$, where a, b, h and k are constants

$$b_{yx} = \frac{r\sigma_y}{\sigma_x} = r \cdot \frac{k\sigma_v}{h\sigma_u} = \frac{k}{h}\left(\frac{r\sigma_v}{\sigma_u}\right) = \frac{k}{h} b_{vu}$$

Similarly, $b_{xy} = \dfrac{h}{k} b_{uv}$.

Thus, b_{yx} and b_{xy} are both independent of a and b but not of h and k.

Property V. *The correlation co-efficient and the two regression co-efficients have same sign.*

Proof. Regression co-efficient of y on $x = b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$

Regression co-efficient of x on $y = b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$

Since σ_x and σ_y are both positive; b_{yx}, b_{xy} and r have same sign.

3.46 ANGLE BETWEEN TWO LINES OF REGRESSION

If θ is the acute angle between the two regression lines in the case of two variables x and y, show that

$$\tan\theta = \frac{1-r^2}{r} \cdot \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}, \quad \text{where } r, \sigma_x, \sigma_y \text{ have their usual meanings.}$$

Explain the significance of the formula when $r = 0$ and $r = \pm 1$.

[U.P.T.U. 2007, 2015; G.B.T.U. (C.O.) 2011]

Proof. Equations to the lines of regression of y on x and x on y are

$$y - \bar{y} = \frac{r\sigma_y}{\sigma_x}(x - \bar{x}) \quad \text{and} \quad x - \bar{x} = \frac{r\sigma_x}{\sigma_y}(y - \bar{y})$$

Their slopes are $m_1 = \dfrac{r\sigma_y}{\sigma_x}$ and $m_2 = \dfrac{\sigma_y}{r\sigma_x}$.

$$\therefore \quad \tan\theta = \pm\frac{m_2 - m_1}{1 + m_2 m_1} = \pm\frac{\dfrac{\sigma_y}{r\sigma_x} - \dfrac{r\sigma_y}{\sigma_x}}{1 + \dfrac{\sigma_y^2}{\sigma_x^2}}$$

$$= \pm \frac{1-r^2}{r} \cdot \frac{\sigma_y}{\sigma_x} \cdot \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2} = \pm \frac{1-r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}$$

Since $r^2 \le 1$ and σ_x, σ_y are positive.

$\therefore$ +ve sign gives the acute angle between the lines.

Hence $$\boxed{\tan\theta = \frac{1-r^2}{r} \cdot \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}}$$

when $r = 0$, $\theta = \frac{\pi}{2}$ $\therefore$ The two lines of regression are perpendicular to each other.

Hence the estimated value of y is the same for all values of x and vice-versa.

When $r = \pm 1$, $\tan\theta = 0$ so that $\theta = 0$ or π

Hence the lines of regression coincide and there is perfect correlation between the two variates x and y.

EXAMPLES

Example 1. *If the regression coefficients are 0.8 and 0.2, what would be the value of coefficient of correlation?*

Sol. We know that,

$$r^2 = b_{yx} \cdot b_{xy} = 0.8 \times 0.2 = 0.16$$

Since r has the same sign as both the regression coefficients b_{yx} and b_{xy}

Hence $r = \sqrt{0.16} = 0.4$.

Example 2. *Calculate linear regression coefficients from the following:*

x	$\rightarrow$	1	2	3	4	5	6	7	8
y	$\rightarrow$	3	7	10	12	14	17	20	24

Sol. Linear regression coefficients are given by

$$b_{yx} = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma x^2 - (\Sigma x)^2} \quad \text{and} \quad b_{xy} = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma y^2 - (\Sigma y)^2}$$

Let us prepare the following table:

x	y	x^2	y^2	xy
1	3	1	9	3
2	7	4	49	14
3	10	9	100	30
4	12	16	144	48
5	14	25	196	70
6	17	36	289	102
7	20	49	400	140
8	24	64	576	192
$\Sigma x = 36$	$\Sigma y = 107$	$\Sigma x^2 = 204$	$\Sigma y^2 = 1763$	$\Sigma xy = 599$

Here, $n = 8$

$$\therefore \qquad b_{yx} = \frac{(8 \times 599) - (36 \times 107)}{(8 \times 204) - (36)^2} = \frac{940}{336} = 2.7976$$

and

$$b_{xy} = \frac{(8 \times 599) - (36 \times 107)}{(8 \times 1763) - (107)^2} = \frac{940}{2655} = 0.3540$$

Example 3. *The following table gives age (x) in years of cars and annual maintenance cost (y) in hundred rupees:*

X:	*1*	*3*	*5*	*7*	*9*
Y:	*15*	*18*	*21*	*23*	*22*

Estimate the maintenance cost for a 4 year old car after finding the regression equation.

Sol.

x	y	xy	x^2
1	15	15	1
3	18	54	9
5	21	105	25
7	23	161	49
9	22	198	81
$\Sigma x = 25$	$\Sigma y = 99$	$\Sigma xy = 533$	$\Sigma x^2 = 165$

Here, $n = 5$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{25}{5} = 5, \; \bar{y} = \frac{\Sigma y}{n} = \frac{99}{5} = 19.8$$

$$\therefore \qquad b_{yx} = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma x^2 - (\Sigma x)^2} = \frac{(5 \times 533) - (25 \times 99)}{(5 \times 165) - (25)^2} = 0.95$$

Regression line of y on x is given by

$$y - \bar{y} = b_{yx}\,(x - \bar{x})$$

$$\Rightarrow \qquad y - 19.8 = 0.95\,(x - 5)$$

$$\Rightarrow \qquad y = 0.95x + 15.05$$

When $x = 4$ years, $\quad y = (0.95 \times 4) + 15.05 = 18.85$ hundred rupees = ₹ 1885.

Example 4. *In a partially destroyed laboratory record of an analysis of a correlation data, the following results only are legible:*

Variance of x = 9

Regression equations: 8x – 10y + 66 = 0, 40x – 18y = 214.

What were (a) the mean values of x and y (b) the standard deviation of y and the co-efficient of correlation between x and y? [U.P.T.U. 2008, 2009; U.K.T.U. 2010]

Sol. (*a*) **Since both the lines of regression pass through the point** $(\bar{x}, \bar{y})$ therefore, we have

$$8\bar{x} - 10\bar{y} + 66 = 0 \qquad \ldots(1)$$

$$40\bar{x} - 18\bar{y} - 214 = 0 \qquad \ldots(2)$$

Multiplying (1) by 5, $40\bar{x} - 50\bar{y} + 330 = 0$ ...(3)

Subtracting (3) from (2), $32\bar{y} - 544 = 0$ $\therefore$ $\bar{y} = 17$

$\therefore$ From (1), $8\bar{x} - 170 + 66 = 0$ or $8\bar{x} = 104$ $\therefore$ $\bar{x} = 13$

Hence, $\bar{x} = 13,\ \bar{y} = 17$

(*b*) Variance of $x = \sigma_x^2 = 9$ (given)

$\therefore$ $\sigma_x = 3$

The equations of lines of regression can be written as

$$y = 0.8x + 6.6 \quad \text{and} \quad x = 0.45y + 5.35$$

$\therefore$ The regression co-efficient of y on x is $\dfrac{r\sigma_y}{\sigma_x} = 0.8$ ...(4)

The regression co-efficient of x on y is $\dfrac{r\sigma_x}{\sigma_y} = 0.45$ ...(5)

Multiplying (4) and (5), $r^2 = 0.8 \times 0.45 = 0.36$ $\therefore$ $r = 0.6$

(+ve sign with square root is taken because regression co-efficients are +ve).

From (4), $\sigma_y = \dfrac{0.8\sigma_x}{r} = \dfrac{0.8 \times 3}{0.6} = 4.$

Example 5. *The regression lines of y on x and x on y are respectively $y = ax + b$, $x = cy + d$. Show that*

$$\frac{\sigma_y}{\sigma_x} = \sqrt{\frac{a}{c}},\ \bar{x} = \frac{bc + d}{1 - ac} \ \textit{and} \ \bar{y} = \frac{ad + b}{1 - ac}.$$

[U.P.T.U. (C.O.) 2009, U.P.T.U. (MCA) 2008]

Sol. The regression line of y on x is

$$y = ax + b \qquad ...(1)$$

$\therefore$ $b_{yx} = a$

The regression line of x on y is

$$x = cy + d \qquad ...(2)$$

$\therefore$ $b_{xy} = c$

We know that, $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$ ...(3)

and $b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$ ...(4)

Dividing eqn. (3) by (4), we get

$$\frac{b_{yx}}{b_{xy}} = \frac{\sigma_y^{\,2}}{\sigma_x^{\,2}} \Rightarrow \frac{a}{c} = \frac{\sigma_y^{\,2}}{\sigma_x^{\,2}} \Rightarrow \frac{\sigma_y}{\sigma_x} = \sqrt{\frac{a}{c}}$$

Since both the regression lines pass through the point $(\bar{x}, \bar{y})$ therefore,

$$\bar{y} = a\bar{x} + b \quad \text{and} \quad \bar{x} = c\bar{y} + d$$

$$\Rightarrow \quad a\bar{x} - \bar{y} = -b \qquad ...(5)$$

$$\bar{x} - c\bar{y} = d \qquad ...(6)$$

Multiplying equation (6) by a and then subtracting from (5), we get

$$(ac - 1)\ \bar{y} = -ad - b \quad \Rightarrow \quad \bar{y} = \frac{ad+b}{1-ac}$$

Similarly, we get $\quad \bar{x} = \dfrac{bc+d}{1-ac}.$

Example 6. *For two random variables, x and y with the same mean, the two regression equations are $y = ax + b$ and $x = \alpha y + \beta$. Show that $\dfrac{b}{\beta} = \dfrac{1-a}{1-\alpha}$. Find also the common mean.*

(G.B.T.U. 2010)

Sol. Here, $\quad b_{yx} = a,\ b_{xy} = \alpha$

Let the common mean be m, then regression lines are

$$y - m = a\,(x - m)$$

$$\Rightarrow \quad y = ax + m\,(1 - a) \qquad ...(1)$$

and

$$x - m = \alpha(y - m)$$

$$\Rightarrow \quad x = \alpha y + m\,(1 - \alpha) \qquad ...(2)$$

Comparing (1) and (2) with the given equations.

$$b = m\,(1 - a),\ \beta = m\,(1 - \alpha)$$

$$\therefore \quad \frac{b}{\beta} = \frac{1-a}{1-\alpha}$$

Since regression lines pass through $(\bar{x}, \bar{y})$

$$\therefore \quad \bar{x} = \alpha\bar{y} + \beta \quad \text{and} \quad \bar{y} = a\bar{x} + b \text{ will hold.}$$

$$\Rightarrow \quad m = am + b,\ \ m = \alpha m + \beta$$

$$\Rightarrow \quad am + b = \alpha m + \beta$$

$$\Rightarrow \quad m = \frac{\beta - b}{a - \alpha}.$$

Example 7. (*i*) *Obtain the line of regression of y on x for the data given below:*

x:	*1.53*	*1.78*	*2.60*	*2.95*	*3.42*
y:	*33.50*	*36.30*	*40.00*	*45.80*	*53.50.*

(*ii*) *The following data regarding the heights (y) and weights (x) of 100 college students are given:*

$$\Sigma x = 15000, \quad \Sigma x^2 = 2272500, \quad \Sigma y = 6800, \quad \Sigma y^2 = 463025 \text{ and } \Sigma xy = 1022250.$$

Find the equation of regression line of height on weight.

Sol. (*i*) The line of regression of y on x is given by

$$y - \bar{y} = b_{yx}(x - \bar{x}) \qquad ...(1)$$

where b_{yx} is the coefficient of regression given by

$$b_{yx} = \frac{n\ \Sigma xy - \Sigma x\ \Sigma y}{n\ \Sigma x^2 - (\Sigma x)^2} \qquad \text{...(2)}$$

Now we form the table as,

x	y	x^2	xy
1.53	33.50	2.3409	51.255
1.78	36.30	2.1684	64.614
2.60	40.00	6.76	104
2.95	45.80	8.7025	135.11
3.42	53.50	11.6964	182.97
$\Sigma x = 12.28$	$\Sigma y = 209.1$	$\Sigma x^2 = 32.6682$	$\Sigma xy = 537.949$

Here, $n = 5$

$$b_{yx} = \frac{(5 \times 537.949) - (12.28 \times 209.1)}{(5 \times 32.6682) - (12.28)^2} = 9.726$$

Also, mean $\bar{x} = \frac{\Sigma x}{n} = \frac{12.28}{5} = 2.456$ and $\bar{y} = \frac{\Sigma y}{n} = \frac{209.1}{5} = 41.82$

$\therefore$ From (1), we get

$$y - 41.82 = 9.726(x - 2.456) = 9.726x - 23.887$$

$$y = 17.932 + 9.726x$$

(*ii*) $$\bar{x} = \frac{\Sigma x}{n} = \frac{15000}{100} = 150, \quad \bar{y} = \frac{\Sigma y}{n} = \frac{6800}{100} = 68$$

Regression coefficient of y on x,

$$b_{yx} = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma x^2 - (\Sigma x)^2} = \frac{(100 \times 1022250) - (15000 \times 6800)}{(100 \times 2272500) - (15000)^2} = 0.1$$

Regression line of height (y) on weight (x) is given by

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$\Rightarrow$ $$y - 68 = 0.1(x - 150)$$

$\Rightarrow$ $$y = 0.1x + 53.$$

Example 8. *For 10 observations on price (x) and supply (y), the following data were obtained (in appropriate units):*

$$\Sigma x = 130, \quad \Sigma y = 220, \quad \Sigma x^2 = 2288, \quad \Sigma y^2 = 5506 \text{ and } \Sigma xy = 3467$$

Obtain the two lines of regression and estimate the supply when the price is 16 units.

Sol. Here, $n = 10$, $\bar{x} = \frac{\Sigma x}{n} = 13$ and $\bar{y} = \frac{\Sigma y}{n} = 22$

Regression coefficient of y on x is

$$b_{yx} = \frac{n\ \Sigma xy - \Sigma x\ \Sigma y}{n\ \Sigma x^2 - (\Sigma x)^2} = \frac{(10 \times 3467) - (130 \times 220)}{(10 \times 2288) - (130)^2} = 1.015$$

$\therefore$ Regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 22 = 1.015(x - 13)$$

$$\Rightarrow \quad y = 1.015x + 8.805 \qquad \text{...(1)}$$

Regression coefficient of x on y is

$$b_{xy} = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma y^2 - (\Sigma y)^2} = \frac{(10 \times 3467) - (130 \times 220)}{(10 \times 5506) - (220)^2} = 0.9114$$

Regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 13 = 0.9114(y - 22)$$

$$x = 0.9114y - 7.0508 \qquad \text{...(2)}$$

Since we are to estimate supply (y) when price (x) is given therefore we are to use regression line of y on x here.

When $x = 16$ units,

$$y = 1.015(16) + 8.805 = 25.045 \text{ units.}$$

Example 9. *The following results were obtained from records of age (x) and systolic blood pressure (y) of a group of 10 men:*

	x	y
Mean	*53*	*142*
Variance	*130*	

and $\Sigma(x - \bar{x})(y - \bar{y}) = 1220$

Find the appropriate regression equation and use it to estimate the blood pressure of a man whose age is 45.

Sol. Given

Mean $\quad \bar{x} = 53$ and $\bar{y} = 142$; $\quad$ Variance $\quad \sigma_x^{\,2} = 130$

$n = 10$ and $\Sigma(x - \bar{x})(y - \bar{y}) = 1220$

Since we are to estimate blood pressure (y) of a 45 years old man, we will find regression line of y on x.

Regression coefficient

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\,\sigma_x\,\sigma_y}\left(\frac{\sigma_y}{\sigma_x}\right)$$

$$= \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\,\sigma_x^2} = \frac{1220}{(10)(130)} = 0.93846$$

Regression line of y on x is given by

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$\Rightarrow \quad y - 142 = 0.93846\,(x - 53)$$

$$\Rightarrow \quad y = 0.93846\,x + 92.26162$$

When $\quad x = 45, \quad y = 134.49$

Hence the required blood pressure = 134.49.

Example 10. *The following results were obtained from marks in Applied Mechanics and Engineering Mathematics in an examination:*

	Applied Mechanics (x)	*Engineering Mathematics (y)*
Mean	*47.5*	*39.5*
Standard Deviation	*16.8*	*10.8*

$r = 0.95$.

Find both the regression equations. Also estimate the value of y for x = 30.

Sol. $\bar{x} = 47.5$, $\bar{y} = 39.5$

$\sigma_x = 16.8$, $\sigma_y = 10.8$ and $r = 0.95$.

Regression coefficients are

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = 0.95 \times \frac{10.8}{16.8} = 0.6107$$

and $$b_{xy} = r\frac{\sigma_x}{\sigma_y} = 0.95 \times \frac{16.8}{10.8} = 1.477.$$

Regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$\Rightarrow \quad y - 39.5 = 0.6107\,(x - 47.5) = 0.6107x - 29.008$$

$$y = 0.6107x + 10.49 \quad ...(1)$$

Regression line of x on y is

$$x - \bar{x} = b_{xy}\,(y - \bar{y})$$

$$\Rightarrow \quad x - 47.5 = 1.477\,(y - 39.5)$$

$$x = 1.477y - 10.8415 \quad ...(2)$$

Putting $x = 30$ in equation (1), we get

$$y = (0.6107)(30) + 10.49 = 28.81.$$

Example 11. *The equations of two regression lines, obtained in a correlation analysis of 60 observations are:*

$$5x = 6y + 24 \text{ and } 1000y = 768x - 3608.$$

What is the correlation coefficient? Show that the ratio of coefficient of variability of x to that of y is $\frac{5}{24}$. What is the ratio of variances of x and y?

Sol. Regression line of x on y is

$$5x = 6y + 24$$

$$\Rightarrow \quad x = \frac{6}{5}y + \frac{24}{5}$$

$$\therefore \quad b_{xy} = \frac{6}{5} \quad ...(1)$$

Regression line of y on x is

$$1000y = 768x - 3608$$

$$\Rightarrow \quad y = 0.768x - 3.608$$

$$\therefore \quad b_{yx} = 0.768 \quad ...(2)$$

From (1), $$r\,\frac{\sigma_x}{\sigma_y} = \frac{6}{5} \qquad ...(3)$$

From (2), $$r\,\frac{\sigma_y}{\sigma_x} = 0.768 \qquad ...(4)$$

Multiplying equations (3) and (4), we get

$$r^2 = 0.9216 \quad \Rightarrow \quad r = 0.96 \qquad ...(5)$$

Dividing (4) by (3), we get

$$\frac{{\sigma_x}^2}{{\sigma_y}^2} = \frac{6}{5 \times 0.768} = 1.5625.$$

Taking square root, we get

$$\frac{\sigma_x}{\sigma_y} = 1.25 = \frac{5}{4} \qquad ...(6)$$

Since the regression lines pass through the point $(\bar{x}, \bar{y})$, we have

$$5\bar{x} = 6\bar{y} + 24$$

$$1000\bar{y} = 768\bar{x} - 3608.$$

Solving the above equations for $\bar{x}$ and $\bar{y}$, we get $\bar{x} = 6$, $\bar{y} = 1$.

Co-efficient of variability of $x = \dfrac{\sigma_x}{\bar{x}}$

Co-efficient of variability of $y = \dfrac{\sigma_y}{\bar{y}}$.

$\therefore$ Required ratio $= \dfrac{\sigma_x}{\bar{x}} \times \dfrac{\bar{y}}{\sigma_y} = \dfrac{\bar{y}}{\bar{x}}\left(\dfrac{\sigma_x}{\sigma_y}\right) = \dfrac{1}{6} \times \dfrac{5}{4} = \dfrac{5}{24}$. | Using (6)

Example 12. *A panel of two judges, A and B, graded seven TV serial performances by awarding marks independently as shown in the following table:*

Performance	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Marks by A	*46*	*42*	*44*	*40*	*43*	*41*	*45*
Marks by B	*40*	*38*	*36*	*35*	*39*	*37*	*41*

The eighth TV performance which judge B could not attend, was awarded 37 marks by judge A. If the judge B had also been present, how many marks would be expected to have been awarded by him to the eighth TV performance?

Use regression analysis to answer this question.

Sol. Let the marks awarded by judge A be denoted by x and the marks awarded by judge B be denoted by y.

Here, $n = 7$; $$\bar{x} = \frac{\Sigma x}{n} = \frac{46 + 42 + 44 + 40 + 43 + 41 + 45}{7} = 43$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{40 + 38 + 36 + 35 + 39 + 37 + 41}{7} = 38$$

Let us form the table as

x	y	xy	x^2
46	40	1840	2116
42	38	1596	1764
44	36	1584	1936
40	35	1400	1600
43	39	1677	1849
41	37	1517	1681
45	41	1845	2025
$\Sigma x = 301$	$\Sigma y = 266$	$\Sigma xy = 11459$	$\Sigma x^2 = 12971$

Regression coefficient,

$$b_{yx} = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma x^2 - (\Sigma x)^2} = \frac{(7 \times 11459) - (301 \times 266)}{(7 \times 12971) - (301)^2} = 0.75$$

Regression line of y on x is given by

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 38 = 0.75(x - 43)$$

$\Rightarrow$ $$y = 0.75x + 5.75$$

when $x = 37$, $$y = 0.75(37) + 5.75 = 33.5 \text{ marks}$$

Hence, if judge B had also been present, 33.5 marks would be expected to have been awarded to the eighth TV performance.

Example 13. *Two variables x and y have zero means, the same variance* σ^2 *and zero correlation, show that:*

$$u = x \cos \alpha + y \sin \alpha \quad \text{and} \quad v = x \sin \alpha - y \cos \alpha$$

have the same variance σ^2 *and zero correlation.* (U.P.T.U. 2007)

Sol. We are given that

$$r(x, y) = 0 \quad \Rightarrow \quad \text{Cov}(x, y) = 0, \quad \sigma_x^2 = \sigma_y^2 = \sigma^2$$

We have,
$$\begin{aligned}
\sigma_u^2 &= V(x \cos \alpha + y \sin \alpha) \\
&= \cos^2 \alpha\, V(x) + \sin^2 \alpha\, V(y) + 2 \sin \alpha \cos \alpha\, \text{Cov}(x, y) \\
&= (\cos^2 \alpha + \sin^2 \alpha)\, \sigma^2 && | \because \ \text{Cov}(x, y) = 0 \\
&= \sigma^2
\end{aligned}$$

Similarly, $$\sigma_v^2 = \sigma^2$$

$$\begin{aligned}
\text{Cov}(u, v) &= E[(u - \bar{u})(v - \bar{v})] \\
&= E[(x \cos \alpha + y \sin \alpha - \bar{x} \cos \alpha - \bar{y} \sin \alpha) \\
&\qquad (x \sin \alpha - y \cos \alpha - \bar{x} \sin \alpha + \bar{y} \cos \alpha)] \\
&= E[(x \cos \alpha + y \sin \alpha)(x \sin \alpha - y \cos \alpha)] && | \because \ \bar{x} = 0 = \bar{y} \\
&= [E(x^2) - E(y^2)] \sin \alpha \cos \alpha + E(xy)(\sin^2 \alpha - \cos^2 \alpha) \\
&= 0 && | \because \ \sigma_x^2 = \sigma_y^2 = \sigma^2 \text{ and } E(xy) = 0
\end{aligned}$$

$\therefore$ $$r = \frac{\text{Cov}(u, v)}{\sigma_u\, \sigma_v} = 0.$$

ASSIGNMENT

1. (*i*) Discuss regression and its importance. Given the following data:

x:	1	5	3	2	1	1	7	3
y:	6	1	0	0	1	2	1	5

Find a regression line of x on y. (*U.P.T.U. 2008*)

(*ii*) In a study between the amount of rainfall and the quantity of air pollution removed the following data were collected:

Daily rainfall: *(in .01 cm)*	4.3	4.5	5.9	5.6	6.1	5.2	3.8	2.1
Pollution removed: *(mg / m*^{3}*)*	12.6	12.1	11.6	11.8	11.4	11.8	13.2	14.1

Find the regression line of y on x.

(*iii*) Find the two lines of regression and coefficient of correlation for the data given below:
$n = 18$, $\Sigma x = 12$, $\Sigma y = 18$, $\Sigma x^2 = 60$, $\Sigma y^2 = 96$, $\Sigma xy = 48$ [*U.P.T.U. (MCA) 2009*]

(*iv*) From the data given, find the equation of lines of regression of x on y and y on x. Also calculate the correlation co-efficient.

x:	2	4	6	8	10
y:	5	7	9	8	11

(*U.P.T.U. 2011*)

2. (*i*) Can Y = 5 + 2.8 X and X = 3 – 0.5 Y be the estimated regression equations of Y on X and X on Y respectively? Explain your answer with suitable theoretical arguments.

(*ii*) Find the co-efficient of correlation when the two regression equations are

$$X = -0.2\,Y + 4.2, \quad Y = -0.8\,X + 8.4$$

3. (*i*) If F is the pull required to lift a load W by means of a pulley block, fit a linear law of the form F = mW + c connecting F and W, using the data

W:	50	70	100	120
F:	12	15	21	25

where F and W are in kg wt. Compute F when W = 150 kg wt. (*U.P.T.U. 2007*)

(*ii*) A simply supported beam carries a concentrated load P (kg) at its mid-point. The following table gives maximum deflection y (cm) corresponding to various values of P:

P:	100	120	140	160	180	200
y:	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $y = a + b$P. Also find the value of maximum deflection when P = 150 kg.

4. (*i*) Find both the lines of regression of following data:

x:	5.60	5.65	5.70	5.81	5.85
y:	5.80	5.70	5.80	5.79	6.01

(*ii*) Obtain regression line of x on y for the given data:

x:	1	2	3	4	5	6
y:	5.0	8.1	10.6	13.1	16.2	20.0

[*U.P.T.U. (MCA) 2007*]

(*iii*) Given that:

x:	1	3	5	7	8	10
y:	8	12	15	17	18	20

Find the equations of both lines of regression. [*U.P.T.U. (C.O.) 2008*]

5. (*i*) The two regression equations of the variables x and y are $x = 19.13 - 0.87\,y$ and $y = 11.64 - 0.50x$. Find (*a*) mean of x's (*b*) mean of y's and (*c*) correlation coefficient between x and y.

(*ii*) Two random variables have the regression lines with equations $3x + 2y = 26$ and $6x + y = 31$. Find the mean values and the correlation coefficient between x and y.

[*G.B.T.U. (MBA) 2011*]

(*iii*) In a partially destroyed laboratory data, only the equations giving the two lines of regression of y on x and x on y are available and are respectively

$$7x - 16y + 9 = 0, \quad 5y - 4x - 3 = 0.$$

Calculate the coefficient of correlation, $\overline{x}$ and $\overline{y}$.

(*iv*) The regression equations calculated from a given set of observations for two random variables are

$$x = -\,0.4y + 6.4 \quad \text{and} \quad y = -\,0.6x + 4.6$$

Calculate (*i*) $\overline{x}$ (*ii*) $\overline{y}$ (*iii*) r.

(*v*) Two lines of regression are given by

$$x + 2y - 5 = 0 \quad \text{and} \quad 2x + 3y - 8 = 0 \quad \text{and} \quad \sigma_x^{\,2} = 12,$$

Calculate:

(*a*) the mean values of x and y (*b*) variance of y

(*c*) the coefficient of correlation between x and y. [*U.P.T.U. (MCA) 2008, G.B.T.U. (C.O.) 2011*]

6. An analyst for a company was studying travelling expenses (y) in ₹ and duration (x) of these trips for 102 sales trip. He has found relation between x and y linear and data as follows:

$\Sigma x = 510$, $\Sigma y = 7140$, $\Sigma x^2 = 4150$, $\Sigma xy = 54900$, $\Sigma y^2 = 740200$

Calculate (*i*) Two regression lines

(*ii*) A given trip has to take 7 days. How much money should be allowed so that they will not run short of money?

7. Assuming that we conduct an experiment with 8 fields planted with corn, four fields having no nitrogen fertiliser and four fields having 80 kgs of nitrogen fertilizer. The resulting corn yields are shown in table in bushels per acre :

Field:	1	2	3	4	5	6	7	8
Nitrogen (kgs) x:	0	0	0	0	80	80	80	80
Corn yield y: (acre)	120	360	60	180	1280	1120	1120	760

(*a*) Compute a linear regression equation of y on x.

(*b*) Predict corn yield for a field treated with 60 kgs of fertilizer.

8. If the coefficient of correlation between two variables x and y is 0.5 and the acute angle between their lines of regression is $\tan^{-1}\left(\dfrac{3}{5}\right)$, show that $\sigma_x = \dfrac{1}{2}\sigma_y$. (*U.P.T.U. 2009*)

9. Given N = 50, Mean of $y = 44$, Variance of x is $\dfrac{9}{16}$ of the variance of y.

Regression equation of x on y is $3y - 5x = -\,180$

Find (*i*) Mean of x (*ii*) Coeff. of correlation between x and y.

10. The means of a bivariate frequency distribution are at (3, 4) and $r = 0.4$. The line of regression of y on x is parallel to the line $y = x$. Find the two lines of regression and estimate value of x when $y = 1$.

11. The following results were obtained in the analysis of data on yield of dry bark in ounces (y) and age in years (x) of 200 cinchona plants:

	x	y
Average:	9.2	16.5
Standard deviation:	2.1	4.2

Correlation coefficient = 0.84

Construct the two lines of regression and estimate the yield of dry bark of a plant of age 8 years.

12. A panel of judges A and B graded 7 debators and independently awarded the following marks:

Debator:	1	2	3	4	5	6	7
Marks by A:	40	34	28	30	44	38	31
Marks by B:	32	39	26	30	38	34	28

An eighth debator was awarded 36 marks by judge A while judge B was not present. If judge B were also present, how many marks would you expect him to award to the eighth debator assuming that the same degree of relationship exists in their judgement?

Answers

1. (*i*) $72x = -20y + 247$ (*ii*) $y = -0.6842x + 15.5324$

(*iii*) $y = 0.6923x + 0.53846$; $x = 0.4615y + 0.2051$

(*iv*) $x = 1.3y - 4.4$, $y = .65x + 4.1$; $r = .9192$

2. (*i*) No (*ii*) $r = -0.4$

3. (*i*) F = 0.18793W + 2.27595; F = 30.4654 kg wt.

(*ii*) $y = 0.04765 + 0.004071$ P; $y = 0.6583$ cm

4. (*i*) Regression line of y on x: $y = 0.74306x + 1.56821$

Regression line of x on y: $x = 0.63602y + 2.0204$

(*ii*) $x = 0.34195y - 0.660355$ (*iii*) $y = 1.3012x + 7.6265$; $x = 0.75y - 5.5833$.

5. (*i*) (*a*) 15.935 (*b*) 3.67 (*c*) – 0.659

(*ii*) $\bar{x} = 4$, $\bar{y} = 7$, $r = -0.5$ (*iii*) $r = 0.7395$, $\bar{x} = -0.1034$, $\bar{y} = 0.5172$

(*iv*) $\bar{x} = 6$, $\bar{y} = 1$, $r = -0.48989$

(*v*) (*a*) $\bar{x} = 1$, $\bar{y} = 2$ (*b*) 4 (*c*) $-\dfrac{\sqrt{3}}{2}$

6. (*i*) $y = 12x + 10$, $x = 0.07986y - 0.59068$ (*ii*) ₹ 94

7. (*a*) $y = 11.125x + 180$ (*b*) 847.5 acre

9. (*i*) 62.4 (*ii*) 0.8

10. $y = x + 1$; $x = 0.16y + 2.36$; $x = 2.52$

11. $y = 1.68x + 1.044$, $x = 0.42y + 2.27$; $y = 14.484$

12. 33 marks.

3.47 POLYNOMIAL FIT: NON-LINEAR REGRESSION

Let
$$y = a + bx + cx^2 \qquad ...(1)$$
be a second degree parabolic curve of regression of y on x to be fitted for the data (x_i, y_i), $i = 1, 2, ..., n$.

Residual at $x = x_i$ is

$$E_i = y_i - f(x_i) = y_i - a - bx_i - cx_i^2$$

Now, let
$$U = \sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i - cx_i^2)^2$$

By principle of Least squares, U should be minimum for the best values of a, b and c.

For this,
$$\frac{\partial U}{\partial a} = 0, \frac{\partial U}{\partial b} = 0 \text{ and } \frac{\partial U}{\partial c} = 0$$

$$\frac{\partial U}{\partial a} = 0 \quad \Rightarrow \quad 2\sum_{i=1}^{n} (y_i - a - bx_i - cx_i^2)(-1) = 0$$

$$\Rightarrow \quad \boxed{\Sigma y = na + b\Sigma x + c\Sigma x^2} \qquad ...(1)$$

$$\frac{\partial U}{\partial b} = 0 \quad \Rightarrow \quad 2\sum_{i=1}^{n} (y_i - a - bx_i - cx_i^2)(-x_i) = 0$$

$$\Rightarrow \quad \boxed{\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3} \qquad ...(2)$$

$$\frac{\partial U}{\partial c} = 0 \quad \Rightarrow \quad 2\sum_{i=1}^{n} (y_i - a - bx_i - cx_i^2)(-x_i^2) = 0$$

$$\Rightarrow \quad \boxed{\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4} \qquad ...(3)$$

Equations (1), (2) and (3) are the normal equations for fitting a second degree parabolic curve of regression of y on x. Here n is the no. of pairs of values of x and y.

EXAMPLES

Example 1. (a) *Fit a second degree parabola to the following data:*

x	0.0	1.0	2.0
y	1.0	6.0	17.0

(b) *Fit a second degree curve of regression of y on x to the following data:*

x	1.0	2.0	3.0	4.0
y	6.0	11.0	18.0	27

(c) *Fit a second degree parabola in the following data:*

x	0.0	1.0	2.0	3.0	4.0
y	1.0	4.0	10.0	17.0	30.0

Sol. The equation of second degree parabola is given by

$$y = a + bx + cx^2 \quad ...(1)$$

Normal equations are

$$\Sigma y = ma + b\Sigma x + c\Sigma x^2 \quad ...(2)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad ...(3)$$

and

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad ...(4)$$

(*a*) Here, $m = 3$. Table is as follows:

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	6	1	1	1	6	6
2	17	4	8	16	34	68
Total 3	24	5	9	17	40	74

Substituting in eqns. (2), (3) and (4), we get

$$24 = 3a + 3b + 5c \quad ...(5)$$

$$40 = 3a + 5b + 9c \quad ...(6)$$

$$74 = 5a + 9b + 17c \quad ...(7)$$

Solving eqns. (5), (6) and (7), we get $a = 1, b = 2, c = 3$

Hence the required second degree parabola is $y = 1 + 2x + 3x^2$

(*b*) Here, $m = 4$. Table is as follows:

x	y	x^2	x^3	x^4	xy	x^2y
1	6	1	1	1	6	6
2	11	4	8	16	22	44
3	18	9	27	81	54	162
4	27	16	64	256	108	432
$\Sigma x = 10$	$\Sigma y = 62$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma xy = 190$	$\Sigma x^2y = 644$

Substituting values in eqns. (2), (3) and (4), we get

$$62 = 4a + 10b + 30c \quad ...(8)$$

$$190 = 10a + 30b + 100c \quad ...(9)$$

$$644 = 30a + 100b + 354c \quad ...(10)$$

Solving equations (8), (9) and (10), we get $a = 3, b = 2, c = 1$

Hence the required second degree parabola is $y = 3 + 2x + x^2$

(*c*) Here, $m = 5$. Table is as follows:

x	y	x^2	x^3	x^4	xy	x^2y
0.0	1.0	0	0	0	0	0
1.0	4.0	1	1	1	4	4
2.0	10.0	4	8	16	20	40
3.0	17.0	9	27	81	51	153
4.0	30.0	16	64	256	120	480
$\Sigma x = 10$	$\Sigma y = 62$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma xy = 195$	$\Sigma x^2y = 677$

Substituting values in eqns. (2), (3) and (4), we get

$$62 = 5a + 10b + 30c \quad ...(11)$$

$$195 = 10a + 30b + 100c \quad ...(12)$$

$$677 = 30a + 100b + 354c \quad ...(13)$$

Solving eqns. (11), (12) and (13), we get $a = 1.2, b = 1.1$ and $c = 1.5$

Hence the required second degree parabola is $y = 1.2 + 1.1x + 1.5x^2$.

Example 2. *Fit a second degree parabola to the following data taking y as dependent variable:*

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Sol. Normal equations to fit a second degree parabola of the form $y = a + bx + cx^2$ are

$$\left.\begin{aligned} \Sigma y &= ma + b\Sigma x + c\Sigma x^2 \\ \Sigma xy &= a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \\ \text{and} \quad \Sigma x^2 y &= a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \end{aligned}\right\} \quad ...(1)$$

Here, $m = 9$

x	y	x^2	x^3	x^4	xy	x^2y
1	2	1	1	1	2	2
2	6	4	8	16	12	24
3	7	9	27	81	21	63
4	8	16	64	256	32	128
5	10	25	125	625	50	250
6	11	36	216	1296	66	396
7	11	49	343	2401	77	539
8	10	64	512	4096	80	640
9	9	81	729	6561	81	729
$\Sigma x = 45$	$\Sigma y = 74$	$\Sigma x^2 = 285$	$\Sigma x^3 = 2025$	$\Sigma x^4 = 15333$	$\Sigma xy = 421$	$\Sigma x^2y = 2771$

Putting in (1), we get

$$74 = 9a + 45b + 285c$$

$$421 = 45a + 285b + 2025c$$

$$2771 = 285a + 2025b + 15333c$$

Solving the above equations, we get $a = -1, \quad b = 3.55, \quad c = -0.27$

Hence the required equation of second degree parabola is $y = -1 + 3.55x - 0.27x^2$.

Example 3. *Employ the method of least squares to fit a parabola $y = a + bx + cx^2$ in the data:* (x, y): $(-1, 2)$, $(0, 0)$, $(0, 1)$, $(1, 2)$

Sol. Normal equations to the parabola $y = a + bx + cx^2$ are

$$\Sigma y = ma + b\Sigma x + c\Sigma x^2 \quad ...(1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad ...(2)$$

and $$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad ...(3)$$

Here, $m = 4$. The table is as follows:

x	y	x^2	x^3	x^4	xy	x^2y
– 1	2	1	– 1	1	– 2	2
0	0	0	0	0	0	0
0	1	0	0	0	0	0
1	2	1	1	1	2	2
$\Sigma x = 0$	$\Sigma y = 5$	$\Sigma x^2 = 2$	$\Sigma x^3 = 0$	$\Sigma x^4 = 2$	$\Sigma xy = 0$	$\Sigma x^2y = 4$

Substituting these values in equations (1), (2) and (3); we get

$$5 = 4a + 2c \quad \ldots(4)$$

$$0 = 2b \quad \ldots(5)$$

and

$$4 = 2a + 2c \quad \ldots(6)$$

Solving (4), (5) and (6), we get $a = 0.5$, $b = 0$ and $c = 1.5$

Hence the required second degree parabola is $y = 0.5 + 1.5x^2$

Example 4. *Fit a second degree parabola to the following data by Least Squares method:*

x	*1*	*2*	*3*	*4*	*5*
y	*1090*	*1220*	*1390*	*1625*	*1915*

[U.P.T.U. (MCA) 2009, U.P.T.U. 2007; U.K.T.U. 2010]

Sol. Here $m = 5$ (odd)

Let $u = x - 3, \quad v = y - 1220$

x	y	u	v	u^2	u^2v	uv	u^3	u^4
1	1090	– 2	– 130	4	– 520	260	– 8	16
2	1220	– 1	0	1	0	0	– 1	1
3	1390	0	170	0	0	0	0	0
4	1625	1	405	1	405	405	1	1
5	1915	2	695	4	2780	1390	8	16
Total		$\Sigma u = 0$	$\Sigma v = 1140$	$\Sigma u^2 = 10$	$\Sigma u^2v = 2665$	$\Sigma uv = 2055$	$\Sigma u^3 = 0$	$\Sigma u^4 = 34$

Putting these values in normal equations, we get

$$1140 = 5a' + 10c', \quad 2055 = 10b', \quad 2655 = 10a' + 34c'$$

$$\Rightarrow \quad a' = 173, \quad b' = 205.5, \quad c' = 27.5$$

$$\therefore \quad v = 173 + 205.5u + 27.5u^2 \quad \ldots(1)$$

Put $u = x - 3$ and $v = y - 1220$

From (1), $y - 1220 = 173 + 205.5\,(x - 3) + 27.5\,(x - 3)^2$

$$\Rightarrow \quad y = 27.5x^2 + 40.5x + 1024.$$

3.48 MULTIPLE LINEAR REGRESSION

Now we proceed to discuss the case where the dependent variable is a function of two or more linear or non-linear independent variables. Consider such a linear function as

$$y = a + bx + cz \quad \ldots(1)$$

The sum of the squares of residual is

$$U = \sum_{i=1}^{n} (y_i - a - bx_i - cz_i)^2 \qquad \ldots(2)$$

Differentiating U partially w.r.t. a, b, c ; we get

$$\frac{\partial U}{\partial a} = 0 \Rightarrow 2\sum_{i=1}^{n} (y_i - a - bx_i - cz_i)\,(-1) = 0$$

$$\frac{\partial U}{\partial b} = 0 \Rightarrow 2\sum_{i=1}^{n} (y_i - a - bx_i - cz_i)\,(-x_i) = 0$$

and
$$\frac{\partial U}{\partial c} = 0 \Rightarrow 2\sum_{i=1}^{n} (y_i - a - bx_i - cz_i)\,(-z_i) = 0$$

which on simplification and omitting the suffix i, yields.

$$\Sigma y = ma + b\Sigma x + c\Sigma z$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma xz$$
$$\Sigma yz = a\Sigma z + b\Sigma xz + c\Sigma z^2$$

Solving the above three equations, we get values of a, b and c. Consequently, we get the linear function $y = a + bx + cz$ called **regression plane.**

EXAMPLES

Example 1. *Obtain a regression plane by using multiple linear regression to fit the data given below:*

x	1	2	3	4
z	0	1	2	3
y	12	18	24	30

[U.P.T.U. MCA (C.O.) 2008]

Sol. Let $y = a + bx + cz$ be the required regression plane where a, b, c are the constants to be determined by following equations:

$$\Sigma y = ma + b\Sigma x + c\Sigma z$$
$$\Sigma yx = a\Sigma x + b\Sigma x^2 + c\Sigma zx$$
and
$$\Sigma yz = a\Sigma z + b\Sigma zx + c\Sigma z^2$$

Here, $m = 4$

x	z	y	x^2	z^2	yx	zx	yz
1	0	12	1	0	12	0	0
2	1	18	4	1	36	2	18
3	2	24	9	4	72	6	48
4	3	30	16	9	120	12	90
$\Sigma x = 10$	$\Sigma z = 6$	$\Sigma y = 84$	$\Sigma x^2 = 30$	$\Sigma z^2 = 14$	$\Sigma yx = 240$	$\Sigma zx = 20$	$\Sigma yz = 156$

Substitution yields, $84 = 4a + 10b + 6c$

$240 = 10a + 30b + 20c$

and $156 = 6a + 20b + 14c$

Solving, we get $a = 10, b = 2, c = 4$

Hence the required regression plane is $y = 10 + 2x + 4z$

Example 2. *Find the multiple linear regression of X_1 on X_2 and X_3 from the data relating to three variables:*

X_1	4	6	7	9	13	15
X_2	15	12	8	6	4	3
X_3	30	24	20	14	10	4

Sol. Let $X_1 = a + bX_2 + cX_3$ be the required regression plane where a, b, c are the constants, determined by following normal equations

$$\Sigma X_1 = ma + b\Sigma X_2 + c\Sigma X_3$$
$$\Sigma X_1X_2 = a\Sigma X_2 + b\Sigma X_2^2 + c\Sigma X_2X_3$$
$$\Sigma X_1X_3 = a\Sigma X_3 + b\Sigma X_2X_3 + c\Sigma X_3^2$$

Here, $m = 6$

X_1	X_2	X_3	X_1X_2	X_2^2	X_2X_3	X_1X_3	X_3^2
4	15	30	60	225	450	120	900
6	12	24	72	144	288	144	576
7	8	20	56	64	160	140	400
9	6	14	54	36	84	126	196
13	4	10	52	16	40	130	100
15	3	4	45	9	12	60	16
Total 54	48	102	339	494	1034	720	2188

Substituting the values, we get

$$54 = 6a + 48b + 102c$$
$$339 = 48a + 102b + 1034c$$
$$720 = 102a + 1034b + 2188c$$

On solving, we get, $a = 16.413$, $b = -0.00536$, $c = -0.4335$

Hence $X_1 = 16.413 - 0.00536X_2 - 0.4335X_3$

ASSIGNMENT

1. Fit a parabola of the form $y = a + bx + cx^2$ to the data:

x	1	2	3	4
y	1.7	1.8	2.3	3.2

by the method of least squares. (*U.P.T.U. 2009*)

2. Find the best values of a_0, a_1, a_2 so that the parabola $y = a_0 + a_1x + a_2x^2$ fits the data:

x	1	1.5	2	2.5	3	3.5	4
y	1.1	1.2	1.5	2.6	2.8	3.3	4.1

[*U.P.T.U. (C.O.) 2008*]

3. (*i*) Fit a second degree parabola to the following data:

x	1	2	3	4	5
y	25	28	33	39	46

[*U.P.T.U. (C.O.) 2011*]

(*ii*) Fit a second degree parabola to the following data:

x	1	2	3	4	5	6	7	8	9	10
y	124	129	140	159	228	289	315	302	263	210

(*U.P.T.U. 2009*)

4. Fit a second degree parabola to the following data taking x as the independent variable:

(*i*)

x	0	1	2	3	4
y	1	5	10	22	38

(*ii*)

x	1	2	3	4	5	6	7	8	9
y	3	7	8	9	11	12	13	14	15

(*U.P.T.U. 2007*)

5. The profit of a certain company in X^{th} year of its life are given by:

x	1	2	3	4	5
y	1250	1400	1650	1950	2300

Taking $u = x - 3$ and $v = \dfrac{y - 1650}{50}$, show that the parabola of second degree of v on u is $v + 0.086 = 5.3\,u + 0.643u^2$ and deduce that the parabola of second degree of y on x is

$$y = 1144 + 72x + 32.15x^2$$

6. (*i*) The corresponding values of x and y are given below:

x	87	84	79	64	47	37
y	292	283	270	235	197	181

Fit a parabola of the form $y = ax^2 + bx + c$. Also find the value of y for $x = 80$ correct upto third place of decimal. (*U.P.T.U. 2006*)

(*ii*) Determine the constants a, b and c by the method of least squares such that $y = ax^2 + bx + c$ fits the following data:

x	2	4	6	8	10
y	4.01	11.08	30.12	81.89	222.62

7. The velocity V of a liquid is known to vary with temperature T, according to a quadratic law $V = a + bT + cT^2$. Find the best values of a, b and c for the following table:

T	1	2	3	4	5	6	7
V	2.31	2.01	1.80	1.66	1.55	1.47	1.41

[*G.B.T.U. (MCA) 2010*]

8. The following table gives the results of the measurements of train resistances, V is the velocity in miles per hour, R is the resistance in pounds per ton:

V	20	40	60	80	100	120
R	5.5	9.1	14.9	22.8	33.3	46

If R is related to V by the relation $R = a + bV + cV^2$; find a, b and c by using the method of least squares.

9. Find the multiple linear regression of X_1 on X_2 and X_3 from the data relating to three variables:

X_1	7	12	17	20
X_2	4	7	9	12
X_3	1	2	5	8

(*U.P.T.U. 2009*)

10. Fit a second degree parabola to the following data:

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	8	13	5

(*U.P.T.U. 2015*)

Answers

1. $y = 2 - 0.5\,x + 0.2\,x^2$ **2.** $y = 0.45714 + 0.39286\,x + 0.12857\,x^2$

3. (*i*) $y = 22.8 + 1.44x + 0.64x^2$ (*ii*) $y = 18.866 + 66.1576\,x - 4.3333\,x^2$

4. (*i*) $y = 1.43 + 0.24x + 2.21x^2$ (*ii*) $y = 1.5238 + 2.38398\,x - 0.10173\,x^2$

6. (*i*) $y = 0.010626822\,x^2 + 0.908257322\,x + 132.2040143$; 272.876

(*ii*) $a = 5.358035714$, $b = -38.89492857$, $c = 67.56$

7. $V = 2.5928 - 0.3258\,T + 0.02274\,T^2$ **8.** $R = 4.35 + 0.00241\,V + 0.0028705\,V^2$

9. $X_1 = 0.6441 + 1.661X_2 + 0.0169\,X_3$ **10.** $y = -1.619 + 4.031\,x - 0.339\,x^2$.

3.49 THEORETICAL PROBABILITY DISTRIBUTIONS

Generally, frequency distribution are formed from the observed or experimental data. However, frequency distribution of certain populations can be deduced mathematically by fitting theoretical probability distribution under certain assumptions.

Frequency distributions can be classified under two heads:

(*i*) Observed Frequency Distributions.

(*ii*) Theoretical or Expected Frequency Distributions.

Observed frequency distributions are based on actual observation and experimentation. If certain hypothesis is assumed, it is sometimes possible to derive mathematically what the frequency distribution of certain universe should be. Such distributions are called **Theoretical Distributions**.

Theoretical probability distributions are of two types:

(*i*) Discrete probability distribution. Binomial, poisson, geometric, negative binomial, hypergeometric, multinomial, multivariate hypergeometric distributions.

(*ii*) Continuous probability distributions.

Uniform, normal Gamma, exponential, χ^2, Beta, bivariate normal, t, F-distributions.

Here, we will study three important theoretical probability distributions:

1. Binomial Distribution (or Bernoulli's Distribution)
2. Poisson's Distribution
3. Normal Distribution.

3.50 BINOMIAL PROBABILITY DISTRIBUTION [*G.B.T.U. 2010, 2013*]

It was discovered by a Swiss Mathematician Jacob James Bernoulli in the year 1700.

This distribution is concerned with trials of a repretitive nature in which only the occurrence or non-occurrence, success or failure, acceptance or rejection, yes or no of a particular event is of interest.

For convenience, we shall call the occurrence of the event 'a success' and its non-occurrence 'a failure'.

Let there be n independent trials in an experiment. Let a random variable X denote the number of successes in these n trials. Let p be the probability of a success and q that of a failure in a single trial so that $p + q = 1$. Let the trials be independent and p be constant for every trial.

Let us find the probability of r successes in n trials.

r successes can be obtained in n trials in ${}^nC_r$ ways.

$$\therefore \quad P(X = r) = {}^nC_r P(\underbrace{S\,S\,S \ldots\ldots S}_{r \text{ times}} \quad \underbrace{F\,F\,F \ldots\ldots F}_{(n-r) \text{ times}})$$

$$= {}^nC_r \underbrace{P(S)\,P(S) \ldots\ldots P(S)}_{r \text{ factors}} \quad \underbrace{P(F)\,P(F) \ldots\ldots P(F)}_{(n-r) \text{ factors}}$$

$$= {}^nC_r \underbrace{p\;p\;p \ldots\ldots p}_{r \text{ factors}} \quad \underbrace{q\;q\;q \ldots\ldots q}_{(n-r) \text{ factors}}$$

$$= {}^nC_r p^r q^{n-r} \qquad \ldots(1)$$

Hence $\boxed{\mathbf{P(X = r) = {}^nC_r\, p^r\, q^{n-r}}}$ where **p + q = 1 and r = 0, 1, 2, ..., n.**

The distribution (1) is called the *binomial probability distribution* and X is called the *binomial variate*.

Note 1. $P(X = r)$ is usually written as $P(r)$.

Note 2. The successive probabilities $P(r)$ in (1) for $r = 0, 1, 2, \ldots\ldots, n$ are

$${}^nC_0 q^n,\ {}^nC_1 q^{n-1} p,\ {}^nC_2 q^{n-2} p^2,\ \ldots\ldots,\ {}^nC_n p^n$$

which are the successive terms of the binomial expansion of $(q + p)^n$. That is why this distribution is called "binomial" distribution.

Note 3. n and p occurring in the binomial distribution are called the *parameters* of the distribution.

Note 4. In a binomial distribution:

(*i*) n, the number of trials is finite.

(*ii*) each trial has only two possible outcomes usually called success and failure.

(*iii*) all the trials are independent.

(*iv*) p (and hence q) is constant for all the trials.

3.51 RECURRENCE OR RECURSION FORMULA FOR THE BINOMIAL DISTRIBUTION

In a binomial distribution,

$$P(r) = {}^nC_r\, q^{n-r} p^r = \frac{n!}{(n-r)!\,r!}\, q^{n-r} p^r$$

$$P(r+1) = {}^nC_{r+1}\, q^{n-r-1} p^{r+1} = \frac{n!}{(n-r-1)!\,(r+1)!}\, q^{n-r-1} p^{r+1}$$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{(n-r)!}{(n-r-1)!} \times \frac{r!}{(r+1)!} \times \frac{p}{q}$$

$$= \frac{(n-r)\times(n-r-1)!}{(n-r-1)!} \times \frac{r!}{(r+1)\times r!} \times \frac{p}{q} \times = \left(\frac{n-r}{r+1}\right) . \frac{p}{q}$$

$$\Rightarrow \quad \boxed{P(r+1) = \frac{n-r}{r+1} . \frac{p}{q}\ P(r)}$$

which is the required recurrence formula. Applying this formula successively, we can find P(1), P(2), P(3), ..., if P(0) is known.

3.52 MEAN AND VARIANCE OF THE BINOMIAL DISTRIBUTION

[*U.P.T.U. 2008, G.B.T.U. 2012*]

For the binomial distribution, $P(r) = {}^nC_r\, q^{n-r} p^r$

$$\text{Mean } \mu = \sum_{r=0}^{n} rP(r) = \sum_{r=0}^{n} r\ .\ {}^nC_r q^{n-r} p^r$$

$$= 0 + 1\ .\ {}^nC_1\, q^{n-1} p + 2\ .\ {}^nC_2\, q^{n-2} p^2 + 3\ .\ {}^nC_3\, q^{n-3} p^3 + \ldots\ldots + n\ .\ {}^nC_n\, p^n$$

$$= nq^{n-1} p + 2\ .\ \frac{n(n-1)}{2\,.\,1}\, q^{n-2}p^2 + 3\ .\ \frac{n(n-1)(n-2)}{3\,.\,2\,.\,1}\, q^{n-3} p^3 + \ldots\ldots + n.p^n$$

$$= nq^{n-1}p + n(n-1)\, q^{n-2}p^2 + \frac{n(n-1)(n-2)}{2\,.\,1}\, q^{n-3}p^3 + \ldots\ldots + np^n$$

$$= np[{}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 q^{n-2}p + {}^{n-1}C_2 q^{n-3}p^2 + \ldots + {}^{n-1}C_{n-1} p^{n-1}]$$

$$= np(q+p)^{n-1} = np \qquad (\because \quad p + q = 1)$$

Hence the mean of the binomial distribution in np.

$$\text{Variance } \sigma^2 = \sum_{r=0}^{n} r^2 P(r) - \mu^2 = \sum_{r=0}^{n} [r + r(r-1)]\ P(r) - \mu^2$$

$$= \sum_{r=0}^{n} rP(r) + \sum_{r=0}^{n} r(r-1)\ P(r) - \mu^2 = \mu + \sum_{r=2}^{n} r(r-1)\ {}^nC_r q^{n-r} p^r - \mu^2$$

(since the contribution due to $r = 0$ and $r = 1$ is zero)

$= \mu + [2\,.1\,.\,{}^nC_2 q^{n-2}p^2 + 3.2\,.\,{}^nC_3 q^{n-3}p^3 + \ldots + n(n-1)\,{}^nC_n p^n] - \mu^2$

$= \mu + \left[2\,.\,1\,.\,\dfrac{n(n-1)}{2\,.\,1}\,q^{n-2}p^2 + 3\,.\,2\,.\,\dfrac{n(n-1)(n-2)}{3\,.\,2\,.\,1}\,q^{n-3}p^3 + \ldots + n(n-1)p^n\right] - \mu^2$

$= \mu + [n(n-1)q^{n-2}p^2 + n(n-1)(n-2)q^{n-3}p^3 + \ldots + n(n-1)p^n] - \mu^2$

$= \mu + n(n-1)\,p^2[q^{n-2} + (n-2)q^{n-3}p + \ldots + p^{n-2}] - \mu^2$

$= \mu + n\,(n-1)p^2\,[{}^{n-2}C_0 q^{n-2} + {}^{n-2}C_1 q^{n-3}p + \ldots + {}^{n-2}C_{n-2}p^{n-2}] - \mu^2$

$= \mu + n(n-1)p^2(q+p)^{n-2} - \mu^2 = \mu + n(n-1)p^2 - \mu^2$ $\quad [\because \; q + p = 1]$

$= np + n(n-1)p^2 - n^2p^2 = np[1-p] = npq.$ $\quad [\because \; \mu = np]$

Hence the variance of the binomial distribution is npq.

Standard deviation of the binomial distribution is $\sqrt{npq}$.

3.53 MOMENT GENERATING FUNCTION OF BINOMIAL DISTRIBUTION

1. About origin

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx}\;{}^nC_x\;p^x\;q^{n-x} = \sum_{x=0}^{n} {}^nC_x\;(pe^t)^x\;q^{n-x} = (q + pe^t)^n$$

2. About mean *[G.B.T.U. 2012, U.P.T.U. 2008, 2015]*

$$M_{x-np}(t) = E[e^{t(x-np)}]$$
$$= e^{-npt}\,E(e^{tx}) = e^{-npt}\,M_x(t) = e^{-npt}\,(q + pe^t)^n$$
$$= (qe^{-pt} + pe^{t-pt})^n = (qe^{-pt} + pe^{qt})^n \qquad |\because \; 1 - p = q$$

3.54 MOMENTS ABOUT MEAN OF BINOMIAL DISTRIBUTION

$$M_{x-np}(t) = (qe^{-pt} + pe^{qt})^n$$

$$= \left[q\left(1 - pt + \frac{p^2t^2}{2!} - \frac{p^3t^3}{3!} + \ldots\right) + p\left(1 + qt + \frac{q^2t^2}{2!} + \frac{q^3t^3}{3!} + \ldots\right)\right]^n$$

$$= \left[(q+p) + \frac{t^2}{2!}\,pq\,(q+p) + \frac{t^3}{3!}\,pq\,(q^2 - p^2) + \frac{t^4}{4!}\,pq\,(q^3 + p^3) + \ldots\right]^n$$

$$= \left[1 + \left\{\frac{t^2}{2!}\,.\,pq + \frac{t^3}{3!}\,pq\,(q-p) + \frac{t^4}{4!}\,qp\,(1 - 3pq) + \ldots\right\}\right]^n$$

$$= \left[1 + {}^nC_1\left\{\frac{t^2}{2!}\,.\,pq + \frac{t^3}{3!}\,pq\,(q-p) + \frac{t^4}{4!}\,pq\,(1-3pq) + \ldots\right\}\right.$$
$$\left. + {}^nC_2\left\{\frac{t^2}{2!}\,.\,pq + \frac{t^3}{3!}\,pq\,(q-p) + \ldots\right\}^2 + \ldots\right]$$

Now,

$$\mu_2 = \text{coefficient of } \frac{t^2}{2!} = npq$$

$$\mu_3 = \text{coefficient of } \frac{t^3}{3!} = npq\,(q - p)$$

$$\mu_4 = \text{coefficient of } \frac{t^4}{4!} = npq\,(1 - 3\,pq) + 3n(n - 1)\,p^2q^2$$

$$= 3n^2p^2q^2 + npq\,(1 - 6pq)$$

Hence,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq}$$

$$\therefore \qquad \gamma_1 = \frac{1-2p}{\sqrt{npq}}$$

and

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq}$$

$$\therefore \qquad \gamma_2 = \frac{1-6pq}{npq}$$

Note 1. $\gamma_1 = \dfrac{1-2p}{\sqrt{npq}}$ gives a **measure of skewness** of the binomial distribution. If $p < \frac{1}{2}$, skewness is positive, if $p > \frac{1}{2}$, skewness is negative and if $p = \frac{1}{2}$, it is zero.

$\beta_2 = 3 + \dfrac{1-6pq}{npq}$ gives a **measure of the kurtosis** of the binomial distribution.

Note 2. If n independent trials constitute one experiment and this experiment is repeated N times then the frequency of r successes is $N \cdot {}^nC_r\, p^r\, q^{n-r}$.

3.55 APPLICATIONS OF BINOMIAL DISTRIBUTION

1. In problems concerning no. of defectives in a sample production line.
2. In estimation of reliability of systems.
3. No. of rounds fired from a gun hitting a target.
4. In Radar detection.

EXAMPLES

Example 1. (*i*) *Comment on the following statement:*

For a Binomial distribution, mean is 6 and variance is 9.

(*ii*) *A die is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and variance of the number of success.*

Sol. (*i*) $\mu = np = 6$ ...(1)

$\sigma^2 = npq = 9$ ...(2)

Dividing (2) by (1), we get

$$q = \frac{9}{6} = 1.5$$

which is impossible as $0 \le q \le 1$

$\therefore$ The above statement is **False.**

(*ii*) Prob. of getting success (1 or 6) on a toss $= \frac{2}{6} = \frac{1}{3} = p$

$\therefore \quad q = 1 - \frac{1}{3} = \frac{2}{3}$

No. of tosses of a die, $n = 3$

(*i*) Mean $= np = 3\left(\frac{1}{3}\right) = 1.$ (*ii*) Variance $= npq = (3)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2}{3}.$

Example 2. *If 10% of the bolts produced by a machine are defective, determine the probability that out of 10 bolts choosen at random*

(*i*) *1* (*ii*) *None* (*iii*) *at most 2 bolts will be defective.*

Sol. Here, $p(\text{defective}) = \frac{10}{100} = \frac{1}{10}$ (given)

$\therefore \quad q(\text{non-defective}) = 1 - \frac{1}{10} = \frac{9}{10}$

Also, $n = 10$, (n is no. of bolts chosen). (given)

The probability of r defective bolts out of n bolts chosen at random is given by

$$P(r) = {}^nC_r\, p^r\, q^{n-r} \quad ...(1)$$

(*i*) Here $r = 1$,

$$\therefore \quad P(1) = {}^{10}C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^{10-1} \quad | \text{ Using (1)}$$

$$= 10\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^9 = (.9)^9 = 0.3874 \quad ...(2)$$

(*ii*) Here $r = 0$

$$\therefore \quad P(0) = {}^{10}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{10-0} = \left(\frac{9}{10}\right)^{10} = 0.3486 \quad ...(3) \;| \text{ Using (1)}$$

(*iii*) Prob. that at most 2 bolts will be defective $= P(r \le 2) = P(0) + P(1) + P(2)$ ...(4)

$$\text{Now,} \quad P(2) = {}^{10}C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{10-2} \quad | \text{ Using (1)}$$

$$= 45\left(\frac{1}{100}\right)(0.43046) = 0.1937$$

$\therefore$ From (4), Required Probability $= P(0) + P(1) + P(2)$

$= 0.3486 + 0.3874 + 0.1937 = 0.9297.$

Example 3. *A binomial variable X satisfies the relation 9P* ($X = 4$) $= P(X = 2)$ *when* $n = 6$. *Find the value of the parameter p and* $P(X = 1)$.

Sol. We know that

$$P(X = r) = {}^nC_r\, p^r\, q^{n-r} \quad ...(1)$$

$$\therefore \quad P(X = 4) = {}^6C_4\, p^4\, q^2 = 15p^4q^2$$

and $P(X = 2) = {}^6C_2 p^2\, q^4 = 15\, p^2\, q^4$ | Since $n = 6$

The given relation is

$$9\, P(X = 4) = P(X = 2) \quad \Rightarrow \quad 9(15p^4\, q^2) = 15p^2q^4$$

$\Rightarrow \quad 9p^2 = q^2 = (1-p)^2 \qquad | \because \; p + q = 1$

$\Rightarrow \quad 9p^2 = 1 + p^2 - 2p$

$\Rightarrow \quad 8p^2 + 2p - 1 = 0 \quad \Rightarrow \quad (4p-1)(2p+1) = 0$

$\therefore \quad \boxed{p = \frac{1}{4}} \qquad | \because \; p$ cannot be negative

Now, $\quad P(X=1) = {}^6C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^5 = .3559. \qquad \left| \because \; q = 1 - p = \frac{3}{4}\right.$

Example 4. *Fit a binomial distribution to the following frequency data:*

x :	*0*	*1*	*2*	*3*	*4*
f :	*30*	*62*	*46*	*10*	*2*

Sol. The table is as follows:

x	*f*	*fx*
0	30	0
1	62	62
2	46	92
3	10	30
4	2	8
	$\Sigma f = 150$	$\Sigma fx = 192$

Mean of observations $= \frac{\Sigma fx}{\Sigma f} = \frac{192}{150} = 1.28$

$\Rightarrow \quad np = 1.28$

$\Rightarrow \quad 4p = 1.28 \qquad$ (n is no. of trials)

$\Rightarrow \quad p = 0.32$

$\therefore \quad q = 1 - p = 1 - 0.32 = 0.68$

Also, $\quad N = 150 \qquad | \because \; N = \Sigma f$

Hence the binomial distribution is $= N(q+p)^n = 150\,(0.68 + 0.32)^4$.

Example 5. *A student is given a true-false examination with 8 questions. If he corrects at least 7 questions, he passes the examination. Find the probability that he will pass given that he guesses all questions.*

Sol. Here, n = no. of questions asked = 8

$$p = \frac{1}{2}, q = \frac{1}{2} \qquad | \text{ Since the question can either be true or false}$$

Probability that he will pass

$$= P(r \geq 7) = P(7) + P(8)$$

$$= {}^8C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{8-7} + {}^8C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{8-8} = 8\left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^1 + 1 \cdot \left(\frac{1}{2}\right)^8$$

$$= \left(\frac{1}{2}\right)^8 (8+1) = \frac{9}{256} = .03516.$$

Example 6. *During war, 1 ship out of 9 was sunk on an average in making a certain voyage. What was the probability that exactly 3 out of a convoy of 6 ships would arrive safely?*

Sol. p, the probability of a ship arriving safely $= 1 - \frac{1}{9} = \frac{8}{9}$; $q = \frac{1}{9}$, $n = 6$

The probability that exactly 3 ships arrive safely $= P(r = 3) = {}^6C_3 \left(\frac{1}{9}\right)^3 \left(\frac{8}{9}\right)^3 = \frac{10240}{9^6}$.

Example 7. *A policeman fires 6 bullets on a dacoit. The probability that the dacoit will be killed by a bullet is 0.6. What is the probability that dacoit is still alive?*

Sol. Here n = no. of bullets fired = 6, $p = 0.6$, $q = 1 - p = 0.4$

Probability that dacoit is still alive (not killed)

$$= P(r = 0) = {}^nC_0\, p^0\, q^{n-0} = {}^6C_0\, (.6)^0\, (.4)^6 = (.4)^6 = .004096.$$

Example 8. *If the probability of hitting a target is 10% and 10 shots are fired independently. What is the probability that the target will be hit at least once?*

Sol. Here, $p = \frac{10}{100} = \frac{1}{10}$, $q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$, $n = 10$

Probability that the target will be hit at least once

$$= P(r \geq 1) = 1 - P(r = 0)$$

$$= 1 - [{}^nC_0\, p^0\, q^n] = 1 - \left[{}^{10}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{10}\right] = 0.6513.$$

Example 9. *Out of 800 families with 4 children each, how many families would be expected to have* (*i*) *2 boys and 2 girls* (*ii*) *at least one boy* (*iii*) *no girl* (*iv*) *atmost two girls? Assume equal probabilities for boys and girls.* (U.P.T.U. 2014)

Sol. Since probabilities for boys and girls are equal,

p = probability of having a boy $= \frac{1}{2}$; q = probability of having a girl $= \frac{1}{2}$

$n = 4$ N = 800

(*i*) The expected number of families having 2 boys and 2 girls

$$= 800\ {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 800 \times 6 \times \frac{1}{16} = 300.$$

(*ii*) The expected number of families having *at least one boy*

$$= 800 \left[{}^4C_1 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) + {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 + {}^4C_3 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4 \left(\frac{1}{2}\right)^4\right]$$

$$= 800 \times \frac{1}{16}\ [4 + 6 + 4 + 1] = 750$$

(*iii*) The expected number of families having no girl *i.e.*, having 4 boys

$$= 800\ .\ {}^4C_4 \left(\frac{1}{2}\right)^4 = 50.$$

(*iv*) The expected number of families having *atmost two girls i.e.*, having *at least 2 boy*

$$= 800 \left[{}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 + {}^4C_3 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4 \left(\frac{1}{2}\right)^4\right] = 800 \times \frac{1}{16}\ [6 + 4 + 1] = 550.$$

Example 10. *Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six?*

Sol. p = the chance of getting 5 or 6 with one die $= \frac{2}{6} = \frac{1}{3}$

$$q = 1 - \frac{1}{3} = \frac{2}{3},\ n = 6,\ \text{N} = 729$$

since dice are in sets of 6 and there are 729 sets.

The expected number of times *at least three* dice showing five or six

$$= \text{N} \,.\, \text{P}(r \geq 3)$$

$$= 729\,[\text{P}(3) + \text{P}(4) + \text{P}(5) + \text{P}(6)]$$

$$= 729 \left[{}^6\text{C}_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^3 + {}^6\text{C}_4 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^4 + {}^6\text{C}_5 \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5 + {}^6\text{C}_6 \left(\frac{1}{3}\right)^6 \right]$$

$$= \frac{729}{3^6}\,[160 + 60 + 12 + 1] = 233.$$

Example 11. *The probability of a man hitting a target is $\frac{1}{3}$. How many times must he fire so that the probability of his hitting the target at least once is more than 90%?*

Sol. $$p = \frac{1}{3}$$

The probability of not hitting the target in n trials is q^n.

Therefore, to find the smallest n for which the probability of hitting at least once is more than 90%, we have

$$1 - q^n > 0.9$$

$$\Rightarrow \quad 1 - \left(\frac{2}{3}\right)^n > 0.9$$

$$\Rightarrow \quad \left(\frac{2}{3}\right)^n < 0.1$$

The smallest n for which the above inequality holds true is 6 hence he must fire 6 times.

Example 12. *In a bombing action, there is 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completely. How many bombs are required to be dropped to give a 99% chance or better of completely destroying the target?*

Sol. We have, $p = \frac{50}{100} = \frac{1}{2}$

Since the probability must be greater than 0.99, if n bombs are dropped, we have

$${}^n\text{C}_2 \left(\frac{1}{2}\right)^n + {}^n\text{C}_3 \left(\frac{1}{2}\right)^n + {}^n\text{C}_4 \left(\frac{1}{2}\right)^n + \ldots\ldots + {}^n\text{C}_n \left(\frac{1}{2}\right)^n \geq 0.99$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^n [{}^n\text{C}_2 + {}^n\text{C}_3 + {}^n\text{C}_4 + \ldots\ldots + {}^n\text{C}_n] \geq 0.99$$

$$\frac{2^n - n - 1}{2^n} \geq 0.99$$

$$\Rightarrow \qquad 1 - \frac{1+n}{2^n} \geq 0.99$$

$$\Rightarrow \qquad \frac{1+n}{2^n} \leq 0.01$$

$$\Rightarrow \qquad 2^n \geq 100n + 100$$

By trial, $n = 11$ satisfies the inequality.

Hence 11 bombs are required to be dropped.

ASSIGNMENT

1. (*i*) Ten coins are tossed simultaneously. Find the probability of getting at least seven heads.

(*ii*) A die is thrown five times. If getting an odd number is a success, find the probability of getting at least four successes. (*M.T.U. 2012*)

2. (*a*) The probability of any ship of a company being destroyed on a certain voyage is 0.02. The company owns 6 ships for the voyage. What is the probability of :

(*i*) losing one ship (*ii*) losing atmost two ships (*iii*) losing none?

(*b*) Assume that on the average one telephone number out of fifteen called between 2 P.M. and 3 P.M. on week-days is busy. What is the probability that if 6 randomly selected telephone numbers are called (*i*) not more than 3 (*ii*) at least 3 of them will be busy?

3. (*i*) The incidence of occupational disease in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of six workers chosen at random, four or more will suffer from the disease?

(*ii*) The probability that a man aged 60 will live to be 70 is 0.65. What is the probability that out of 10 men, now 60, at least 7 will live to be 70?

4. (*i*) If the mean of a binomial distribution is 3 and the variance is $\frac{3}{2}$, find the probability of obtaining at least 4 successes.

(*ii*) In a binomial distribution, for $n = 5$ if $P(x = 1) = 0.4096$ and $P(x = 2) = 0.2048$, then find the value of p.

(*iii*) The sum and product of the mean and variance of a binomial distribution are $\frac{25}{3}$ and $\frac{50}{3}$ respectively. Find the distribution. (*U.P.T.U. 2007*)

(*iv*) If the probability of a defective bolt is 0.1, find (*a*) The mean (*b*) The standard deviation for the distribution in a total of 400 bolts.

(*v*) If the moment generating function of a random variable X is $\left(\frac{1}{3} + \frac{2}{3}e^t\right)^5$, find $P(X = 2)$.

5. (*a*) The probability that a bomb dropped from a plane will strike the target is $\frac{1}{5}$. If six bombs are dropped, find the probability that (*i*) exactly two will strike the target, (*ii*) at least two will strike the target.

(*b*) Four persons in a group of 20 are graduates. If 4 persons are selected at random from 20, find the probability that

(*i*) all are graduates (*ii*) at least one is a graduate.

6. A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn, one by one, with replacement, what is the probability that

(*i*) none is white (*ii*) all are white

(*iii*) at least one is white (*iv*) only 2 are white?

7. (*i*) In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles ?

(*ii*) If on an average one ship in every ten is wrecked, find the probability that out of 5 ships expected to arrive, 4 at least will arrive safely.

8. The prob. that a bulb produced by a factory will fuse after a use of 150 days is 0.05. Find the prob. that out of 5 such bulbs [*M.T.U.* (*B. Pharma*) *2011*]

(*i*) None (*ii*) Atmost one

(*iii*) More than one (*iv*) At least one will fuse after 150 days of use.

9. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(*i*) all the five cards are spades (*ii*) only three are spades (*iii*) none is spade?

10. Manish takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of 11 steps, he is one step away from the starting point.

[**Hint.** Manish will take either 6 steps forward and 5 backward or 5 steps forward and 6 backward].

11. (*a*) In 800 families with 5 children each, how many families would be expected to have (*i*) 3 boys and 2 girls, (*ii*) 2 boys and 3 girls, (*iii*) no girl (*iv*) at the most two girls. (Assume probabilities for boys and girls to be equal.)

(*b*) Out of 800 families with 5 children each, how many would you expect to have (*i*) 3 boys (*ii*) 5 girls (*iii*) either 2 or 3 boys? Assume equal probabilities for boys and girls. (*M.T.U. 2012*)

(*c*) Out of 320 families with 5 children each, what percentage would be expected to have (*i*) 2 boys and 3 girls (*ii*) at least one boy? Assume equal probability for boys and girls. (*G.B.T.U. 2011*)

12. In 100 sets of ten tosses of an unbiased coin, in how many cases do you expect to get

(*i*) 7 heads and 3 tails (*ii*) at least 7 heads?

13. The following data are the number of seeds germinating out of 10 on damp filter for 80 sets of seeds. Fit a binomial distribution to this data:

x :	0	1	2	3	4	5	6	7	8	9	10	Total
f :	6	20	28	12	8	6	0	0	0	0	0	80

[**Hint.** Here $n = 10$, N = 80, Mean = $\frac{\Sigma fx}{\Sigma f} = 2.175 \quad \therefore \quad np = 2.175$ etc.]

14. Fit a binomial distribution for the following data and compare the theoretical frequencies with the actual ones :

x :	0	1	2	3	4	5
f :	2	14	20	34	22	8

15. Fit a binomial distribution to the data given in the following table:

(*i*)

x :	0	1	2	3	4
f :	24	41	28	5	2

(*M.T.U. 2012*)

(*ii*)

x :	0	1	3	4
f :	28	62	10	4

(*U.K.T.U. 2011*)

16. (*i*) Assuming half the population of a town consumes chocolates so that the chance of an individual being consumer is $\frac{1}{2}$ and that 100 investigators each take 10 individuals to see whether they are consumers, how many investigators would you expect to report that 3 people or less were consumers? (*U.P.T.U. 2006*) (*ii*) Assuming that 20% of the population of a city are literate, so that the chance of an individual being literate is $\frac{1}{5}$ and assuming that 100 investigators each take 10 individuals to see whether they are literate, how many investigators would you expect to report 3 or less were literate?

17. Following results were obtained when 100 batches of seeds were allowed to germinate on damp filter paper in a laboratory : $\beta_1 = \frac{1}{15}, \beta_2 = \frac{89}{30}$. Determine the Binomial distribution. Calculate the expected frequency for $x = 8$ assuming $p > q$.

18. A coffee connoisseur claims that he can distinguish between a cup of instant coffee and a cup of percolator coffee 75% of the time. It is agreed that his claim will be accepted if he correctly identifies at least 5 of the 6 cups. Find his chances of having the claim (*i*) accepted (*ii*) rejected when he does have the ability he claims.

19. A multiple-choice test consists of 8 questions with 3 answers to each question of which only one is correct. A student answers each question by rolling a balanced die and checking the first answer if he gets 1 or 2, the second answer if he gets 3 or 4 and the third answer if he gets 5 or 6. To get a distinction, the student must secure at least 75% correct answers. If there is no negative marking, what is the probability that the student secures a distinction?

20. An irregular six-faced die is thrown and the expectation that in 10 throws, it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws each, would you expect it to give no even number?

Answers

1. (*i*) $\frac{11}{64}$ (*ii*) $\frac{3}{16}$

2. (*a*) (*i*) 0.1085, (*ii*) 0.9997, (*iii*) 0.8858 (*b*) (*i*) 0.9997 (*ii*) 0.005

3. (*i*) $\frac{53}{3125}$ (*ii*) 0.5137

4. (*i*) $\frac{11}{32}$, (*ii*) $\frac{1}{5}$, (*iii*) $\left(\frac{2}{3}+\frac{1}{3}\right)^{15}$

(*iv*) (*a*) 40 (*b*) 6 (*v*) 0.1646

5. (*a*) (*i*) 0.246 (*ii*) 0.345 (*b*) (*i*) 0.0016 (*ii*) 0.5904

6. (*i*) $\frac{81}{256}$, (*ii*) $\frac{1}{256}$, (*iii*) $\frac{175}{256}$, (*iv*) $\frac{27}{128}$ **7.** (*i*) $\frac{5}{2}\left(\frac{5}{6}\right)^9$, (*ii*) 0.91854

8. (*i*) $\left(\frac{19}{20}\right)^5$ (*ii*) $\frac{6}{5}\left(\frac{19}{20}\right)^4$ (*iii*) $1-\frac{6}{5}\left(\frac{19}{20}\right)^4$ (*iv*) $1-\left(\frac{19}{20}\right)^5$

9. (*i*) $\left(\frac{1}{4}\right)^5$ (*ii*) $90\left(\frac{1}{4}\right)^5$ (*iii*) $\left(\frac{3}{4}\right)^5$ **10.** 0.36787

11. (*a*) (*i*) 250 (*ii*) 250 (*iii*) 25 (*iv*) 400

(*b*) (*i*) 250 (*ii*) 25 (*iii*) 500

(*c*) (*i*) 31.25% (*ii*) 96.875%

12. (*i*) 12 nearly (*ii*) 17 nearly **13.** $80\,(0.7825 + 0.2175)^{10}$

14. $100\,(0.432 + 0.568)^5$ **15.** (*i*) $100\,(0.7 + 0.3)^4$, (*ii*) $104\,(0.7404 + 0.2596)^4$

16. (*i*) 17 (*ii*) 88 **17.** $100\left(\frac{1}{4}+\frac{3}{4}\right)^{20}$, 0.075168752

18. (*i*) 0.534 (*ii*) 0.466

19. 0.0197 **20.** 1.

POISSON DISTRIBUTION

3.56 POISSON DISTRIBUTION AS A LIMITING CASE OF BINOMIAL DISTRIBUTION

[*U.P.T.U. 2007 ; G.B.T.U. 2010, 2013 ; M.T.U. 2013, 2014*]

Poisson distribution was discovered by S.D. Poisson in the year 1837.

If the parameters n and p of a binomial distribution are known, we can find the distribution. But in situations where n is very large and p is very small, application of binomial distribution is very labourious. However, if we assume that as $n \to \infty$ and $p \to 0$ such that np always remains finite, say λ, we get the Poisson approximation to the binomial distribution.

Now, for a binomial distribution

$$P(X = r) = {}^nC_r q^{n-r} p^r$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \times (1-p)^{n-r} \times p^r$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \times \left(1-\frac{\lambda}{n}\right)^{n-r} \times \left(\frac{\lambda}{n}\right)^r \qquad \Big| \text{Since } np = \lambda \quad \therefore \quad p = \frac{\lambda}{n}$$

$$= \frac{\lambda^r}{r!} \times \frac{n(n-1)(n-2)\ldots(n-r+1)}{n^r} \times \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\ldots\ldots\left(\frac{n-r+1}{n}\right) \times \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{r-1}{n}\right) \times \frac{\left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^r}$$

As $n \to \infty$, each of the $(r-1)$ factors

$$\left(1-\frac{1}{n}\right), \left(1-\frac{2}{n}\right), \ldots\ldots, \left(1-\frac{r-1}{n}\right) \quad \text{tends to 1. Also } \left(1-\frac{\lambda}{n}\right)^r \text{ tends to 1.}$$

Since $\underset{x \to \pm\infty}{\text{Lt}} \left(1+\frac{1}{x}\right)^x = e$, the Naperian base. $\therefore \quad \left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda} \longrightarrow e^{-\lambda}$ as $n \to \infty$

Hence in the limiting case when $n \to \infty$, we have

$$\boxed{P(X = r) = \frac{e^{-\lambda}.\lambda^r}{r\,!} \quad (r = 0, 1, 2, 3, ...)} \qquad ...(1)$$

where λ is a finite number $= np$.

(1) represents a probability distribution which is called the *Poisson probability distribution*.

Note 1. λ is called the parameter of the distribution.

Note 2. The sum of the probabilities $P(r)$ for $r = 0, 1, 2, 3, ...$ is 1, since

$$P(0) + P(1) + P(2) + P(3) + ... = e^{-\lambda} + \frac{\lambda e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2\,!} + \frac{\lambda^3 e^{-\lambda}}{3\,!} + ...$$

$$= e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2\,!} + \frac{\lambda^3}{3\,!} + ...\right) = e^{-\lambda} \,.\, e^{\lambda} = 1.$$

3.57 RECURRENCE FORMULA FOR THE POISSON DISTRIBUTION

[*U.P.T.U. (B. Pharma.) 2009*]

For Poisson distribution, $P(r) = \dfrac{e^{-\lambda}\,\lambda^r}{r\,!}$ and $P(r+1) = \dfrac{e^{-\lambda}\,\lambda^{r+1}}{(r+1)\,!}$

$$\therefore \qquad \frac{P(r+1)}{P(r)} = \frac{\lambda r\,!}{(r+1)\,!} = \frac{\lambda}{r+1}$$

or $\qquad P(r+1) = \dfrac{\lambda}{r+1}\, P(r), \; r = 0, 1, 2, 3, ...$

This is called the *recurrence* or *recursion formula* for the Poisson distribution.

3.58 MEAN AND VARIANCE OF THE POISSON DISTRIBUTION

[*U.P.T.U. 2006*]

For the Poisson distribution, $P(r) = \dfrac{e^{-\lambda}\,\lambda^r}{r\,!}$

$$\text{Mean } \mu = \sum_{r=0}^{\infty} rP(r) = \sum_{r=0}^{\infty} r \,.\, \frac{e^{-\lambda}\,\lambda^r}{r\,!}$$

$$= e^{-\lambda} \sum_{r=1}^{\infty} \frac{\lambda^r}{(r-1)\,!} = e^{-\lambda}\left(\lambda + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2\,!} + ...\right)$$

$$= \lambda e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2\,!} + ...\right) = \lambda e^{-\lambda} \,.\, e^{\lambda} = \lambda$$

Thus, the mean of the Poisson distribution is equal to the parameter λ.

$$\text{Variance } \sigma^2 = \sum_{r=0}^{\infty} r^2 P(r) - \mu^2 = \sum_{r=0}^{\infty} r^2 \,.\, \frac{\lambda^r e^{-\lambda}}{r\,!} - \lambda^2 = e^{-\lambda} \sum_{r=1}^{\infty} \frac{r^2\,\lambda^r}{r\,!} - \lambda^2$$

$$= e^{-\lambda}\left[\frac{1^2 \,.\, \lambda}{1\,!} + \frac{2^2 \,.\, \lambda^2}{2\,!} + \frac{3^2 \,.\, \lambda^3}{3\,!} + \frac{4^2 \,.\, \lambda^4}{4\,!} + ...\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1 + \frac{2\lambda^2}{1!} + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + \ldots\ldots\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1 + \frac{(1+1)\lambda}{1!} + \frac{(1+2)\lambda^2}{2!} + \frac{(1+3)\lambda^3}{3!} + \ldots\ldots\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\ldots\right) + \left(\frac{\lambda}{1!} + \frac{2\lambda^2}{2!} + \frac{3\lambda^3}{3!} + \ldots\ldots\right)\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[e^{\lambda} + \lambda\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\ldots\right)\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\,[e^{\lambda} + \lambda e^{\lambda}] - \lambda^2 = \lambda e^{-\lambda}\,.\,e^{\lambda}\,(1 + \lambda) - \lambda^2 = \lambda(1 + \lambda) - \lambda^2 = \lambda.$$

Hence, the variance of the Poisson distribution is also λ.

Thus, the mean and the variance of the Poisson distribution are each equal to the parameter λ.

Note 1. The mean and the variance of the Poisson distribution can also be derived from those of the binomial distribution in the limiting case when $n \to \infty$, $p \to 0$ and $np = \lambda$.

Mean of Binomial distribution is np.

$\therefore$ Mean of Poisson distribution $= \underset{n \to \infty}{\text{Lt}}\; np = \underset{n \to \infty}{\text{Lt}}\; \lambda = \lambda$

Variance of Binomial distribution is $npq = np\,(1 - p)$

$\therefore$ Variance of Poisson distribution $= \underset{n \to \infty}{\text{Lt}}\; np\,(1 - p) = \underset{n \to \infty}{\text{Lt}}\; \lambda\left(1 - \frac{\lambda}{n}\right) = \lambda.$

Note 2. For Poisson distribution, $\mu_3 = \lambda$ and $\mu_4 = 3\lambda^2 + \lambda$.

Coefficients of skewness and kurtosis are given by

$$\beta_1 = \frac{1}{\lambda} \text{ and } \gamma_1 = \frac{1}{\sqrt{\lambda}}\text{. Also, } \beta_2 = 3 + \frac{1}{\lambda} \text{ and } \gamma_2 = \frac{1}{\lambda}$$

Hence Poisson distribution is always a skewed distribution.

Remark. While fitting the Poisson distribution to a given data, we round the figures to the nearest integer but it should be kept in mind that the total of the observed and the expected frequencies should be same.

3.59 MODE OF POISSON DISTRIBUTION

Let $\qquad \mathrm{P}(x = r) = e^{-\lambda}\,\dfrac{\lambda^r}{r!},\; r = 0, 1, 2, \ldots\ldots, \infty$

The value of r which has a greater probability than any other value is the mode of the Poisson distribution. Thus r is mode if

$$\mathrm{P}(\mathrm{X} = r) \ge \mathrm{P}(\mathrm{X} = r + 1) \text{ and } \mathrm{P}(\mathrm{X} = r) \ge \mathrm{P}(\mathrm{X} = r - 1)$$

$$\Rightarrow \quad \frac{e^{-\lambda}\,.\,\lambda^r}{r!} \ge \frac{e^{-\lambda}\,.\,\lambda^{r+1}}{(r+1)!} \quad \text{and} \quad \frac{e^{-\lambda}\,.\,\lambda^r}{r!} \ge \frac{e^{-\lambda}\,.\,\lambda^{r-1}}{(r-1)!}$$

$$\Rightarrow \quad 1 \ge \frac{\lambda}{r+1} \quad \text{and} \quad \frac{\lambda}{r} \ge 1$$

$$\Rightarrow \quad r \ge \lambda - 1 \quad \text{and} \quad r \le \lambda \quad \textit{i.e.,} \quad \lambda - 1 \le r \le \lambda$$

Case I. If λ is a positive integer, there are two modes $\lambda - 1$ and λ.

Case II. If λ is not a positive integer, there is one mode and is the integral value between $\lambda - 1$ and λ.

3.60 APPLICATIONS OF POISSON DISTRIBUTION

This distribution is applied to problems concerning :

(*i*) Arrival pattern of defective vehicles in a workshop.
(*ii*) Patients in a hospitals.
(*iii*) Telephone calls.
(*iv*) Demand pattern for certain spare parts.
(*v*) Number of fragments from a shell hitting a target.
(*vi*) Emission of radioactive (α) particles.

EXAMPLES

Example 1. *If the variance of the Poisson distribution is 2, find the probabilities for r = 1, 2, 3, 4 from the recurrence relation of the Poisson distribution. Also find P(r ≥ 4).* (M.T.U. 2013)

Sol. λ, the parameter of Poisson distribution = Variance = 2

Recurrence relation for the Poisson distribution is

$$P(r+1) = \frac{\lambda}{r+1} P(r) = \frac{2}{r+1} P(r) \qquad ...(1)$$

Now $$P(r) = \frac{e^{-\lambda} \lambda^r}{r!} \Rightarrow P(0) = \frac{e^{-2} (2)^0}{0!} = e^{-2} = 0.1353$$

Putting $r = 0, 1, 2, 3$ in (1), we get

$$P(1) = 2P(0) = 2 \times 0.1353 = 0.2706 \; ; \qquad P(2) = \frac{2}{2} P(1) = 0.2706$$

$$P(3) = \frac{2}{3} P(2) = \frac{2}{3} \times 0.2706 = 0.1804 \; ; \quad P(4) = \frac{2}{4} P(3) = \frac{1}{2} \times 0.1804 = 0.0902.$$

Now, $P(r \geq 4) = 1 - [P(0) + P(1) + P(2) + P(3)]$

$$= 1 - [0.1353 + 0.2706 + 0.2706 + 0.1804] = 0.1431.$$

Example 2. *Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well-shuffled cards at least once in 104 consecutive trials.* (U.P.T.U. 2015)

Sol. $$p = \frac{1}{52}, \; n = 104$$

$\therefore$ $$\lambda = np = 104 \times \frac{1}{52} = 2$$

Prob. (at least once) $= P(r \geq 1) = 1 - P(0)$

$$= 1 - \frac{e^{-\lambda} . \lambda^0}{0!} = 1 - e^{-2} = 1 - 0.135335 \approx 0.8647.$$

Example 3. (*i*) *Fit a Poisson distribution to the following data and calculate theoretical frequencies.*

Deaths:	*0*	*1*	*2*	*3*	*4*
Frequencies:	*122*	*260*	*15*	*2*	*1*

[U.P.T.U. 2014 ; U.K.T.U. 2010]

(*ii*) *The frequency of accidents per shift in a factory is shown in the following table:*

Accident per shift	*Frequency*
0	*192*
1	*100*
2	*24*
3	*3*
4	*1*
Total	320

Calculate the mean number of accidents per shift. Fit a Poisson distribution and calculate theoretical frequencies.

Sol. (*i*) Mean of given distribution $= \dfrac{\Sigma fx}{\Sigma f}$

$$\Rightarrow \qquad \lambda = \frac{60 + 30 + 6 + 4}{200} = 0.5$$

Required Poisson distribution $= \text{N} \,.\, \dfrac{e^{-\lambda} \,.\, \lambda^r}{r\,!} = 200 \,.\, \dfrac{e^{-.5}\,(.5)^r}{r\,!} = (121.306)\dfrac{(0.5)^r}{r\,!}$

r	*N. P(r)*	*Theoretical frequency*
0	$121.306 \times \dfrac{(0.5)^0}{0\,!} = 121.306$	121
1	$121.306 \times \dfrac{(0.5)^1}{1\,!} = 60.653$	61
2	$121.306 \times \dfrac{(0.5)^2}{2\,!} = 15.163$	15
3	$121.306 \times \dfrac{(0.5)^3}{3\,!} = 2.527$	3
4	$121.306 \times \dfrac{(0.5)^4}{4\,!} = 0.3159$	0
		Total = 200

(*ii*) Mean number of accidents per shift

$$\lambda = \frac{\Sigma fx}{\Sigma f} = \frac{100 + 48 + 9 + 4}{320} = 0.5031$$

$\therefore$ Required Poisson distribution

$$= \text{N} \cdot \frac{e^{-\lambda} \cdot \lambda^r}{r!} = 320 \cdot \frac{e^{-.5031}\,(.5031)^r}{r!} = \frac{(193.48)(.5031)^r}{r!}$$

r	*N. P(r)*	*Theoretical frequency*
0	193.48	194
1	97.34	97
2	24.38	24
3	4.10	4
4	0.51	1
		Total = 320

Example 4. (*i*) *Suppose that a book of 600 pages contains 40 printing mistakes. Assume that these errors are randomly distributed throughout the book and r, the number of errors per page has a Poisson distribution. What is the probability that 10 pages selected at random will be free from errors?*

(*ii*) *Wireless sets are manufactured with 25 solders joints each, on the average 1 joint in 500 is defective. How many sets can be expected to be free from defective joints in a consignment of 10000 sets?*

Sol. (*i*) $p = \frac{40}{600} = \frac{1}{15}, \quad n = 10$

$\therefore$ $\lambda = np = 10\left(\frac{1}{15}\right) = \frac{2}{3}$

$$\text{P}(r) = \frac{e^{-\lambda}\,\lambda^r}{r!} = \frac{e^{-2/3}\,(2/3)^r}{r!}$$

$\therefore$ $\text{P}(0) = \frac{e^{-2/3}\,(2/3)^0}{0!} = e^{-2/3} = 0.51.$

(*ii*) $p = \frac{1}{500}, \; n = 25$

$\therefore$ $\lambda = np = 25 \times \frac{1}{500} = \frac{1}{20} = 0.05$

No. of sets in a consignment, N= 10000

Probability of having no defective joint = $\text{P}(r = 0) = \frac{e^{-.05}\,(0.05)^0}{0!} = 0.9512.$

$\therefore$ The expected no. of sets free from defective joints = 0.9512 × 10000 = 9512.

Example 5. *A manufacturer knows that the condensors he makes contain on an average 1% of defectives. He packes them in boxes of 100. What is the probability that a box picked at random will contain 4 or more faulty condensors?*

Sol. $p = 0.01, \; n = 100$

$\therefore$ $\lambda = np = 1$

$$\text{P}(r) = \frac{e^{-\lambda}\,\lambda^r}{r!} = \frac{e^{-1}}{r!}$$

P(4 or more faulty condensors) = P(4) + P(5) + ... + P(100)

$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$= 1 - \left[\frac{e^{-1}}{0!} + \frac{e^{-1}}{1!} + \frac{e^{-1}}{2!} + \frac{e^{-1}}{3!}\right]$$

$$= 1 - e^{-1}\left[1 + 1 + \frac{1}{2} + \frac{1}{6}\right] = 1 - \frac{8}{3e} = 1 - 0.981 = 0.019.$$

Example 6. (*i*) *If the probabilities of a bad reaction from a certain injection is 0.0002, determine the chance that out of 1000 individuals more than two will get a bad reaction.*

(*ii*) *The probability that a man aged 50 years will die within a year is 0.01125. What is the probability that of 12 such men, at least 11 will reach their 51st birthday?*

(*Given:* $e^{-.135} = 0.87366$)

Sol. (*i*) Here, $p = 0.0002$, $n = 1000$

$\therefore$ $\lambda = np = 1000 \times 0.0002 = 0.2.$

Since the prob. of bad reaction is very small and no. of trials is very high, we use Poisson distribution here.

The prob. that out of 100 individuals, more than 2 will get a bad reaction is

$$= P(r > 2) = 1 - P(r \le 2) = 1 - [P(0) + P(1) + P(2)] \qquad ...(1)$$

Now, $P(0) = \dfrac{e^{-0.2}\,(0.2)^0}{0} = 0.8187$ (Here $r = 0$)

$P(1) = \dfrac{e^{-0.2}\,(0.2)^1}{1} = 0.1637$ (Here $r = 1$)

and $P(2) = \dfrac{e^{-0.2}\,(0.2)^2}{2} = 0.0164.$ (Here $r = 2$)

$\therefore$ From (1), Reqd. probability = 1 – [0.8187 + 0.1637 + 0.0164] = 0.0012.

(*ii*) $p = 0.01125$, $n = 12$

$\therefore$ $\lambda = np = 12 \times 0.01125 = 0.135$

P (at least 11 survive) = P(atmost 1 dies)

$$= P(0) + P(1) = \frac{e^{-\lambda}\,.\lambda^0}{0!} + \frac{e^{-\lambda}\,.\lambda^1}{1!}$$

$$= e^{-.135}\,(1 + 0.135) = 1.135 \times 0.87366 = 0.9916.$$

Example 7. *A car-hire firm has two cars, which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days on which neither car is used and the proportion of days on which some demand is refused* ($e^{-1.5} = 0.2231$).

Sol. Since the number of demands for a car is distributed as a Poisson distribution with mean $\lambda = 1.5$.

$\therefore$ Proportion of days on which neither car is used

= Probability of there being no demand for the car

$$= \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-1.5} = 0.2231$$

Proportion of days on which some demand is refused

= probability for the number of demands to be more than two

$$= 1 - P(x \le 2) = 1 - \left(e^{-\lambda} + \frac{\lambda e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!}\right)$$

$$= 1 - e^{-1.5}\left(1 + 1.5 + \frac{(1.5)^2}{2}\right) = 0.1912625.$$

Example 8. *Suppose the number of telephone calls on an operator received from 9 : 00 to 9 : 05 follow a Poisson distribution with a mean 3. Find the probability that*

(*i*) *The operator will receive no calls in that time interval tomorrow.*

(*ii*) *In the next three days, the operator will receive a total of 1 call in that time interval.* (*Given: $e^{-3} = 0.04978$*)

Sol. Here, $\lambda = 3$

(*i*) $$P(0) = \frac{e^{-\lambda} . \lambda^0}{0!} = e^{-3} = 0.04978$$

(*ii*) Reqd. probability = P(0) P(0) P(1) + P(0) P(1) P(0) + P(1) P(0) P(0)

$$= 3\left\{\frac{e^{-\lambda} . \lambda^0}{0!}\right\}^2 \frac{e^{-\lambda} . \lambda^1}{1!} = 9(e^{-3})^3 = 0.00111.$$

Example 9. *The no. of arrivals of customers during any day follows Poisson distribution with a mean of 5. What is the probability that the total no. of customers on two days selected at random is less than 2?* (*Given: $e^{-10} = 4.54 \times 10^{-5}$*)

Sol. $\lambda = 5$

Arrival of Customers

I day	*II day*	*Total*
0	0	0
0	1	1
1	0	1

Reqd. probability = P(0) P(0) + P(0) P(1) + P(1) P(0)

$$= \frac{e^{-5} .5^0}{0!} . \frac{e^{-5} .5^0}{0!} + \frac{e^{-5} .5^0}{0!} . \frac{e^{-5} .5^1}{1!} + \frac{e^{-5} .5^1}{1!} . \frac{e^{-5} .5^0}{0!}$$

$$= e^{-10} + 2 . 5 . e^{-10} = 11\, e^{-10} = 11 \times 4.54 \times 10^{-5}$$

$$= 4.994 \times 10^{-4}.$$

Example 10. *An insurance company finds that 0.005% of the population dies from a certain kind of accident each year. What is the probability that the company must pay off no more than 3 of 10,000 insured risks against such incident in a given year?*

Sol. $$p = \frac{0.005}{100} = 0.00005, \quad n = 10000$$

$\therefore$ $$\lambda = np = 10000 \times 0.00005 = 0.5$$

Reqd. Probability = $1 - P(r \le 3) = 1 - [P(0) + P(1) + P(2) + P(3)]$

$$= 1 - \left[\frac{e^{-0.5}(0.5)^0}{0!} + \frac{e^{-0.5}(0.5)^1}{1!} + \frac{e^{-0.5}(0.5)^2}{2!} + \frac{e^{-0.5}(0.5)^3}{3!}\right]$$

$$= 1 - e^{-5}\,[1 + 0.5 + 0.125 + 0.021] = 0.0016.$$

Example 11. *In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Calculate the approximate number of packets containing no defective, one defective and two defective blades in a consignment of 10,000 packets.* (*Given:* $e^{-0.02} = 0.9802$) [U.P.T.U. 2009]

Sol. $p(\text{defective}) = 0.002$

$n = 10$ (no. of blades in a packet)

$\therefore \quad \lambda = np = 10 \times 0.002 = 0.02$

No. of packets in the consignment, N = 10,000.

(*i*) Probability of having no defective = $P(0) = \dfrac{e^{-0.02}\,(0.02)^0}{0\,!} = 0.9802$ | Here $r = 0$

Approximate no. of packets having zero defective in the consignment = 0.9802×10000
$= 9802$

(*ii*) Probability of having one defective = $P(1) = \dfrac{e^{-0.02}\,(0.02)^1}{1\,!} = 0.9802 \times 0.02 = 0.019604$

Approximate no. of packets having one defective in the consignment
$= 0.019604 \times 10000 \approx 196.$

(*iii*) Probability of having two defective blades

$$P(2) = \frac{e^{-0.02}\,(0.02)^2}{2\,!} = \frac{(0.980198) \times (0.0004)}{2} = 0.000196.$$

$\therefore$ Approximate no. of packet having two defectives in the consignment
$= 0.000196 \times 10000 = 1.96 \approx 2.$

Example 12. (*i*) *Six coins are tossed 6400 times. Using the Poisson distribution, determine the approximate probability of getting six heads x times.* [U.P.T.U. (C.O.) 2008]

(*ii*) *A Poisson distribution has a double mode at x = 3 and x = 4. What is the probability that x will have one or the other of these two values?* [U.P.T.U. (C.O.) 2008]

Sol. (*i*) Probability of getting one head with one coin = $\dfrac{1}{2}$.

$\therefore$ The probability of getting six heads with six coins = $\left(\dfrac{1}{2}\right)^6 = \dfrac{1}{64}$

$\therefore$ Average number of six heads with six coins in 6400 throws = $np = 6400 \times \dfrac{1}{64} = 100$

$\therefore$ The mean of the Poisson distribution = 100.

Approximate probability of getting six heads x times when the distribution is Poisson

$$= \frac{\lambda^x e^{-\lambda}}{x\,!} = \frac{(100)^x \,.\, e^{-100}}{x\,!}\ .$$

(*ii*) Since 2 modes are given when λ is an integer, modes are $\lambda - 1$ and λ.

$\therefore \quad \lambda - 1 = 3 \quad \Rightarrow \quad \lambda = 4$

Probability (when $r = 3$) $= \dfrac{e^{-4}\,(4)^3}{3\,!}$

Probability (when $r = 4$) $= \dfrac{e^{-4}\,(4)^4}{4\,!}$

$\therefore$ Required Probability = P(r = 3 or 4) = P(r = 3) + P(r = 4)

$$= \frac{e^{-4}\,(4)^3}{3\,!} + \frac{e^{-4}\,(4)^4}{4\,!} = \frac{64}{3}\,e^{-4} = 0.39073.$$

Example 13. *For a Poisson distribution with mean m, show that*

$$\mu_{r+1} = mr\,\mu_{r-1} + m\,\frac{d\mu_r}{dm} \text{ where, } \mu_r = \sum_{x=0}^{\infty} (x-m)^r\,\frac{e^{-m}\,.\,m^x}{x\,!}. \qquad \text{(U.P.T.U. 2007)}$$

Sol. $$\mu_r = \sum_{x=0}^{\infty} (x-m)^r\,.\,\frac{e^{-m}\,.\,m^x}{x\,!}$$

$$\frac{d\mu_r}{dm} = \sum_{x=0}^{\infty}\left[\frac{-e^{-m}}{x\,!}\,.\,m^x\,(x-m)^r + \frac{e^{-m}}{x\,!}\{xm^{x-1}\,(x-m)^r - r(x-m)^{r-1}\,.\,m^x\}\right]$$

$$\Rightarrow \quad m\frac{d\mu_r}{dm} = \sum_{x=0}^{\infty}\frac{e^{-m}}{x\,!}\,m^x\,(x-m)^{r+1} - rm\sum_{x=0}^{\infty}\frac{e^{-m}}{x\,!}\,m^x\,(x-m)^{r-1} = \mu_{r+1} - mr\,\mu_{r-1}$$

$$\Rightarrow \quad \mu_{r+1} = m\,\frac{d\mu_r}{dm} + mr\,\mu_{r-1}.$$

Example 14. *Show that in a Poisson distribution with unit mean, mean deviation about mean is* $\left(\dfrac{2}{e}\right)$ *times the standard deviation.* (G.B.T.U. 2012)

Sol. Here, $\lambda = 1$

$$\therefore \quad P(X = x) = \frac{e^{-1}\,.\,(1)^x}{x\,!} = \frac{e^{-1}}{x\,!}\ ;\ x = 0, 1, 2, \ldots$$

Mean deviation about mean 1 is

$$= \sum_{x=0}^{\infty} |\,x-1\,|\,p(x) = e^{-1}\sum_{x=0}^{\infty}\frac{|\,x-1\,|}{x\,!} = e^{-1}\left[1 + \frac{1}{2\,!} + \frac{2}{3\,!} + \frac{3}{4\,!} + \ldots\right] \qquad \ldots(1)$$

we have, $$\frac{n}{(n+1)\,!} = \frac{\overline{n+1} - 1}{(n+1)\,!} = \frac{1}{n\,!} - \frac{1}{(n+1)\,!}$$

$\therefore$ From (1), Mean deviation about mean $= e^{-1}\left[1 + \left(1 - \dfrac{1}{2\,!}\right) + \left(\dfrac{1}{2\,!} - \dfrac{1}{3\,!}\right) + \ldots\right]$

$$= e^{-1}\,(1+1) = \frac{2}{3} \times 1 = \frac{2}{e} \times \text{S.D.} \qquad |\text{ Since Variance = mean = 1}$$

ASSIGNMENT

1. If X is a Poisson variate such that P(X = 2) = 9P(X = 4) + 90P(X = 6), find the standard deviation.

2. If a random variable has a Poisson distribution such that P(1) = P(2), find
(*i*) mean of the distribution (*ii*) P(4).

3. Suppose that X has a Poisson distribution. If P(X = 2) = $\frac{2}{3}$ P(X = 1) find, (*i*) P(X = 0) (*ii*) P(X = 3).

4. A certain screw making machine produces on average 2 defective screws out of 100, and packs them in boxes of 500. Find the probability that a box contains 15 defective screws.

5. (*i*) The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is the probability that in a group of 7, five or more will suffer from it?
(*ii*) The experience shows that 4 industrial accidents occur in a plant on an average per month. Calculate the probabilities of less than 3 accidents in a certain month. Use Poisson distribution. (Given : e^{-4} = 0.01832). [*M.T.U.* (*MBA*) *2011*]

6. (*i*) Suppose a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed throughout the book, what is the probability that 10 pages, selected at random, will be free from errors ?
(*ii*) Assume that the probability of an individual coalminer being killed in a mine accident during a year is $\frac{1}{2400}$. Use Poisson's distribution to calculate the probability that in a mine employing 200 miners there will be at least one fatal accident in a year.

7. (*i*) A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantee that not more than 10 pins will be defective, what is the approximate prob. that a box will fail to meet the guaranteed quality?
(*ii*) An insurance company insures 4000 people against loss of both eyes in a car accident. Based on previous data it was assumed 10 persons out of 1,00,000 will have such type of injury in car accident. What is probability that more than 2 of the insured will collect on their policy in a given year? (*M.T.U. 2013*)

8. Records show that the probability is 0.00002 that a car will have a flat tyre while driving over a certain bridge. Use Poisson distribution to find the probability that among 20,000 cars driven over the bridge, not more than one will have a flat tyre.

9. Between the hours of 2 and 4 P.M., the average no. of phone calls per minute coming into the switch board of a company is 2.5. Find the probability that during a particular minute, there will be no phone call at all. [Given : e^{-2} = 0.13534 and $e^{-0.5}$ = 0.60650.]

10. (*i*) Fit a Poisson distribution to the following data given the number of yeast cells per square for 400 squares :

No. of cells per sq.	:	0	1	2	3	4	5	6	7	8	9	10
No. of squares	:	103	143	98	42	8	4	2	0	0	0	0

It is given that $e^{-1.3225}$ = 0.2665.

(*ii*) Data was collected over a period of 10 years, showing number of deaths from horse kicks in each of the 200 army corps. The distribution of deaths was as follows:

No. of deaths	:	0	1	2	3	4	Total
Frequency	:	109	65	22	3	1	200

Fit a Poisson distribution to the data and calculate the theoretical frequencies.
[*M.T.U.* (*B. Pharma*) *2011* ; *M.T.U.* (*MBA*) *2011*]

(*iii*) The following table gives the no. of days in a 50 day period during which automobile accidents occured in a city.

No. of accidents	:	0	1	2	3	4
No. of days	:	21	18	7	3	1

Fit a Poisson distribution to the data. (*G.B.T.U. 2011*)

11. The number of accidents in a year involving taxi drivers in a city follows a Poisson distribution with mean equal to 3. Out of 1000 taxi drivers, find approximately the number of drivers with

(*i*) no accidents (*ii*) more than 3 accidents in a year.

12. The distribution of the number of road accidents per day in a city is Poisson with mean 4. Find the number of days out of 100 days when there will be

(*i*) no accident (*ii*) at least 2 accidents

(*iii*) atmost 3 accidents (*iv*) between 2 and 5 accidents.

13. It is given that 2% of the electric bulbs manufactured by a company are defective. Using Poisson distribution, find the probability that a sample of 200 bulbs will contain (*i*) no defective bulb (*ii*) two defective bulbs (*iii*) at the most three defective bulbs. (*G.B.T.U. 2011*)

14. A manager accepts the work submitted by his typist only when there is no mistake in the work. The typist has to type on an average 20 letters per day of about 200 words each. Find the chance of her making a mistake if

(*i*) less than 1% of the letters submitted by her are rejected

(*ii*) on 90% days all the letters submitted by her are accepted.

As the probability of making a mistake is small, you may use Poisson distribution. (Take $e = 2.72$)

[**Hint:** (*i*) $e^{-200p} \geq 0.99$ (*ii*) $e^{-4000p} = 0.90$]

15. In a certain factory manufacturing razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Use suitable distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 20,000 packets. (*M.T.U. 2014*)

Answers

1. 1 **2.** (*i*) 2 (*ii*) $\dfrac{2}{3e^2}$ **3.** (*i*) $e^{-(4/3)}$ (*ii*) $\dfrac{32}{81}e^{-(4/3)}$ **4.** 0.035

5. (*i*) 0.0008 (*ii*) 0.23816 **6.** (*i*) 0.4795 (*ii*) 0.08 **7.** (*i*) 0.0136875 (*ii*) 0.007926

8. 0.938 **9.** 0.08208

10. (*i*) Theoretical frequencies are 107, 141, 93, 41, 13, 4, 1, 0, 0, 0, 0.

(*ii*) $108.67 \times \dfrac{(0.61)^r}{r\,!}$; 109, 66, 20, 4 and 1 (*iii*) $(50)\ \dfrac{e^{-0.9}\,(0.9)^r}{r\,!}$; 20, 18, 8, 3 and 1.

11. (*i*) 50 (*ii*) 353

12. (*i*) 2 (*ii*) 91 (*iii*) 43 (*iv*) 39

13. (*i*) 0.018315 (*ii*) 0.146525 (*iii*) 0.43347

14. (*i*) $p \leq 0.0000506$ (*ii*) $p = 0.0000263$ **15.** 19604, 392, 4

3.61 NORMAL DISTRIBUTION [*U.P.T.U. 2007*]

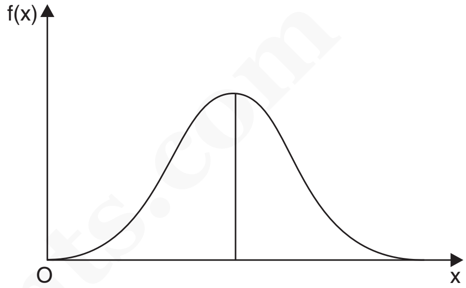
The normal distribution is a continuous distribution. It can be derived from the binomial distribution in the limiting case when n, the number of trials is very large and p, the probability of a success, is close to $\frac{1}{2}$. The general equation of the normal distribution is given by

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

where the variable x can assume all values from $-\infty$ to $+\infty$. μ and σ, called the parameters of the distribution, are respectively the mean and the standard deviation of the distribution and $-\infty < \mu < \infty$, $\sigma > 0$. x is called the normal variate and $f(x)$ is called probability density function of the normal distribution.

If a variable x has the normal distribution with mean μ and standard deviation σ, we briefly write $x : N(\mu, \sigma^2)$.

The graph of the normal distribution is called the *normal curve*. It is bell-shaped and symmetrical about the mean μ. The two tails of the curve extend to $+\infty$ and $-\infty$ towards the positive and negative directions of the x-axis respectively and gradually approach the x-axis without ever metting it. The curve is unimodal and the mode of the normal distribution coincides with its mean μ. The line $x = \mu$ divides the area under the normal curve above x-axis into two equal parts. Thus, the median of the distribution also coincides with its mean and mode. The area under the normal curve between any two given ordinates $x = x_1$ and $x = x_2$ represents the probability of values falling into the given interval. The total area under the normal curve above the x-axis is 1.



3.62 BASIC PROPERTIES OF THE NORMAL DISTRIBUTION

The probability density function of the normal distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

(*i*) $f(x) \geq 0$ (*ii*) $\int_{-\infty}^{\infty} f(x)\, dx = 1,$

i.e., the total area under the normal curve above the x-axis is 1.

(*iii*) The normal distribution is symmetrical about its mean.

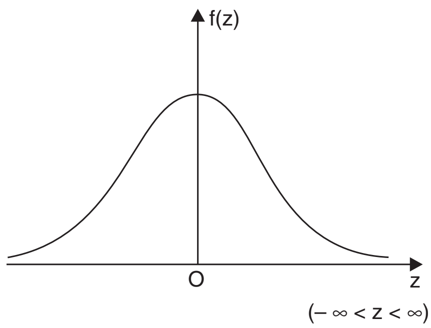
(*iv*) It is a unimodal distribution. The mean, mode and median of this distribution coincide.

3.63 STANDARD FORM OF THE NORMAL DISTRIBUTION

If X is a normal random variable with mean μ and standard deviation σ, then the random variable $Z = \dfrac{X-\mu}{\sigma}$ has the normal distribution with mean 0 and standard deviation 1. The random variable Z is called the *standardized* (or *standard*) normal random variable.

The probability density function for the normal distribution in standard form is given by

$$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}}$$



It is free from any parameter. This helps us to compute areas under the normal probability curve by making use of standard tables.

Note 1. If $f(z)$ is the probability density function for the normal distribution, then

$$P(z_1 \le Z \le z_2) = \int_{z_1}^{z_2} f(z)\, dz = F(z_2) - F(z_1), \quad \text{where} \quad F(z) = \int_{-\infty}^{z} f(z)\, dz = P(Z \le z)$$

The function $F(z)$ defined above is called the *distribution function* for the normal distribution.

Note 2. The probabilities $P(z_1 \le Z \le z_2)$, $P(z_1 < Z \le z_2)$, $P(z_1 \le Z < z_2)$ and $P(z_1 < Z < z_2)$ are all regarded to be the same.

Note 3. $F(-z_1) = 1 - F(z_1)$.

3.64 NORMAL DISTRIBUTION AS A LIMITING FORM OF BINOMIAL DISTRIBUTION (when $p = q$)

(U.P.T.U. 2007)

Let N $(q + p)^n$ be the binomial distribution. If $p = q$ then $p = q = \frac{1}{2}$ (since $p + q = 1$) and consequently the binomial distribution is symmetrical. Let n be an even integer say $2k$, k being an integer. Since $n \to \infty$, the frequencies of r and $r + 1$ successes can be written in following forms:

$$f(r) = N.\ {}^{2k}C_r \left(\frac{1}{2}\right)^{2k}$$

$$f(r+1) = N.\ {}^{2k}C_{r+1} \left(\frac{1}{2}\right)^{2k}$$

$$\therefore \quad \frac{f(r+1)}{f(r)} = \frac{{}^{2k}C_{r+1}}{{}^{2k}C_r} = \frac{2k-r}{r+1}$$

The frequency of r successes will be greater than the frequency of $(r + 1)$ successes if

$$f(r) > f(r+1)$$

$$\Rightarrow \quad \frac{f(r+1)}{f(r)} < 1$$

$$\Rightarrow \quad 2k - r < r + 1$$

$$\Rightarrow \quad r > k - \frac{1}{2} \qquad ...(1)$$

In a similar way, the frequency of r successes will be greater than the frequencies of $(r - 1)$ successes if $r < k + \frac{1}{2}$ ...(2)

In view of (1) and (2), we observe that if $k - \frac{1}{2} < r < k + \frac{1}{2}$ the frequency corresponding to r successes will be the greatest. Clearly, $r = k$ is the value of the success corresponding to which the frequency is maximum. Suppose it is y_0. Then, we have

$$y_0 = N.\ {}^{2k}C_k \left(\frac{1}{2}\right)^{2k} = N.\ \frac{2k!}{k!\,k!}\left(\frac{1}{2}\right)^{2k}$$

Let y_x be the frequency of $k + x$ successes then, we have

$$y_x = \text{N} \,.\, {}^{2k}\text{C}_{k+x} \left(\frac{1}{2}\right)^{2k} = \text{N} \,.\left(\frac{1}{2}\right)^{2k} \cdot \frac{2k\,!}{(k+x)\,!\,(k-x)\,!}$$

Now,
$$\frac{y_x}{y_0} = \frac{k\,!\,k!}{(k+x)\,!\,(k-x)\,!} = \frac{k\,(k-1)\,(k-2)\ldots(k-x+1)}{(k+x)\,(k+x-1)\ldots(k+1)}$$

$$= \frac{\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\ldots\left\{1-\frac{x-1}{k}\right\}}{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\ldots\left(1+\frac{x}{k}\right)}$$

Taking log on both sides,

$$\log\frac{y_x}{y_0} = \left[\log\left(1-\frac{1}{k}\right) + \log\left(1-\frac{2}{k}\right) + \ldots + \log\left(1-\frac{x-1}{k}\right)\right]$$

$$-\left[\log\left(1+\frac{1}{k}\right) + \log\left(1+\frac{2}{k}\right) + \ldots + \log\left(1+\frac{x}{k}\right)\right] \quad \ldots(3)$$

Now, writing expression for each term and neglecting higher powers of $\frac{x}{k}$ (very small quantity), we get from (3),

$$\log\frac{y_x}{y_0} = -\frac{1}{k}\{1+2+3+\ldots+(x-1)\} - \frac{1}{k}\{1+2+3+\ldots+(x-1)+x\}$$

$$= -\frac{2}{k}\{1+2+3+\ldots+(x-1)\} - \frac{x}{k}$$

$$= -\frac{2}{k}\,\frac{(x-1)\,x}{2} - \frac{x}{k} = -\frac{x^2}{k}$$

$\therefore$
$$y_x = y_0\, e^{-x^2/k}$$

$\Rightarrow$
$$\boxed{y_x = y_0\, e^{-x^2/2\sigma^2}} \qquad \Big|\because\ \sigma^2 = npq = \frac{n}{4} = \frac{k}{2}$$

which is **normal distribution.**

3.65 MEAN AND VARIANCE OF NORMAL DISTRIBUTION (*U.P.T.U. 2015*)

1. The A.M. of a continuous distribution $f(x)$ is given by

$$\text{A.M. } (\bar{x}) = \frac{\int_{-\infty}^{\infty} x\, f(x)\, dx}{\int_{-\infty}^{\infty} f(x)\, dx} \qquad \text{| By definition}$$

Consider the normal distribution with μ, σ as the parameters then

$$\bar{x} = \int_{-\infty}^{\infty} x.\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad \Bigg| \text{ Since } \int_{-\infty}^{\infty} f(x)\,dx = \text{area under normal curve} = 1$$

Put $\dfrac{x-\mu}{\sigma} = z$ so that $x = \mu + \sigma z \quad \therefore \quad dx = \sigma dz$

so,

$$\bar{x} = \int_{-\infty}^{\infty} (\mu + \sigma z)\,\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\,(\sigma\, dz)$$

$$= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{-\frac{1}{2}z^2}\, dz$$

$$= \mu + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, d\left(\frac{z^2}{2}\right) \qquad \Bigg| \because \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz = 1$$

$$= \mu + \frac{\sigma}{\sqrt{2\pi}} \left(\frac{e^{-z^2/2}}{-1}\right)_{-\infty}^{\infty}$$

$$\boxed{\bar{x} = \mu}$$

2. By definition,

$$\text{Variance} = \int_{-\infty}^{\infty} (x - \bar{x})^2\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx + \bar{x}^2 \int_{-\infty}^{\infty} f(x)\, dx - 2\bar{x} \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2\, f(x)\, dx + \bar{x}^2 - 2\,\bar{x}\,\bar{x} \qquad \Bigg| \because \int_{-\infty}^{\infty} f(x)\, dx = 1 \text{ and } \int_{-\infty}^{\infty} x\, f(x)\, dx = \bar{x}$$

$$= \int_{-\infty}^{\infty} x^2\, f(x)\, dx - \bar{x}^2 \qquad \text{...(1)}$$

Now, Let $\text{I} = \displaystyle\int_{-\infty}^{\infty} x^2\, f(x)\, dx = \int_{-\infty}^{\infty} x^2\, \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} dx$

Put $\dfrac{x-\bar{x}}{\sigma} = z$ so that $x = \bar{x} + \sigma z \;\therefore\; dx = \sigma\, dz$

Hence,

$$\text{I} = \int_{-\infty}^{\infty} (\bar{x} + \sigma z)^2\, \frac{1}{\sigma\sqrt{2\pi}}\, e^{-z^2/2}\, \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}}\left[\sigma^2 \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\, dz + \bar{x}^2 \int_{-\infty}^{\infty} e^{-z^2/2}\, dz + 2\sigma\bar{x} \int_{-\infty}^{\infty} z\, e^{-z^2/2}\, dz\right]$$

$$= \frac{-\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \, d(e^{-z^2/2}) + \bar{x}^2 \, . \, 1 + 2\sigma\bar{x} \, . \, 0$$

$$= -\frac{\sigma^2}{\sqrt{2\pi}} \left(z\, e^{-z^2/2} \right)_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \, dz + \bar{x}^2$$

$$= 0 + \sigma^2 \, . \, 1 + \bar{x}^2 = \sigma^2 + \bar{x}^2$$

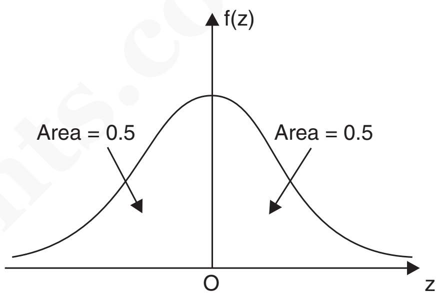
$\therefore$ From (1), $\boxed{\text{Variance} = \sigma^2 + \bar{x}^2 - \bar{x}^2 = \sigma^2}$

$\therefore$ The standard deviation of the normal distribution is σ.

3.66 AREA UNDER THE NORMAL CURVE

By taking $z = \dfrac{x - \mu}{\sigma}$, standard normal curve is formed. The total area under this curve is 1.

The area under the curve is divided into two equal parts by $z = 0$. The area between the ordinate $z = 0$ and any other ordinate can be noted from the supplied table. It should be noted that mean for the normal distribution is 0.



3.67 APPLICATIONS OF NORMAL DISTRIBUTION

De Moivre made the discovery of this distribution in 1733.

This distribution has an important application in the theory of errors made by chance in experimental measurements. Its more applications are in computation of hit probability of a shot and in statistical inference in almost every branch of science.

EXAMPLES

Example 1. *A sample of 100 dry battery cells tested to find the length of life produced the following results:*

$$\bar{x} = 12 \text{ hours}, \ \sigma = 3 \text{ hours}.$$

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life

(*i*) *more than 15 hours* (*ii*) *less than 6 hours* (*iii*) *between 10 and 14 hours?*

Sol. Here x denotes the length of life of dry battery cells.

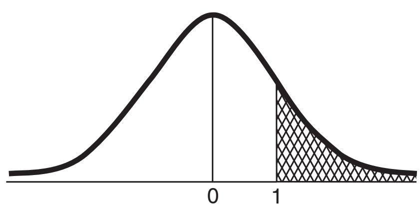
Also $z = \dfrac{x - \bar{x}}{\sigma} = \dfrac{x - 12}{3}$.

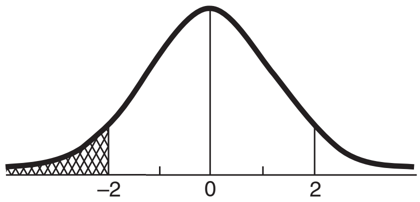
(*i*) When $x = 15$, $z = 1$

$\therefore$ $P(x > 15) = P(z > 1)$

$= P(0 < z < \infty) - P(0 < z < 1)$

$= .5 - 0.3413 = 0.1587 = 15.87\%$.





(*ii*) When $x = 6$, $z = -2$

$\therefore \quad P(x < 6) = P(z < -2)$

$= P(z > 2) = P(0 < z < \infty) - P(0 < z < 2)$

$= 0.5 - 0.4772 = 0.0228 = 2.28\%.$

(*iii*) When $x = 10$, $z = -\dfrac{2}{3} = -0.67$

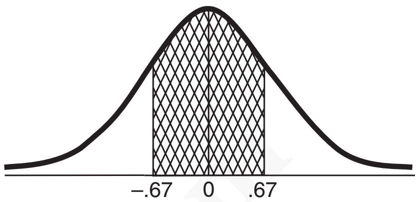
When $x = 14$, $z = \dfrac{2}{3} = 0.67$

$P(10 < x < 14)$

$= P(-0.67 < z < 0.67)$

$= 2P(0 < z < 0.67) = 2 \times 0.2485$

$= 0.4970 = 49.70\%.$



Example 2. *In a sample of 1000 cases, the mean of a certain test is 14 and S.D. is 2.5. Assuming the distribution to be normal, find*

(*i*) *how many students score between 12 and 15?*

(*ii*) *how many score above 18?*

(*iii*) *how many score below 8?*

(*iv*) *how many score 16?*

Sol. (*i*)
$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{12 - 14}{2.5} = -0.8$$
$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{15 - 14}{2.5} = 0.4$$

Area lying between – 0.8 and 0.4

= Area between 0 to 0.8 + Area between 0 to 0.4

= 0.2881 + 0.1554 = 0.4435

Reqd. no. of students = 1000 × 0.4435 = 444 (app.)

(*ii*)
$$z = \frac{18 - 14}{2.5} = 1.6$$

Area right to 1.6 = 0.5 – (Area between 0 and 1.6) = 0.5 – 0.4452 = 0.0548

Reqd. no. of students = 1000 × 0.0548 = 54.8 ≈ 55 (app.)

(*iii*)
$$z = \frac{8 - 14}{2.5} = -2.4$$

Area left to – 2.4 = 0.5 – (Area between 0 and 2.4) = 0.5 – 0.4918 = 0.0082

∴ Reqd. no. of students = 1000 × 0.0082 = 8.2 ≈ 8 (app.).

(*iv*)
$$z_1 = \frac{15.5 - 14}{2.5} = 0.6$$
$$z_2 = \frac{16.5 - 14}{2.5} = 1$$

Area between 0.6 and 1 = 0.3413 – 0.2257 = 0.1156

∴ Reqd. no. of students = 1000 × 0.1156 = 115.6 ≈ 116 (app.).

Example 3. *Assume mean height of soldiers to be 68.22 inches with a variance of 10.8 inches square. How many soldiers in a regiment of 1,000 would you expect to be over 6 feet tall, given that the area under the standard normal curve between z = 0 and z = 0.35 is 0.1368 and between z = 0 and z = 1.15 is 0.3746.* [G.B.T.U. (C.O.) 2011]

Sol. x = 6 feets = 72 inches

$$\therefore \quad z = \frac{x-\mu}{\sigma} = \frac{72-68.22}{\sqrt{10.8}} = 1.15$$

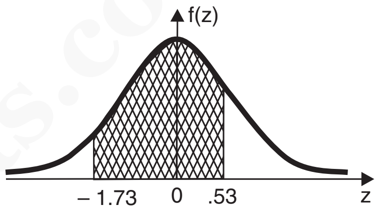
$$P(x > 72) = P(z > 1.15) = 0.5 - P(0 \le z \le 1.15)$$
$$= 0.5 - 0.3746 = 0.1254$$

$\therefore$ Expected no. of soldiers = 1000 × 0.1254 = 125.4 ≈ 125 (app.).

Example 4. *A large number of measurement is normally distributed with a mean 65.5″ and S.D. of 6.2″. Find the percentage of measurements that fall between 54.8″ and 68.8″.*

Sol. Mean μ = 65.5 inches, S.D. σ = 6.2 inches

x_1 = 54.8 inches, x_2 = 68.8 inches



$$\therefore \quad z_1 = \frac{x_1-\mu}{\sigma} = \frac{54.8-65.5}{6.2} = -1.73$$

and $$z_2 = \frac{x_2-\mu}{\sigma} = \frac{68.8-65.5}{6.2} = 0.53$$

Now, $$P(-1.73 \le z \le 0.53) = P(-1.73 \le z \le 0) + P(0 \le z \le 0.53)$$
$$= P(0 \le z \le 1.73) + P(0 \le z \le 0.53)$$
$$= 0.4582 + 0.2019 = 0.6601 \qquad | \text{ By table}$$

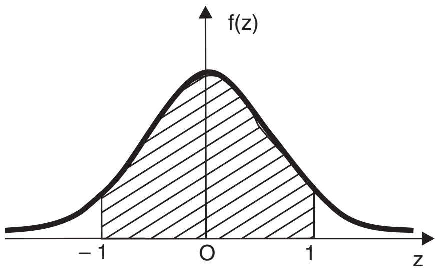
$\therefore$ Reqd. percentage of measurements = 66.01%.

Example 5. *A manufacturer knows from experience that the resistance of resistors he produces is normal with mean* μ = *100 ohms and standard deviation* σ = *2 ohms. What percentage of resistors will have resistance between 98 ohms and 102 ohms?*

Sol. $\mu = 100\ \Omega, \quad \sigma = 2\ \Omega, \quad x_1 = 98\ \Omega, \quad x_2 = 102\ \Omega$

$$\therefore \quad z_1 = \frac{x_1-\mu}{\sigma} = \frac{98-100}{2} = -1$$

and $$z_2 = \frac{x_2-\mu}{\sigma} = \frac{102-100}{2} = 1.$$



Now, $$P(98 < x < 102) = P(-1 < z < 1)$$
$$= P(-1 \le z \le 0) + P(0 \le z \le 1)$$
$$= P(0 \le z \le 1) + P(0 \le z \le 1)$$
$$= 0.3413 + 0.3413 = 0.6826.$$

$\therefore$ Percentage of resistors having resistance between 98 Ω and 102 Ω = 68.26%.

Example 6. *In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution. It is given that if* $f(t) = \frac{1}{\sqrt{2\pi}}\int_0^t e^{-\frac{1}{2}x^2}\, dx$ *then f(0.5) = 0.19 and f(1.4) = 0.42.* (M.T.U. 2013)

Sol. Let μ and σ be the mean and S.D. respectively.

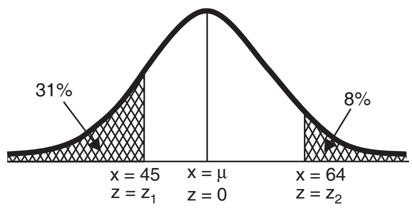
31% of the items are under 45.

$\Rightarrow$ Area to the left of the ordinate $x = 45$ is 0.31

When $x = 45$, let $z = z_1$

$P(z_1 < z < 0) = 0.5 - 0.31 = 0.19$

From the tables, the value of z corresponding to this area is 0.5



$\therefore \quad z_1 = -0.5 [z_1 < 0]$

When $x = 64$, let $z = z_2$

$P(0 < z < z_2) = 0.5 - 0.08 = 0.42$

From the tables, the value of z corresponding to this area is 1.4.

$\therefore \quad z_2 = 1.4$

Since $\quad z = \dfrac{x - \mu}{\sigma}$

$\therefore \quad -0.5 = \dfrac{45 - \mu}{\sigma}$ and $1.4 = \dfrac{64 - \mu}{\sigma}$

$\Rightarrow \quad 45 - \mu = -0.5\sigma$ ...(1)

and $\quad 64 - \mu = 1.4\sigma$ ...(2)

Subtracting $\quad -19 = -1.9\sigma \quad \therefore \quad \sigma = 10$

From (1), $\quad 45 - \mu = -0.5 \times 10 = -5 \quad \therefore \quad \mu = 50.$

Example 7. *The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are insured, how many pairs would be expected to need replacement after 12 months?* $\left[\text{Given that } P(z \geq 2) = 0.0228 \text{ and } z = \dfrac{x - \mu}{\sigma}\right]$.

Sol. Mean $(\mu) = 8$, Standard Deviation $(\sigma) = 2$

Number of pairs of shoes = 5000, Total months $(x) = 12$

when $x = 12$, $\quad z = \dfrac{x - \mu}{\sigma} = \dfrac{12 - 8}{2} = 2$

Area $(z \geq 2) = 0.0228$

Number of pairs whose life is more than 12 months = 5000 × 0.0228 = 114

Pair of shoes needing replacement after 12 months = 5000 – 114 = 4886.

Example 8. *The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 cm and the standard deviation is 0.005 cm. The purpose for which these washers are intended allows a minimum tolerance in the diameter of 0.496 to 0.508 cm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine. Assume the diameters are normally distributed.*

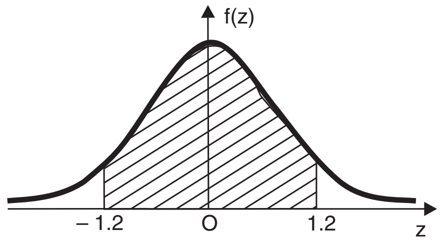
Sol. Given: Mean $\mu = 0.502$ cm, S.D. $\sigma = 0.005$ cm, $x_1 = 0.496$ cm, $x_2 = 0.508$ cm.

Now, $\quad z_1 = \dfrac{x_1 - \mu}{\sigma} = \dfrac{0.496 - 0.502}{0.005} = -1.2$

$z_2 = \dfrac{x_2 - \mu}{\sigma} = \dfrac{0.508 - 0.502}{0.005} = 1.2$

Area for non-defective washers

$$= P(-1.2 \le z \le 1.2)$$
$$= P(-1.2 \le z \le 0) + P(0 \le z \le 1.2)$$
$$= P(0 \le z \le 1.2) + P(0 \le z \le 1.2)$$
$$= 0.3849 + 0.3849 = 0.7698$$
$$= 76.98\%.$$



$\therefore$ Percentage of defective washers = 100 – 76.98
= 23.02%.

Example 9. *Assuming that the diameters of 1000 brass plugs taken consecutively from a machine, form a normal distribution with mean 0.7515 cm and standard deviation 0.002 cm, how many of the plugs are likely to be rejected if the approved diameter is 0.752 ± 0.004 cm.*

Sol. Tolerance limits of the diameter of non-defective plugs are

0.752 – 0.004 = 0.748 cm. and 0.752 + 0.004 = 0.756 cm.

Standard normal variable, $z = \dfrac{x - \mu}{\sigma}$

If $x_1 = 0.748$, $z_1 = \dfrac{0.748 - 0.7515}{0.002} = -1.75$

If $x_2 = 0.756$, $z_2 = \dfrac{0.756 - 0.7515}{0.02} = 2.25$

Area from ($z_1 = -1.75$) to ($z_2 = 2.25$)

$$= P(-1.75 \le z \le 2.25) = P(-1.75 \le z \le 0) + P(0 \le z \le 2.25)$$
$$= P(0 \le z \le 1.75) + P(0 \le z \le 2.25) = 0.4599 + 0.4878 = 0.9477$$

Number of plugs which are likely to be rejected = 1000 × (1 – 0.9477) = 1000 × .0523 = 52.3

Hence approximately 52 plugs are likely to be rejected.

Example 10. *If the heights of 300 students are normally distributed with mean 64.5 inches and standard deviation 3.3 inches, find the height below which 99% of the students lie.*

Sol. Mean μ = 64.5 inches, S.D. σ = 3.3 inches

Area between 0 and $\dfrac{x - 64.5}{3.3} = 0.99 - 0.5 = 0.49$

From the table, for the area 0.49, $z = 2.327$

The corresponding value of x is given by

$$\frac{x - 64.5}{3.3} = 2.327$$

$\Rightarrow$ $x - 64.5 = 7.68$

$\Rightarrow$ $x = 7.68 + 64.5 = 72.18$ inches.

Hence 99% students are of height less than 6 ft. 0.18 inches.

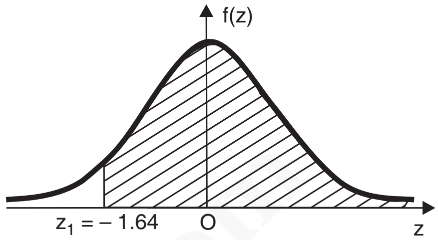
Example 11. *The income of a group of 10,000 persons was found to be normally distributed with mean ₹ 750 p.m. and standard deviation of ₹ 50. Show that, of this group, about 95% had income exceeding ₹ 668 and only 5% had income exceeding ₹ 832. Also find the lowest income among the richest 100.*

Sol. Given: $\mu = 750, \sigma = 50$

Standard normal variable, $z = \dfrac{x - \mu}{\sigma}$

(*i*) If $x_1 = 668$, $z_1 = \dfrac{x_1 - \mu}{\sigma} = \dfrac{668 - 750}{50} = -1.64$

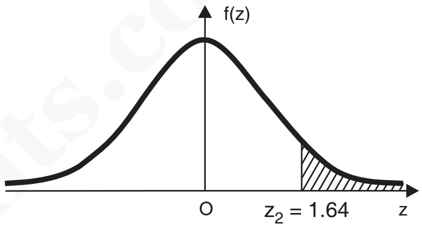
$$\begin{aligned} P(x_1 > 668) &= P(z_1 > -1.64) \\ &= 0.5 + P(-1.64 \le z \le 0) \\ &= 0.5 + P(0 \le z \le 1.64) \\ &= 0.5 + 0.4495 \\ &= 0.9495 \end{aligned}$$



∴ Required percentage of persons having income exceeding ₹ 668 = 94.95% ≈ 95% (approx.)

(*ii*) If $x_2 = 832$, $z_2 = \dfrac{x_2 - \mu}{\sigma} = \dfrac{832 - 750}{50} = 1.64$

$$\begin{aligned} P(x_2 > 832) &= P(z_2 > 1.64) \\ &= 0.5 - P(0 \le z \le 1.64) \\ &= 0.5 - 0.4495 = 0.0505 \end{aligned}$$



∴ Required percentage of persons having income exceeding ₹ 832 = 5.05% ≈ 5% (approx.)

(*iii*) Let x be the lowest income among the richest 100 persons *i.e.,* 1% of 10,000.

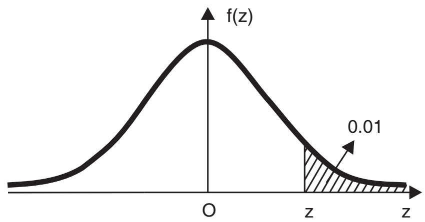
Thus, area between O and $z = 0.49$ (see figure) by Normal distribution table,

$$z = 2.33$$

Thus, $\dfrac{x - \mu}{\sigma} = 2.33$

$\Rightarrow$ $\dfrac{x - 750}{50} = 2.33$

$\Rightarrow$ $x = 866.5$



Hence ₹ 866.5 is the minimum income among the richest 100 persons.

Example 12. *255 metal rods were cut roughly 6 inches over size. Finally the lengths of the over size amount, were measured exactly and grouped with 1 inch intervals, there being in all 12 groups* $\frac{1}{2}'' - 1\frac{1}{2}''$, $1\frac{1}{2}'' - 2\frac{1}{2}''$, ... , $11\frac{1}{2}'' - 12\frac{1}{2}''$.

The frequency distribution for the 255 lengths was as follows:

Length (inches) Central value	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
Frequency	*2*	*10*	*19*	*25*	*40*	*44*	*41*	*28*	*25*	*15*	*5*	*1*

Fit a normal curve to this data.

Sol. The equation of the normal curve for N observations is

$$y = \frac{N}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad ...(1)$$

x	f	$u = x - 6$	fu	fu^2
1	2	– 5	– 10	50
2	10	– 4	– 40	160
3	19	– 3	– 57	171
4	25	– 2	– 50	100
5	40	– 1	– 40	40
6	44	0	0	0
7	41	1	41	41
8	28	2	56	112
9	25	3	75	225
10	15	4	60	240
11	5	5	25	125
12	1	6	6	36
Total	255		66	1300

Mean, $\mu = a + \dfrac{\Sigma fu}{\Sigma f} = 6 + \dfrac{66}{255} = 6.259$

Variance, $\sigma^2 = \dfrac{\Sigma fu^2}{\Sigma f} - \left(\dfrac{\Sigma fu}{\Sigma f}\right)^2 = \dfrac{1300}{225} - \left(\dfrac{66}{255}\right)^2 = 5.031$

$\therefore$ $\sigma = 2.243$

Thus, we have N = 255, Mean, $\mu = 6.259''$, S.D. $\sigma = 2.243''$

Hence the fitted curve is

$$y = \frac{255}{2.243\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-6.259}{2.243}\right)^2} \qquad | \text{ From (1)}$$

$$= \frac{113.68}{\sqrt{2\pi}}\, e^{-0.099(x-6.259)^2}.$$

Example 13. *Show that the area under the normal curve is unity.*

Sol. Area under the normal curve is given by

$$A = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

Put $\dfrac{x-\mu}{\sigma} = z$ so $dx = \sigma\, dz$

$$\therefore \quad A = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-z^2/2}\, (\sigma dz) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-z^2/2}\, dz$$

Now, $$A \cdot A = A^2 = \left(\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x^2/2}\, dx\right)\left(\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-y^2/2}\, dy\right)$$

$$= \frac{2}{\pi} \int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{x^2+y^2}{2}\right)}\, dx\, dy \qquad | \text{ where } x \text{ and } y \text{ are dummy variables}$$

Put $x = r \cos \theta$, $y = r \sin \theta$ so that $J = r$ changing to polar coordinates,

$$A^2 = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\infty} e^{-r^2/2}\, r\, dr\, d\theta = \int_0^{\infty} e^{-r^2/2}\, d\left(\frac{r^2}{2}\right) = 1$$

$\therefore$ A = Area under the normal curve = 1

Example 14. *Prove that for normal distribution, the mean deviation from the mean equals to $\frac{4}{5}$ of the standard deviation approximately.* (U.P.T.U. 2009)

Sol. Let μ and σ be the mean and standard deviation of the normal distribution. Then by definition,

Mean deviation from the mean

$$= \int_{-\infty}^{\infty} |x - \mu|\, f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x - \mu|\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma\, |z|\, e^{-\frac{1}{2}z^2}\, \sigma\, dz \qquad \left|\begin{array}{l} \text{when } \dfrac{x-\mu}{\sigma} = z \\ \Rightarrow \quad dx = \sigma\, dz \end{array}\right.$$

$$= \sigma \sqrt{\frac{2}{\pi}} \int_0^{\infty} z\, e^{-z^2/2}\, dz$$

$$= \sigma \sqrt{\frac{2}{\pi}} \left[- e^{-z^2/2}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\, \sigma = 0.7979\, \sigma \approx 0.8\sigma \approx \frac{4}{5}\, \sigma$$

ASSIGNMENT

1. In a test on 2000 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours, estimate the number of bulbs likely to burn for

(*i*) more than 2150 hours (*ii*) less than 1950 hours

(*iii*) more than 1920 hours but less than 2160 hours. (*U.P.T.U. 2008*)

2. An aptitude test for selecting officers in a bank is conducted on 1000 candidates. The average score is 42 and the standard deviation of score is 24. Assuming normal distribution for the scores, find

(*i*) the number of candidates whose scores exceed 60

(*ii*) the number of candidates whose scores lie between 30 and 60.

3. (*i*) In a normal distribution exactly normal, 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution? (*G.B.T.U. 2010*)

(*ii*) In a normal distribution, 0.0107 of the items lie below 42 and 0.0446 of the items lie above 82. What is the mean and standard deviation of the normal distribution?

[*U.P.T.U. (MBA) 2009*]

4. If Z is a standard normal variable, find the following probabilities: [*G.B.T.U. (MBA) 2010*]

(*i*) P(Z < 1.2) (*ii*) P(Z > – 1.2) (*iii*) P(– 1.2 < Z < 1.3).

5. An aptitude test was conducted on 900 employees of the Metro Tyres Limited in which the mean score was found to be 50 units and standard deviation was 20. On the basis of this information, you are required to answer the following questions:

(*i*) What was the number of employees whose mean score was less than 30?

(*ii*) What was the number of employees whose mean score exceeded 70?

(*iii*) What was the number of employees whose mean score were between 30 and 70?

$\frac{x-\mu}{\sigma}$	0.25	0.50	0.70	1.00	1.25	1.50
Area	0.0987	0.1915	0.2734	0.3413	0.3944	0.4332

[*U.P.T.U. (MBA) 2009*]

6. (*a*) Students of a class were given a mechanical aptitude test. Their marks were found to be normally distributed with mean 60 and standard deviation 5. What percent of students scored?

(*i*) more than 60 marks? (*ii*) less than 56 marks? (*iii*) between 45 and 65 marks?

(*b*) 2000 students appeared in an examination. Distribution of marks is assumed to be normal with mean $\mu = 30$ and $\sigma = 6.25$. How many students are expected to get marks?

(*i*) between 20 and 40 (*ii*) less than 35 and (*iii*) above 50.

[*U.P.T.U. (MBA) 2012*]

(*c*) Suppose the weight W of 600 male students are normally distributed with mean $\mu = 70$ kg and standard deviation $\sigma = 5$ kg. Find number of students with weight

(*i*) between 69 and 74 kg (*ii*) more than 76 kg. (*G.B.T.U. 2013*)

7. (*a*) In an intelligence test administered to 1000 students, the average score was 42 and standard deviation 24. Find:

(*i*) the expected number of students scoring more than 50.

(*ii*) the number of students scoring between 30 and 54.

(*iii*) the value of score exceeded by top 100 students. [*G.B.T.U. (MBA) 2010*]

(*b*) The average monthly sales of 5000 firms are normally distributed. Its mean and standard deviation are ₹ 36000 and ₹ 10000 respectively. Find:

(*i*) the no. of firms having sales over ₹ 40000.

(*ii*) the no. of firms having sales between ₹ 30000 and ₹ 40000.

[Given area under normal curve from 0 to z for Z (0.4) = 0.1554 and Z (0.6) = 0.2257]

[*G.B.T.U. (MBA) 2010*]

(*c*) The daily wages of 1000 workers are distributed around a mean of ₹ 140 and with a standard deviation of ₹ 10. Estimate the number of workers whose daily wages will be

(*i*) between ₹ 140 and ₹ 144 (*ii*) less than ₹ 126

(*iii*) more than ₹ 160. (*G.B.T.U. 2012*)

8. (*a*) Records kept by the goods inwards department of a large factory show that the average no. of lorries arriving each week is 248. It is known that the distribution approximates to be normal with a standard deviation of 26.

If this pattern of arrival continues, what percentage of weeks can be expected to have number of arrivals of:

(*i*) less than 229 per week? (*ii*) more than 280 per week?

(*b*) Pipes for tobacco are being packed in fancy plastic boxes. The length of the pipe is normally distributed with $\mu = 5''$ and $\sigma = 0.1''$. The internal length of the boxes is 5.2″. What is the probability that the box would be small for the pipe?

[Given that : $\phi(1.8) = 0.9641$, $\phi(2) = 0.9772$, $\phi(2.5) = 0.9938$]

(*c*) A manufacturer of envelopes knows that the weight of the envelopes is normally distributed with mean 1.9 gm and variance 0.01 square gm. Find how many envelopes weighing

(*i*) 2 gm or more

(*ii*) 2.1 gm or more, can be expected in a given packet of 1000 envelopes? (*U.P.T.U. 2015*)

[Given: if t is the normal variable, then $\phi(0 \le t \le 1) = 0.3413$ and $\phi(0 \le t \le 2) = 0.4772$]

9. (*a*) East-East Airlines has the policy of employing only Indian woman whose height is between 62 inches and 69 inches. If the height of Indian women is approximately normally distributed with a mean of 64 inches and a standard deviation of 3 inches, Out of the 1000 applications received find the number of applicants that would be

(*i*) too tall (*ii*) too short (*iii*) of acceptable height.

(*b*) The mean height of 500 students is 151 cm and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students have heights between 120 and 155 cm?

(*c*) The monthly mess bill of a student who is staying in the hostel follows a normal distribution with a mean of ₹ 2000 and a standard deviation of ₹ 185. What is the probability that in the next month, his bill will go above ₹ 2400? [*U.P.T.U. (MBA) 2009*]

10. In an examination taken by 500 candidates, the average and the standard deviation of marks obtained (normally distributed) are 40% and 10%. Find approximately

(*i*) how many will pass, if 50% is fixed as a minimum?

(*ii*) what should be the minimum if 350 candidates are to pass?

(*iii*) how many have scored marks above 60%?

11. The income distribution of workers in a certain factory was found to be normal with mean ₹ 500 and standard deviation equal to ₹ 50. There were 228 persons getting above ₹ 600. How many workers were there in all?

(Area under the standard normal curve between height at 0 and 2 is 0.4772).

12. In a certain examination, the percentage of passes and distinction were 46 and 9 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively. Assume the distribution of marks to be normal.

$$\left[\textbf{Hint. } \frac{40-\mu}{\sigma} = 0.1 \quad \text{and} \quad \frac{75-\mu}{\sigma} = 1.34.\right]$$

13. The marks obtained by a no. of students for a certain subject are assumed to be approximately normally distributed with mean value 65 and S.D. of 5. If 3 students are taken at random from this set, what is the probability that exactly 2 of them will have marks over 70?

14. In an examination, it is laid down that a student passes if he secures 30% or more marks. He is placed in the first, second or third division according as he secures 60% or more marks, between 45% and 60% marks and marks between 30% and 45% respectively. He gets distinction incase he secures 80% or more marks. It is noticed from the result that 10% of the students failed in the examination, whereas 5% of them obtained distinction.

Calculate the percentage of students placed in the second division. (Assume normal distribution of marks).

[**Hint.** $P(X < 30) = 0.10$, $P(X \ge 80) = 0.05$]

15. Determine the minimum marks a student must get in order to receive an A grade if the top 10% of the students are awarded A grades in an examination where the mean mark is 72 and standard deviation is 9.

16. (*i*) When the mean of marks was 50% and S.D. 5% then 60% of the students failed in an examination. Determine the 'grace' marks to be awarded in order to show that 70% of the students passed. Assume that the marks are normally distributed.

$$\left[\textbf{Hint. } \frac{x_1 - 0.5}{0.05} = 0.25, \quad \frac{x_2 - 0.5}{0.05} = -0.52\right]$$

(*ii*) In a university examination of a particular year, 60% of the students failed when mean of the marks was 50% and S.D. 5%. University decided to relax the conditions of passing by lowering the pass marks to show its result 70%. Find the minimum marks for a student to pass supposing the marks to be normally distributed and no change in the performance of students takes place.

17. How does a normal distribution differ from a binomial distribution? What are the important properties of normal distribution? [*M.T.U. (MBA) 2012*]

18. If the skulls are classified as A, B and C according as the length-breadth index is under 75, between 75 and 80 or over 80, find approximately (assuming that the distribution is normal) the mean and standard deviation of a series in which A are 58%, B are 38% and C are 4%, being given that if $f(t) = \frac{1}{\sqrt{2\pi}} \int_0^t e^{-(x^2/2)}\, dx$ then $f(0.20) = 0.08$ and $f(1.75) = 0.46$.

[**Hint:** $P(X < 75) = 0.58$, $P(X > 80) = 0.04$]

19. The following table gives frequencies of occurrence of a variable X between certain limits:

Variable X	*Frequency*
Less than 40	30
40 or more but less than 50	33
50 and more	37

The distribution is exactly normal. Find the distribution and also obtain the fequency between X = 50 and X = 60.

20. The marks X obtained in Mathematics by 1000 students are normally distributed with mean 78% and standard deviation 11%.

Determine:

(*i*) how many students got marks above 90%?

(*ii*) What was the highest marks obtained by the lowest 10% of students?

(*iii*) Within what limits did the middle 90% of the students lie?

Answers

1. (*i*) 67 (*ii*) 134 (*iii*) 1909 **2.** (*i*) 227 (*ii*) 465

3. (*i*) $\bar{x} = 50.3$, $\sigma = 10.33$ (*ii*) $\mu = 65$, $\sigma = 10$

4. (*i*) 0.8849 (*ii*) 0.8849 (*iii*) 0.7881

5. (*i*) 143 (*ii*) 143 (*iii*) 614

6. (*a*) (*i*) 50%, (*ii*) 21.2%, (*iii*) 84% (*b*) (*i*) 1781, (*ii*) 1576, (*iii*) 1 (*c*) (*i*) 220, (*ii*) 69

7. (*a*) (*i*) 371, (*ii*) 383, (*iii*) 72.72 (*b*) (*i*) 1723 (*ii*) 1906 (*c*) (*i*) 155, (*ii*) 81, (*iii*) 23

8. (*a*) (*i*) 23% (*ii*) 11% (*b*) 0.0228 (*c*) (*i*) 159 (*ii*) 23

9. (*a*) (*i*) 48 (*ii*) 251 (*iii*) 701 (*b*) 294 (*c*) 0.0154

10. (*i*) 79 (*ii*) 35% (*iii*) 11 **11.** 10,000 **12.** 37.2

13. 0.06357 **14.** 34% **15.** 84 marks

16. (*i*) 3.85 (*ii*) 47.4. **18.** $\mu = 74.35$, $\sigma = 3.23$ **19.** $\mu = 46.12$, $\sigma = 11.76$, 25

20. (*i*) 138 (*ii*) 63.92% (*iii*) between 60 and 96.

3.68 POPULATION OR UNIVERSE

An aggregate of objects (animate or inanimate) under study is called **population or universe.** It is thus a collection of individuals or of their attributes (qualities) or of results of operations which can be numerically specified.

A universe containing a finite number of individuals or members is called a **finite inverse.** For example, the universe of the weights of students in a particular class.

A universe with infinite number of members is known as an **infinite universe.** For example, the universe of pressures at various points in the atmosphere.

In some cases, we may be even ignorant whether or not a particular universe is infinite, *e.g.*, the universe of stars.

The universe of concrete objects is an **existent universe.** The collection of all possible ways in which a specified event can happen is called a **hypothetical universe.** The universe of heads and tails obtained by tossing a coin an infinite number of times (provided that it does not wear out) is a hypothetical one.

3.69 SAMPLING

The statistician is often confronted with the problem of discussing universe of which he cannot examine every member *i.e.*, of which complete enumeration is impracticable. For example, if we want to have an idea of the average per capita income of the people of India, enumeration of every earning individual in the country is a very difficult task. Naturally, the question arises : What can be said about a universe of which we can examine only a limited number of members ? This question is the origin of the Theory of Sampling.

A finite subset of a universe is called a **sample.** A sample is thus a small portion of the universe. The number of individuals in a sample is called the **sample size.** The process of selecting a sample from a universe is called **sampling.**

The theory of sampling is a study of relationship existing between a population and samples drawn from the population. The fundamental object of sampling is to get as much information as possible of the whole universe by examining only a part of it. An attempt is thus made through sampling to give the maximum information about the parent universe with the minimum effort.

Sampling is quite often used in our day-to-day practical life. For example, in a shop we assess the quality of sugar, rice or any other commodity by taking only a handful of it from the bag and then decide whether to purchase it or not. A housewife normally tests the cooked products to find if they are properly cooked and contain the proper quantity of salt or sugar, by taking a spoonful of it.

3.70 SAMPLING METHODOLOGIES

Sampling methodologies are classified under two general categories:

1. Probability sampling and
2. Non-probability sampling

In the former, the researcher knows the exact possibility of selecting each member of the population while in the latter, the chance of being included in the sample is not known. A probability sample tends to be more difficult and costly to conduct. However, probability samples are the only type of samples where the results can be generalized from the sample to the population. In addition, probability samples allow the researcher to calculate the precision of the estimates obtained from the sample and to specify the sampling error.

Non-probability samples, incontrast, donot allow the study's findings to be generalized from the sample to the population. When discussing the results of a non-probability sample, the researchers must limit his/her findings to the persons or elements sampled.

This procedure also does not allow the researcher to calculate sampling statistics that provide information about the precision of the results. The advantage of non-probability sampling is the case in which it can be administered.

Non-probability samples tend to be less complicated and less time consuming than probability samples. If the researcher has no intention of generalizing beyond the sample, one of the non-probability sampling methodologies will provide the desired information.

3.71 NON-PROBABILITY SAMPLING

The three common types of non-probability samples are:

(*i*) Convenience Sampling. As the name implies, convenience sampling involves choosing respondents at the convenience of the researcher. Examples of convenience sampling include people-in-the street interviews—the sampling of people to which the researcher has easy access, such as a class of students and studies that use people who have volunteered to be questioned as a result of an advertisement or another type of promotion. A drawback to this methodology is the lack of sampling accuracy. Because the probability of inclusion in the sample is unknown for each respondent, none of the reliability or sampling precision statistics can be calculated. Convenience samples, however, are employed by researchers because the time and cost of collecting information can be reduced.

(*ii*) Quota Sampling [*G.B.T.U. (B. Pharm.) 2010*]

Quota sampling is often confused with stratified and cluster sampling—two probability sampling methodologies. All of these methodologies sample a population that has been subdivided into classes or categories.

The primary differences between the methodologies is that with stratified and cluster sampling, the classes are mutually exclusive and are isolated prior to sampling. Thus, the probability of being selected is known and members of the population selected to be sampled are not arbitrarily disqualified from being included in the results. In quota sampling, the classes cannot be isolated prior to sampling and respondents are categorized into the classes as the survey proceeds. As each class fills or reaches its quota, additional respondents that would have fallen into these classes are rejected or excluded from the results.

An example of a quota sample would be a survey in which the researcher desires to obtain a certain number of respondents from various income categories. Generally, researchers donot know the income of the persons they are sampling until they ask about income. Therefore, the researcher is unable to subdivide the population from which the sample is drawn into mutually exclusive income categories prior to drawing the sample.

(*iii*) Judgemental Sampling. In judgemental or purposive sampling, the researcher employs his or her own expert judgement about who to include in the sample frame. Prior knowledge and research skill are used in selecting the respondents or elements to be sampled.

An example of this type of sample would be a study of potential users of a new recreational facility that is limited to those persons who live within two miles of the new facility. Expert judgement based on past experience indicates that most of the use of this type of facility comes from persons living within two miles. However, by limiting the sample to only this group, usage projections may not be reliable if the usage characteristics of the new facility vary from those previously experienced. As with all non-probability sampling methods, the degree and direction of error introduced by the researcher cannot be measured and statistics that measure the precision of the estimates cannot be calculated.

3.72 PROBABILITY SAMPLING

Five methodologies are most commonly used for conducting probability sampling.

(*i*) Simple Random Sampling. Simple random sampling provides the base from which the other more complex sampling methodologies are derived.

To conduct a simple random sampling, the researcher must first prepare an exhaustive list (sampling frame) of all members of the population of interest. From this list, the sample is drawn so that each person or item has an equal chance of being drawn during each selection round. Samples may be drawn with or without replacement. In practice, however, most simple random sampling for survey research is done without replacement ; that is, a person or item selected for sampling is removed from the population for all subsequent selections. At any draw, the process for a simple random sample without replacement must provide an equal chance of inclusion to any member of the population not already drawn. To draw a simple random sample without introducing researcher bias, computerized sampling programs and random numbers tables are used to impartially select the members of the population to be sampled.

An example of a simple random sample would be a survey of County employees. An exhaustive list of all County employees as of a certain date could be obtained from the Department of Human Resources. If 100 names were selected from this list using a random number table or a computerized sampling program, then a simple random sample would be created. Such a random sampling procedure has the advantage of reducing bias and enables the researcher to estimate sampling errors and the precision of the estimates derived through statistical calculations.

(*ii*) Stratified Random Sampling [*G.B.T.U. (B. Pharm.) 2010*]

Stratified random sampling involves categorizing the members of the population into mutually exclusive and collectively exhaustive groups. An independent simple random sample is then drawn from each group. Stratified sampling techniques can provide more precise estimates if the population being surveyed is more heterogeneous than the categorized groups, can enable the researcher to determine desired levels of sampling precision for each group, and can provide administrative efficiency.

An example of a stratified sample would be a sample conducted to determine the average income earned by families in the United States. To obtain more precise estimates of income, the researcher may want to stratify the sample by geographic region (northeast, mid-Atlantic, etc.) and/or stratify the sample by urban, suburban, and rural groupings. If the differences in income among the regions or groupings are greater than the income differences within the regions or groupings, precision of the estimates is improved. In addition, if the research organization has branch offices located in these regions, the administration of the survey can be decentralized and perhaps conducted in a more cost-efficient manner.

(*iii*) Cluster Sampling. Cluster sampling is similar to stratified sampling because the population to be sampled is subdivided into mutually exclusive groups. However, in cluster sampling, the groups are defined so as to maintain the heterogeneity of the population. It is the researcher's goal to establish clusters that are representative of the population as a whole, although in practice this may be difficult to achieve. After the clusters are established, a simple random sample of the clusters is drawn and the members of the chosen clusters are sampled. If all of the elements (members) of the clusters selected are sampled, then the sampling procedure is defined as **one-stage cluster sampling.** If a random sample of the elements of each selected cluster is drawn, then the sampling procedure is defined as **two-stage cluster sampling.**

Cluster sampling is frequently employed when the researcher is unable to compile a comprehensive list of all the elements in the population of interest. A cluster sample might be used by a researcher attempting to measure the age distribution of persons residing in Mumbai. It would be much more difficult for the researcher to compile a list of every person residing in Mumbai than to compile a list of residential addresses. In this example, each address would represent a cluster of elements (persons) to be sampled. If the elements contained in the clusters are as heterogeneous as the population, then estimates derived from cluster sampling are as precise as those from simple random sampling. However, if the heterogeneity of the clusters is less than that of the population, the estimates will be less precise.

(*iv*) Systematic Sampling. Systematic sampling, a form of one-stage cluster sampling, is often used in place of simple random sampling. In systematic sampling, the researcher selects every n^{th} member after randomly selecting the first through n^{th} element as the starting point. For example, if the researcher decides to sample every 20^{th} member of the population, a 5 percent sample, the starting point for the sample is randomly selected from the first 20 members. A systematic sample is a type of cluster sample because each of the first 20 members of the sampling frame defines a cluster that contains 5 percent of the population.

A researcher may choose to conduct a systematic sample instead of a simple random sample for several reasons. Systematic samples tend to be easier to draw and execute. The researcher does not have to jump backward and forward through the sampling frame to draw the members to be sampled. A systematic sample may spread the members selected for measurement more evenly across the entire population than simple random sampling. Therefore, in some cases, systematic sampling may be more representative of the population and more precise.

One of the most attractive aspects of systematic sampling is that this method can allow the researcher to draw a probability sample without complete prior knowledge of the sampling frame. For example, a survey of visitors to the County's publications desk could be conducted by sampling every 10^{th} visitor after randomly selecting the first through 10^{th} visitor as the starting point. By conducting the sample in this manner, it would not be necessary for the researcher to obtain a comprehensive list of visitors prior to drawing the sample.

As with other types of cluster sampling, systematic sampling is as precise as simple random sampling if the members contained in the clusters are as heterogeneous as the population. If this assumption is not valid, then systematic sampling will be less precise than simple random sampling. In conducting systematic sampling, it is also essential that the researcher does not introduce bias into the sample by selecting an inappropriate sampling interval. For instance, when conducting a sample of financial records, or other items that follow a calendar schedule, the researcher would not want to select "7" as the sampling interval because the sample would then be comprised of observations that were all on the same day of the week. Day-of-the-week influences may cause contamination of the sample, giving the researcher biased results.

(*v*) Multi-Stage Sampling. Multi-stage sampling is like cluster sampling, but involves selecting a sample within each chosen cluster, rather than including all units in the cluster. Thus, multi-stage sampling involves selecting a sample in at least two stages. In the first stage, large groups or clusters are selected. These clusters are designed to contain more population units than are required for the final sample.

In the second stage, population units are chosen from selected clusters to derive a final sample. If more than two stages are used, the process of choosing population units within clusters continues until the final sample is achieved.

An example of multi-stage sampling is where, firstly, electoral sub-divisions (clusters) are sampled from a city or state. Secondly, blocks of houses are selected from within the electoral sub-divisions and, thirdly, individual houses are selected from within the selected blocks of houses.

The advantages of multi-stage sampling are convenience, economy and efficiency. Multi-stage sampling does not require a complete list of members in the target population, which greatly reduces sample preparation cost. The list of members is required only for those clusters used in the final stage. The main disadvantage of multi-stage sampling is the same as for cluster sampling : lower accuracy due to higher sampling error.

3.73 PARAMETERS OF STATISTICS

The statistical constants of the population such as mean, the variance etc. are known as the parameters. The statistical concepts of the sample from the members of the sample to estimate the parameters of the population from which the sample has been drawn is known as *statistic*.

Population mean and variance are denoted by μ and σ^2, while those of the samples are given by $\bar{x}$, s^2.

3.74 STANDARD ERROR

The standard deviation of the sampling distribution of a statistic is known as the **standard error (S.E.).** It plays an important role in the theory of large samples and it forms a basis of the testing of hypothesis. If t is any statistic, for large sample. $z = \dfrac{t - \mathrm{E}(t)}{\mathrm{S.E.}(t)}$ is normally distributed with mean 0 and variance unity.

3.75 TEST OF SIGNIFICANCE

An important aspect of the sampling theory is to study the test of significance which will enable us to decide, on the basis of the results of the sample, whether

(*i*) *the deviation between the observed sample statistic and the hypothetical parameter value or*

(*ii*) *the deviation between two sample statistics*

is significant or might be attributed due to chance or the fluctuations of the sampling.

3.76 TESTING OF STATISTICAL HYPOTHESIS

Step 1. Null hypothesis: [*U.P.T.U. (MCA) 2007*]

For applying the tests of significance, we first set up a hypothesis which is a definite statement about the population parameter called Null Hypothesis. It is denoted by H_0.

Null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true. First, we set up H_0 in clear terms.

Step 2. Alternative hypothesis:

Any hypothesis which is complementary to the null hypothesis (H_0) is called an alternative hypothesis. It is denoted by H_1.

For example, if we want to test the null hypothesis that the population has a specified mean μ_0 then we have

$$H_0 : \mu = \mu_0$$

then the alternative hypothesis will be

(*i*) $H_1 : \mu \neq \mu_0$ (Two tailed alternative hypothesis)

(*ii*) $H_1 : \mu > \mu_0$ (right tailed alternative hypothesis (or) single tailed)

(*iii*) $H_1 : \mu < \mu_0$ (left tailed alternative hypothesis (or) single tailed)

Hence alternative hypothesis helps to know whether the test is two tailed or one tailed. Therefore, we set up H_1 for this decision.

Step 3. Level of significance: [*U.P.T.U. (MCA) 2008, 2007*]

The probability of the value of the variate falling in the critical region is known as level of significance. A region corresponding to a statistic t in the sample space S which amounts to rejection of the null hypothesis H_0 is called as **critical region** or region of rejection while which amounts to acceptance of H_0 is called **acceptance region.** The probability α that a random value of the statistic t belongs to the critical region is known as the **level of significance.**

$$P(t \in w/H_0) = \alpha$$

i.e., the level of significance is the size of the type I error (refer art. 3.77) or the maximum producer's risk.

We select the appropriate level of significance in advance depending on the reliability of the estimates.

Step 4. Test statistic (or test criterion): We compute the test statistic z under the null hypothesis. For larger samples corresponding to the statistic t, the variable $z = \dfrac{t - E(t)}{S.E.(t)}$ is normally distributed with mean 0 and variance 1. The value of z given above under the null hypothesis is known as **test statistic.**

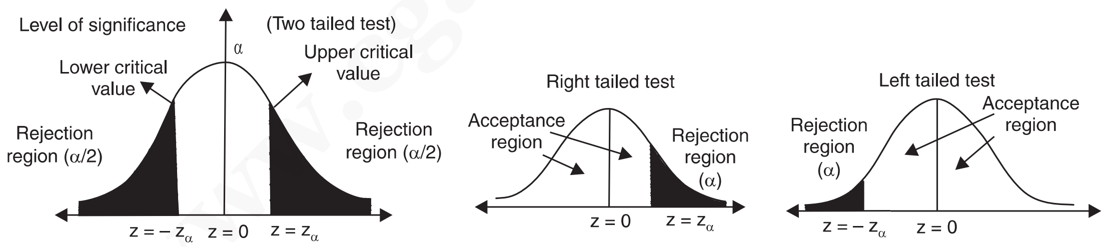
Step 5. Conclusion: We compare the computed value of z with the critical value z_α at level of significance (α). The critical value of z_α of the test statistic at level of significance α for a two tailed test is given by

$$p(|z| > z_\alpha) = \alpha \qquad \ldots(1)$$

i.e., z_α is the value of z so that the total area of the critical region on both tails is α. Since the normal curve is symmetrical, from equation (1), we get

$$p(z > z_\alpha) + p(z < -z_\alpha) = \alpha \; ; \; i.e., \; 2p(z > z_\alpha) = \alpha \; ; \; i.e., \; p(z > z_\alpha) = \alpha/2$$

i.e., the area of each tail is $\alpha/2$.



The critical value z_α is that value such that the area to the right of z_α is $\alpha/2$ and the area to the left of $-z_\alpha$ is $\alpha/2$.

In the case of one tailed test,

$p(z > z_\alpha) = \alpha$ if it is right tailed ; $p(z < -z_\alpha) = \alpha$ if it is left tailed.

The critical value of z for a single tailed test (right or left) at level of significance α is same as the critical value of z for two tailed test at level of significance 2α.

Using the equation, also using the normal tables, the critical value of z at different levels of significance (α) for both single tailed and two tailed test are calculated and listed below. The equations are

$$p(|z| > z_\alpha) = \alpha \; ; \; p(z > z_\alpha) = \alpha \; ; \; p(z < -z_\alpha) = \alpha$$

	Level of significance		
	1% (0.01)	5% (0.05)	10% (0.1)
Two tailed test	$\mid z_\alpha \mid = 2.58$	$\mid z_\alpha \mid = 1.96$	$\mid z_\alpha \mid = 1.645$
Right tailed test	$z_\alpha = 2.33$	$z_\alpha = 1.645$	$z_\alpha = 1.28$
Left tailed test	$z_\alpha = -2.33$	$z_\alpha = -1.645$	$z_\alpha = -1.28$

If $|z| > z_\alpha$, we reject H_0 and conclude that there is significant difference. If $|z| < z_\alpha$, we accept H_0 and conclude that there is no significant difference.

3.77 ERRORS IN SAMPLING

The main aim of the sampling theory is to draw a valid conclusion about the population parameters on the basis of the sample results. In doing this we may commit the following two types of errors:

Type I. Error. [*U.P.T.U. (MCA) 2008, 2007*]

When H_0 is true, we may reject it.

P(Reject H_0 when it is true) = P(Reject H_0/H_0) = α

α is called the size of the type I error also referred to as **producer's risk.**

Type II. Error. When H_0 is wrong, we may accept it P(Accept H_0 when it is wrong) = P(Accept H_0/H_1) = β . β is called the size of the type II error, also referred to as **consumer's risk.**

Note. The values of the test statistic which separates the critical region and acceptance region are called the **critical values** or **significant values.** This value is dependent on (*i*) the level of significance used and (*ii*) the alternative hypothesis, whether it is one tailed or two tailed.

3.78 TEST OF SIGNIFICANCE OF SMALL SAMPLES

When the size of the sample is less than 30, then the sample is called small sample. For such sample it will not be possible for us to assume that the random sampling distribution of a statistic is approximately normal and the values given by the sample data are sufficiently close to the population values and can be used in their place for the calculation of the standard error of the estimate.

3.79 STUDENT'S t-DISTRIBUTION (t-Test)

[*G.B.T.U. (MBA) 2011 ; G.B.T.U. (MCA) 2010*]

This t-distribution is used when sample size is ≤ 30 and the population standard deviation is unknown.

t-statistic is defined as

$$\boxed{t = \frac{\bar{x} - \mu}{S/\sqrt{n}}} \qquad \text{where,} \quad S = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n - 1}}$$

$\bar{x}$ is the mean of sample, μ is population mean. S is the standard deviation of population and n is sample size.

If the standard deviation of the sample 's' is given then t-statistic is defined as

$$\boxed{t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}}$$

Note. The relation between s and S is $ns^2 = (n - 1)S^2$.

3.79.1. The t-Table

The t-table given at the end is the probability integral of t-distribution. The t-distribution has different values for each degrees of freedom and when the degrees of freedom are infinitely large, the t-distribution is equivalent to normal distribution and the probabilities shown in the normal distribution tables are applicable.

3.79.2. Applications of t-Distribution [*G.B.T.U. (MBA) 2011*]

Some of the applications of t-distribution are given below:

1. *To test if the sample mean* $(\bar{x})$ *differs significantly from the hypothetical value* μ *of the population mean.*
2. *To test the significance between two sample means.*
3. *To test the significance of observed partial and multiple correlation coefficients.*

3.79.3. Critical Value of t

The critical value or significant value of t at level of significance α, degrees of freedom γ for two tailed test is given by

$$P[\,|\,t\,| > t_\gamma(\alpha)] = \alpha$$
$$P[\,|\,t\,| \le t_\gamma(\alpha)] = 1 - \alpha$$

The significant value of t at level of significance α, for a single tailed test can be got from those of two tailed test by referring to the values at 2α.

3.80 TEST I : t-TEST OF SIGNIFICANCE OF THE MEAN OF A RANDOM SAMPLE

To test whether the mean of a sample drawn from a normal population deviates significantly from a stated value when variance of the population is unknown.

H_0 : There is no significant difference between the sample mean $\bar{x}$ and the population mean μ *i.e.,* we use the statistic

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} \qquad \text{where} \quad S = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n-1}}$$

with degree of freedom $n - 1$.

At given level of significance α and degrees of freedom $(n - 1)$, we refer to t-table t_α (two tailed or one tailed). If calculated t value is such that $|\,t\,| < t_\alpha$, the null hypothesis is accepted. If $|\,t\,| > t_\alpha$, H_0 is rejected.

3.80.1. Fiducial Limits of Population Mean

If t_α is the value of t at level of significance α at $(n - 1)$ degrees of freedom then,

$$\left|\frac{\bar{x} - \mu}{S/\sqrt{n}}\right| < t_\alpha \text{ for acceptance of } H_0.$$

$$\bar{x} - t_\alpha \, S/\sqrt{n} < \mu < \bar{x} + t_\alpha \, S/\sqrt{n}$$

95% confidence limits (level of significance 5%) are $\bar{x} \pm t_{0.05} S/\sqrt{n}$.

99% confidence limits (level of significance 1%) are $\bar{x} \pm t_{0.01} S/\sqrt{n}$.

EXAMPLES

Example 1. *A random sample of size 16 has 53 as mean. The sum of squares of the deviation from mean is 135. Can this sample be regarded as taken from the population having 56 as mean? Obtain 95% and 99% confidence limits of the mean of the population.*

Sol. Null hypothesis, H_0 : There is no significant difference between the sample mean and hypothetical population mean *i.e.,* $\mu = 56$.

Alternative hypothesis, $H_1 : \mu \neq 56$ (**Two tailed test**)

Test statistic. Under H_0, test statistic is $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$

Given: $\bar{x} = 53$, $\mu = 56$, $n = 16$, $\Sigma(x - \bar{x})^2 = 135$

$$S = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n - 1}} = \sqrt{\frac{135}{15}} = 3$$

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{53 - 56}{3/\sqrt{16}} = -4$$

$|\, t \,| = 4$

d.fv. $= 16 - 1 = 15$.

Conclusion. Since $|\, t \,| = 4 > t_{0.05} = 2.13$ *i.e.,* the calculated value of t is more than the tabulated value, the null hypothesis is rejected. Hence, the sample mean has not come from a population having 56 as mean.

95% confidence limits of the population mean

$$= \bar{x} \pm \frac{S}{\sqrt{n}} t_{0.05} = 53 \pm \frac{3}{\sqrt{16}} (2.13) = 51.4025, 54.5975$$

99% confidence limits of the population mean

$$= \bar{x} \pm \frac{S}{\sqrt{n}} t_{0.01,} = 53 \pm \frac{3}{\sqrt{16}} (2.95) = 50.7875, 55.2125.$$

Example 2. *The lifetime of electric bulbs for a random sample of 10 from a large consignment gave the following data:*

Item	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Life in '000 hrs.	*4.2*	*4.6*	*3.9*	*4.1*	*5.2*	*3.8*	*3.9*	*4.3*	*4.4*	*5.6*

Can we accept the hypothesis that the average lifetime of bulb is 4000 hrs?

Sol. Null hypothesis: H_0: There is no significant difference in the sample mean and population mean. *i.e.,* $\mu = 4000$ hrs.

Alternative hypothesis: $\mu \neq 4000$ hrs (**Two tailed test**)

Test statistic: Under H_0, the test statistic is $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$

x	4.2	4.6	3.9	4.1	5.2	3.8	3.9	4.3	4.4	5.6
$x - \bar{x}$	– 0.2	0.2	– 0.5	– 0.3	0.8	– 0.6	– 0.5	– 0.1	0	1.2
$(x - \bar{x})^2$	0.04	0.04	0.25	0.09	0.64	0.36	0.25	0.01	0	1.44

$$\bar{x} = \frac{\Sigma x}{n} = \frac{44}{10} = 4.4, \qquad \Sigma(x - \bar{x})^2 = 3.12$$

$$S = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n - 1}} = 0.589$$

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{4.4 - 4}{\left(\frac{0.589}{\sqrt{10}}\right)} = 2.123$$

For $\gamma = 9$, $t_{0.05} = 2.26$.

Conclusion. Since the calculated value of t is less than the tabulated value of t at 5% level of significance. $\therefore$ The null hypothesis $\mu = 4000$ hrs is accepted *i.e.*, the average lifetime of bulbs could be 4000 hrs.

Example 3. *A sample of 20 items has mean 42 units and S.D. 5 units. Test the hypothesis that it is a random sample from a normal population with mean 45 units.*

Sol. Null hypothesis: H_0 : There is no significant difference between the sample mean and the population mean. *i.e.,* $\mu = 45$ units

Alternative hypothesis, H_1: $\mu \neq 45$ **(Two tailed test)**

Given: $n = 20$, $\bar{x} = 42$, $s = 5$; $\gamma = 19$ d.f.

Test statistic: Under H_0, the test statistic is

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{42 - 45}{5/\sqrt{19}} = -2.615$$

$$\therefore \qquad |\, t \,| = 2.615$$

The tabulated value of t at 5% level for 19 d.f. is $t_{0.05} = 2.09$.

Conclusion. Since the calculated value $|\, t \,|$ is greater than the tabulated value of t at 5% level of significance, the null hypothesis H_0 is rejected. *i.e.,* there is significant difference between the sample mean and population mean.

i.e., the sample could not have come from this population.

Example 4. *The 9 items of a sample have the following values*

45, 47, 50, 52, 48, 47, 49, 53, 51.

Does the mean of these values differ significantly from the assumed mean 47.5?

Sol. Here, $n = 9$, $\mu = 47.5$, $\bar{x} = \frac{\Sigma x}{n} = 49.1$

x	45	47	50	52	48	47	49	53	51
$x - \bar{x}$	– 4.1	– 2.1	0.9	2.9	– 1.1	– 2.1	– .1	3.9	1.9
$(x - \bar{x})^2$	16.81	4.41	.81	8.41	1.21	4.41	.01	15.21	3.61

$$\Sigma(x - \bar{x})^2 = 54.89,$$

$$S^2 = \frac{\Sigma\,(x - \bar{x})^2}{n - 1} = 6.86$$

$$\therefore \qquad S = 2.619$$

Null hypothesis:

$H_0 : \mu = 47.5$

i.e., there is no significant difference between the sample and population means.

Alternative hypothesis:

$H_1 : \mu \neq 47.5$

Hence we apply **two-tailed test**.

Test statistic: Under H_0, the test statistic is

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{49.1 - 47.5}{(2.619/\sqrt{9})} = 1.8327$$

$t_{0.05} = 2.31$ for $\gamma = 8$

Conclusion: Since $|t|_{\text{calculated}} < t_{\text{tabulated}}$ at 5% level of significance, the null hypothesis H_0 is accepted *i.e.,* there is no significant difference between their means.

ASSIGNMENT

1. Ten individuals are chosen at random from a normal population of students and their marks are found to be 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. In the light of these data, discuss the suggestion that mean mark of the population of students is 66.
2. The following values gives the lengths of 12 samples of Egyptian cotton taken from a consignment : 48, 46, 49, 46, 52, 45, 43, 47, 47, 46, 45, 50. Test if the mean length of the consignment can be taken as 46.
3. A sample of 18 items has a mean 24 units and standard deviation 3 units. Test the hypothesis that it is a random sample from a normal population with mean 27 units.
4. A random sample of 10 boys had the I.Q.'s 70, 120, 110, 101, 88, 83, 95, 98, 107 and 100. Do these data support the assumption of a population mean I.Q. of 160?

3.81 TEST II : t-TEST FOR DIFFERENCE OF MEANS OF TWO SMALL SAMPLES (from a Normal Population)

This test is used to test whether the two samples $x_1, x_2, \ldots\ldots, x_{n_1}, y_1, y_2, \ldots\ldots, y_{n_2}$ of sizes n_1, n_2 have been drawn from two normal populations with mean μ_1 and μ_2 respectively under the assumption that the population variance are equal. $(\sigma_1 = \sigma_2 = \sigma)$.

H_0 : The samples have been drawn from the normal population with means μ_1 and μ_2 *i.e.,* $H_0 : \mu_1 \neq \mu_2$.

Let $\bar{x}, \bar{y}$ be their means of the two samples.

Under this H_0 the test statistic t is given by $t = \dfrac{(\bar{x} - \bar{y})}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$

Degree of freedom is $n_1 + n_2 - 2$.

Note 1. If the two sample's standard deviations s_1, s_2 are given then we have $S^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$.

Note 2. If s_1, s_2 are not given then $S^2 = \dfrac{\Sigma(x_1 - \bar{x}_1)^2 + \Sigma(x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$.

EXAMPLES

Example 1. *Two samples of sodium vapour bulbs were tested for length of life and the following results were got*:

	Size	*Sample mean*	*Sample S.D.*
Type I	*8*	*1234 hrs*	*36 hrs*
Type II	*7*	*1036 hrs*	*40 hrs*

Is the difference in the means significant to generalise that Type I is superior to Type II regarding length of life?

Sol. Null hypothesis,

$H_0 : \mu_1 = \mu_2$ *i.e.,* two types of bulbs have same lifetime.

Alternative hypothesis,

$H_1 : \mu_1 > \mu_2$ *i.e.,* type I is superior to Type II.

Hence we use **right tailed test.**

$$S^2 = \frac{n_1 s_1{}^2 + n_2 s_2{}^2}{n_1 + n_2 - 2} = \frac{8(36)^2 + 7(40)^2}{8 + 7 - 2} = 1659.076$$

$\therefore$ $\quad S = 40.7317$

Test statistic: Under H_0, the test statistic t is given by

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{1234 - 1036}{40.7317\sqrt{\frac{1}{8} + \frac{1}{7}}} = 18.1480$$

$t_{0.05}$ at d.f. $\gamma = n_1 + n_2 - 2 = 13$ is 1.77.

Conclusion. Since calculated $|\ t\ | > t_{\text{tabulated}}$ at 5% level of significance, H_0 is rejected.

$\therefore$ Type I is definitely superior to Type II.

Example 2. *Samples of sizes 10 and 14 were taken from two normal populations with S.D. 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of the two populations are the same at 5% level.*

Sol. We have, $\bar{x}_1 = 20.3$, $\bar{x}_2 = 18.6$, $n_1 = 10$, $n_2 = 14$, $s_1 = 3.5$, $s_2 = 5.2$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = 22.775$$

$\therefore$ $\quad S = 4.772$

Null hypothesis:

$H_0 : \mu_1 = \mu_2$ *i.e.,* the means of the two populations are the same.

Alternative hypothesis:

$H_1 : \mu_1 \neq \mu_2$

Test statistic: Under H_0, the test statistic is

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{20.3 - 18.6}{4.772\sqrt{\frac{1}{10} + \frac{1}{14}}} = 0.8604$$

The tabulated value of t at 5% level of significance for 22 *d.f.* is $t_{0.05} = 2.0739$

Conclusion:

Since $t = 0.8604 < t_{0.05}$, the null hypothesis H_0 is accepted; *i.e.,* there is no significant difference between their means.

Example 3. *The height of 6 randomly chosen sailors in inches are 63, 65, 68, 69, 71 and 72. Those of 9 randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Test whether the sailors are on the average taller than soldiers.*

Sol. Let X_1 and X_2 be the two samples denoting the heights of sailors and soldiers.

$$n_1 = 6, n_2 = 9$$

Null hypothesis, $H_0 : \mu_1 = \mu_2$.

i.e., the mean of both the population are the same.

Alternative hypothesis, $H_1 : \mu_1 > \mu_2$ (**one tailed test**)

Calculation of two sample means:

X_1	63	65	68	69	71	72
$X_1 - \overline{X}_1$	– 5	– 3	0	1	3	4
$(X_1 - \overline{X}_1)^2$	25	9	0	1	9	16

$$\overline{X}_1 = \frac{\Sigma X_1}{n_1} = 68 \; ; \; \Sigma(X_1 - \overline{X}_1)^2 = 60$$

X_2	61	62	65	66	69	70	71	72	73
$X_2 - \overline{X}_2$	– 6.66	– 5.66	– 2.66	1.66	1.34	2.34	3.34	4.34	5.34
$(X_2 - \overline{X}_2)^2$	44.36	32.035	7.0756	2.7556	1.7956	5.4756	11.1556	18.8356	28.5156

$$\overline{X}_2 = \frac{\Sigma X_2}{n_2} = 67.66 \; ; \; \Sigma(X_2 - \overline{X}_2)^2 = 152.0002$$

$$S^2 = \frac{1}{n_1 + n_2 - 2}[\Sigma(X_1 - \overline{X}_1)^2 + \Sigma(X_2 - \overline{X}_2)^2] = 16.3077$$

$$\therefore \qquad S = 4.038$$

Test statistic:

Under H_0,
$$t = \frac{\overline{X}_1 - \overline{X}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{68 - 67.666}{4.038\sqrt{\frac{1}{6} + \frac{1}{9}}} = 0.1569$$

The value of t at 5% level of significance for 13 d.f. is 1.77. (d.f. $= n_1 + n_2 - 2$)

Conclusion. Since $t_{\text{calculated}} < t_{0.05} = 1.77$, the null hypothesis H_0 is accepted.

i.e., there is no significant difference between their average.

i.e., the sailors are not on the average taller than the soldiers.

ASSIGNMENT

1. The mean life of 10 electric motors was found to be 1450 hrs with S.D. of 423 hrs. A second sample of 17 motors chosen from a different batch showed a mean life of 1280 hrs with a S.D. of 398 hrs. Is there a significant difference between means of the two samples?
2. The marks obtained by a group of 9 regular course students and another group of 11 part time course students in a test are given below:

Regular:	56	62	63	54	60	51	67	69	58		
Part time:	62	70	71	62	60	56	75	64	72	68	66

   Examine whether the marks obtained by regular students and part time students differ significantly at 5% and 1% level of significance.
3. A group of 5 patients treated with the medicine A weigh 42, 39, 48, 60 and 41 kgs. A second group of 7 patients from the same hospital treated with medicine B weigh 38, 42, 56, 64, 68, 69 and 62 kgs. Do you agree with the claim that medicine B increases the weight significantly? It is given that the value of t at 10% level of significance for 10 degree of freedom is 1.81. [*G.B.T.U. (B. Pharm.) 2010*]
4. Two independent samples of sizes 7 and 9 have the following values:

Sample A:	10	12	10	13	14	11	10		
Sample B:	10	13	15	12	10	14	11	12	11

   Test whether the difference between the mean is significant.
5. The average no. of articles produced by two machines per day are 200 and 250 with standard deviations 20 and 25 respectively on the basis of records of 25 days production. Can you regard both the machines equally efficient at 5% level of significance?

3.82 CHI-SQUARE (χ^2) TEST [*G.B.T.U. 2010 ; G.B.T.U. MCA (SUM) 2010*]

When a coin is tossed 200 times, the theoretical considerations lead us to expect 100 heads and 100 tails. But in practice, these results are rarely achieved. The quantity χ^2 (a Greek letter, pronounced as chi-square) describes the magnitude of discrepancy between theory and observation. If $\chi^2 = 0$, the observed and expected frequencies completely coincide. The greater the discrepancy between the observed and expected frequencies, the greater is the value of χ^2. Thus χ^2 **affords a measure of the correspondence between theory and observation.**

If O_i ($i = 1, 2, ..., n$) is a set of observed (experimental) frequencies and E_i ($i = 1, 2, ..., n$) is the corresponding set of expected (theoretical or hypothetical) frequencies, then, χ^2 **is defined as**

$$\boxed{\chi^2 = \sum_{i=1}^{n} \left[\frac{(O_i - E_i)^2}{E_i}\right]}$$

where $\Sigma O_i = \Sigma E_i = N$ (total frequency) and degrees of freedom (d.f.) = $(n - 1)$.

Note. (*i*) If $\chi^2 = 0$, the observed and theoretical frequencies agree exactly.

(*ii*) If $\chi^2 > 0$ they do not agree exactly.

3.83 DEGREES OF FREEDOM

While comparing the calculated value of χ^2 with the tabular value, we have to determine the degrees of freedom.

If we have to choose any four numbers whose sum is 50, we can exercise our independent choice for any three numbers only, the fourth being 50 minus the total of the three numbers selected. Thus, though we were to choose any four numbers, our choice was reduced to three because of one condition imposed. There was only one restraint on our freedom and our degrees of freedom were $4 - 1 = 3$. If two restrictions are imposed, our freedom to choose will be further curtailed and degrees of freedom will be $4 - 2 = 2$.

In general, the number of degrees of freedom is the total number of observations less the number of independent constraints imposed on the observations. Degrees of freedom (d.f.) are usually denoted by ν.

Thus, $\nu = n - k$, where k is the number of independent constraints in a set of data of n observations.

Note. (*i*) For a $p \times q$ contingency table (p columns and q rows), $\nu = (p - 1)(q - 1)$

(*ii*) In the case of a contingency table, the expected frequency of any class

$$= \frac{\text{Total of rows in which it occurs} \times \text{Total of columns in which it occurs}}{\text{Total number of observations}}$$

3.84 APPLICATIONS OF CHI-SQUARE TEST

χ^2 test is one of the simplest and the most general test known. It is applicable to a very large number of problems in practice which can be summed up under the following heads:

(*i*) as a test of goodness of fit.

(*ii*) as a test of independence of attributes.

(*iii*) as a test of homogeneity of independent estimates of the population variance.

(*iv*) as a test of the hypothetical value of the population variance σ^2.

(*v*) as a list to the homogeneity of independent estimates of the population correlation coefficient.

3.85 CONDITIONS FOR APPLYING χ^2 TEST

χ^2-test is an approximate test for large values of n. For the validity of chi-square test of goodness of fit between theory and experiment, the following conditions must be satisfied.

(*a*) The sample observations should be independent.

(*b*) The constraints on the cell frequencies, if any, should be linear *e.g.,* $\Sigma n_i = \Sigma \lambda_i$ or $\Sigma O_i = \Sigma E_i$.

(*c*) N, the total number of frequencies should be reasonably large. It is difficult to say what constitutes largeness, but as an arbitrary figure, we may say that **N should be atleast 50,** however, few the cells.

(*d*) No theoretical cell-frequency should be small. Here again, it is difficult to say what constitutes smallness, but 5 should be regarded as the very minimum and **10 is better.** If small theoretical frequencies occur (*i.e.*, < 10), the difficulty is overcome by grouping two or more classes together before calculating (O – E). **It is important to remember that the number of degrees of freedom is determined with the number of classes after regrouping.**

Note 1. If any one of the theoretical frequency is less than 5, then we apply a corrected given by F Yates, which is usually known as 'Yates correction for continuity', we add 0.5 to the cell frequency which is less than 5 and adjust the remaining cell frequency suitably so that the marginal total is not changed.

Note 2. It may be noted that the χ^2-test depends only on the set of observed and expected frequencies and on degrees of freedom (d.f.). It does not make any assumption regarding the parent population from which the observations are taken. Since χ^2 does not involve any population parameters, it is termed as a statistic and the test is known as Non-parametric test or Distribution-free test.

3.86 THE χ^2 DISTRIBUTION

For large sample sizes, the sampling distribution of χ^2 can be closely approximated by a continuous curve known as the chi-square distribution. The probability function of χ^2 distribution is given by

$$f(\chi^2) = c(\chi^2)^{(\nu/2-1)}\, e^{-x^2/2}$$

where $e = 2.71828$, ν = number of degrees of freedom ; c = a constant depending only on ν.

Symbolically, the degrees of freedom are denoted by the symbol ν or by d.f. and are obtained by the rule $\nu = n - k$, where k refers to the number of independent constraints.

In general, when we fit a binomial distribution the number of degrees of freedom is one less than the number of classes ; when we fit a Poisson distribution the degrees of freedom are 2 less than the number of classes, because we use the total frequency and the arithmetic mean to get the parameter of the Poisson distribution. When we fit a normal curve the number of degrees of freedom are 3 less than the number of classes, because in this fitting we use the total frequency, mean and standard deviation.

If the data is given in a series of "n" numbers then degrees of freedom = $n - 1$.

In the case of Binomial distribution d.f. = $n - 1$

In the case of Poisson distribution d.f. = $n - 2$

In the case of Normal distribution d.f. = $n - 3$.

3.87 χ^2 TEST AS A TEST OF GOODNESS OF FIT

χ^2 test enables us to ascertain how well the theoretical distributions such as Binomial, Poisson or Normal etc. fit empirical distributions, *i.e.*, distributions obtained from sample data. If the **calculated value of χ^2 is less than the tabular value** at a specified level (generally 5%) of significance, the **fit is considered to be good** *i.e.*, the divergence between actual and expected frequencies is attributed to fluctuations of simple sampling. If the calculated value of χ^2 is greater than the tabular value, the fit is considered to be poor.

EXAMPLES

Example 1. *In experiments on pea breeding, the following frequencies of seeds were obtained:*

Round and yellow	*Wrinkled and yellow*	*Round and green*	*Wrinkled and green*	*Total*
315	*101*	*108*	*32*	*556*

Theory predicts that the frequencies should be in proportions 9 : 3 : 3 : 1. Examine the correspondence between theory and experiment.

Sol. Null hypothesis:

H_0 : The experimental result support the theory *i.e.,* there is no significant difference between the observed and theoretical frequency.

Under H_0, The theoretical (expected) frequencies can be calculated as follows:

$$E_1 = \frac{556 \times 9}{16} = 312.75 \qquad E_2 = \frac{556 \times 3}{16} = 104.25$$

$$E_3 = \frac{556 \times 3}{16} = 104.25 \qquad E_4 = \frac{556 \times 1}{16} = 34.75$$

To calculate the value of χ^2:

Observed frequency O_i	315	101	108	32
Expected Frequency E_i	312.75	104.25	104.25	34.75
$\frac{(O_i - E_i)^2}{E_i}$	0.016187	0.101319	0.134892	0.217626

$$\chi^2 = \sum \left[\frac{(O_i - E_i)^2}{E_i}\right] = 0.470024$$

Tabular value of χ^2 at 5% level of significance for $n - 1 = 3$ *d.f.* is 7.815 *i.e.*, $\chi^2_{0.05} = 7.815$.

Conclusion: Since the calculated value of χ^2 is less than that of the tabulated value, hence H_0 is accepted. Therefore, the experimental results support the theory.

Example 2. *The following table gives the number of accidents that took place in an industry during various days of the week. Test if accidents are uniformly distributed over the week.*

Day	*Mon*	*Tue*	*Wed*	*Thu*	*Fri*	*Sat*
No. of accidents	*14*	*18*	*12*	*11*	*15*	*14*

Sol. Null hypothesis H_0: The accidents are uniformly distributed over the week.

Under this H_0, the expected frequencies of the accidents on each of these days $= \frac{84}{6} = 14$

Observed frequency O_i	14	18	12	11	15	14
Expected frequency E_i	14	14	14	14	14	14
$(O_i - E_i)^2$	0	16	4	9	1	0

$$\chi^2 = \sum \left[\frac{(O_i - E_i)^2}{E_i}\right] = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{30}{14} = 2.1428.$$

Tabular value of χ^2 at 5% level for (6 – 1 = 5 d.f.) is 11.09.

Conclusion: Since the calculated value of χ^2 is less than the tabulated value, H_0 is accepted *i.e.*, the accidents are uniformly distributed over the week.

Example 3. *A die is thrown 276 times and the results of these throws are given below*:

No. appeared on the die	*1*	*2*	*3*	*4*	*5*	*6*
Frequency	*40*	*32*	*29*	*59*	*57*	*59*

Test whether the die is biased or not.

Sol. Null hypothesis H_0: Die is unbiased.

Under this H_0, the expected frequencies for each digit is $\frac{276}{6} = 46$.

To find the value of χ^2

O_i	40	32	29	59	57	59
E_i	46	46	46	46	46	46
$(O_i - E_i)^2$	36	196	289	169	121	169

$$\chi^2 = \sum \left[\frac{(O_i - E_i)^2}{E_i}\right] = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{980}{46} = 21.30.$$

Tabulated value of χ^2 at 5% level of significance for $(6 - 1 = 5)$ d.f. is 11.09.

Conclusion. Since the calculated value of $\chi^2 = 21.30 > 11.07$ the tabulated value, H_0 is rejected. *i.e.,* die is not unbiased or die is biased.

Example 4. *Records taken of the number of male and female births in 800 families having four children are as follows:* [U.P.T.U. (MCA) 2009]

No. of male births	*0*	*1*	*2*	*3*	*4*
No. of female births	*4*	*3*	*2*	*1*	*0*
No. of families	*32*	*178*	*290*	*236*	*64*

Test whether the data are consistent with the hypothesis that the Binomial law holds and the chance of male birth is equal to that of female birth, namely $p = q = 1/2$.

Sol. Null hypothesis H_0: The data are consistent with the hypothesis of equal probability for male and female births. *i.e.,* $p = q = 1/2$.

We use Binomial distribution to calculate theoretical frequency given by:

$$N(r) = N \times P(X = r) = N \times {}^nC_r\, p^r q^{n-r}$$

where N is the total frequency, $N(r)$ is the number of families with r male children,

p and q are probabilities of male and female births respectively, n is the number of children.

$$N(0) = 800 \times {}^4C_0 \left(\frac{1}{2}\right)^4 = 50, \quad N(1) = 200, N(2) = 300, N(3) = 200 \text{ and } N(4) = 50$$

Observed frequency O_i	32	178	290	236	64
Expected frequency E_i	50	200	300	200	50
$(O_i - E_i)^2$	324	484	100	1296	196
$\frac{(O_i - E_i)^2}{E_i}$	6.48	2.42	0.333	6.48	3.92

$$\chi^2 = \sum \left[\frac{(O_i - E_i)^2}{E_i}\right] = 19.633.$$

Tabulated value of χ^2 at 5% level of significance for $5 - 1 = 4$ d.f. is 9.49.

Conclusion. Since the calculated value of χ^2 is greater than the tabulated value, H_0 is rejected. *i.e.,* the data are not consistent with the hypothesis that the Binomial law holds and that the chance of a male birth is not equal to that of a female birth.

Example 5. *The theory predicts the proportion of beans in the four groups, G_1, G_2, G_3, G_4 should be in the ratio 9 : 3 : 3 : 1. In an experiment with 1600 beans the numbers in the four groups were 882, 313, 287 and 118. Does the experimental result support the theory?*

Sol. Null hypothesis H_0: The experimental result support the theory. *i.e.,* there is no significant difference between the observed and theoretical frequency.

Under H_0, the theoretical frequency can be calculated as follows:

$$E(G_1) = \frac{1600 \times 9}{16} = 900; \quad E(G_2) = \frac{1600 \times 3}{16} = 300;$$

$$E(G_3) = \frac{1600 \times 3}{16} = 300; \quad E(G_4) = \frac{1600 \times 1}{16} = 100$$

To calculate the value of χ^2.

Observed frequency O_i	882	313	287	118
Expected frequency E_i	900	300	300	100
$\frac{(O_i - E_i)^2}{E_i}$	0.36	0.5633	0.5633	3.24

$$\chi^2 = \sum \left[\frac{(O_i - E_i)^2}{E_i} \right] = 4.7266.$$

Tablular value of χ^2 at 5% level of significance for 3 d.f. is 7.815.

Conclusion: Since the calculated value of χ^2 is less than that of the tabulated value, hence H_0 is accepted. *i.e.,* the experimental results support the theory.

Example 6. *The following table shows the distribution of digits in numbers chosen at random from a telephone directory:*

Digits	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
Frequency	*1026*	*1107*	*997*	*966*	*1075*	*933*	*1107*	*972*	*964*	*853*

Test whether the digits may be taken to occur equally frequently in the directory.

[G.B.T.U. (MCA) 2011]

Sol. Null hypothesis:

H_0 : The digits taken in the directory occur equally frequently *i.e.,* there is no significant difference between the observed and expected frequency.

Under H_0, the expected frequency $= \frac{10000}{10} = 1000$

Calculation of χ^2

O_i	1026	1107	997	966	1075	933	1107	972	964	853
E_i	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
$(O_i - E_i)^2$	676	11449	9	1156	5625	4489	11449	784	1296	21609

$$\chi^2 = \sum \left[\frac{(O_i - E_i)^2}{E_i} \right] = \frac{58542}{1000} = 58.542$$

The tabulated value of χ^2 at 5% level of significance for 9 d.f. is 16.919.

Conclusion: Since $\chi^2_{\text{calculated}} > \chi^2_{\text{tabulated}}$, H_0 is rejected *i.e.,* there is significant difference between the observed and theoretical frequencies. Therefore, the digits taken in the directory donot occur equally frequently.

Example 7. *When the first proof of 392 pages of a book of 1200 pages were read, the distribution of printing mistakes were found to be as follows:*

No. of mistakes in a page (x):	*0*	*1*	*2*	*3*	*4*	*5*	*6*
No. of pages (f):	*275*	*72*	*30*	*7*	*5*	*2*	*1*

Fit a poisson distribution to the above data and test the goodness of fit.

Sol. Null Hypothesis, H_0 : Poisson distribution is a good fit to the data.

$$\text{Mean } (\lambda) = \frac{\sum fx}{\sum f} = \frac{189}{392} = 0.4821$$

The frequency of x mistakes per page is given by the poisson law as follows:

$$N(x) = N \,.\, P(x)$$

$$= 392 \left[\frac{e^{-0.4821}\,(0.4821)^x}{x\,!} \right] = \frac{242.05(0.4821)^x}{x\,!} \;;\; 0 \le x \le 6$$

Under H_0, expected frequencies are,

N(0) = 242.05, N(1) = 116.69, N(2) = 28.13, N(3) = 4.52

N(4) = 0.54, N(5) = 0.052, N(6) = 0.0042

The χ^2-table is as follows:

Mistakes per page (x)	*Observed frequency (O_i)*	*Expected frequency (E_i) (correct to one place of decimal)*	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
0	275	242.1	1082.41	4.471
1	72	116.7	1998.09	17.121
2	30	28.1	3.61	0.128
3	7 }	4.5 }		
4	5 } 15	0.5 } 5.1		
5	2 }	0.1 }	98.01	19.217
6	1 }	0 }		
Total	392	392		40.937

$$\chi^2_{\text{cal.}} = \sum \left[\frac{(O_i - E_i)^2}{E_i}\right] = 40.937$$

$$\text{d.f.} = 7 - 1 - 1 - 3 = 2$$

One d.f. is lost because of linear constraint $\Sigma O_i = \Sigma E_i$. One d.f. is lost because the parameter λ has been estimated from the given data and is then used for computing the expected frequencies. 3 d.f. are lost because of grouping the last four expected cell frequencies which were less than 5.

Tabulated value of χ^2 for 2 d.f. at 5% level of significance is 5.991.

Conclusion : Since $\chi^2_{\text{cal.}} > \chi^2_{\text{tab.}}$, the null hypothesis is rejected at 5% level of significance. Hence, we conclude that poisson distribution is not a good fit to the given data.

Example 8. *Fit a Poisson distribution to the following data and best the goodness of fit :*

x :	0	1	2	3	4
f :	109	65	22	3	1

Sol. Null hypothesis, H_0: Poisson distribution is a good fit to the data.

$$\text{Mean } (\lambda) = \frac{\Sigma fx}{\Sigma f} = \frac{122}{200} = 0.61$$

$$N(x) = N \,.\, P(x) = (200)\, \frac{e^{-.61}\,(.61)^x}{x\,!} = \frac{(108.67)\,(0.61)^x}{x\,!}$$

Under H_0, expected frequencies are

$$N(0) = 108.67 \approx 109, \quad N(1) = 66.29 \approx 66, \quad N(2) = 20.22 \approx 20$$
$$N(3) = 4.11 \approx 4, \quad N(4) = 0.63 \approx 1$$

The χ^2-table is as follows:

x	O_i	E_i	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
0	109	109	0	0
1	65	66	1	0.01515
2	22	20	4	0.2
3 4	$\left.\begin{matrix}3\\1\end{matrix}\right\}4$	$\left.\begin{matrix}4\\1\end{matrix}\right\}5$	1	0.2
Total	200	200		0.41515

$$\chi^2_{\text{cal.}} = \sum \left[\frac{(O_i - E_i)^2}{E_i}\right] = 0.41515$$

$$\text{d.f.} = 5 - 1 - 1 - 1 = 2$$

Tabulated value of χ^2 for 2 d.f. at 5% level of significance is 5.991.

Conclusion: Since $\chi^2_{\text{cal.}} < \chi^2_{\text{tab.}}$, the null hypothesis H_0 is accepted at 5% level of significance. Hence we conclude that Poisson distribution is a good fit to the given data.

ASSIGNMENT

1. A sample analysis of examination results of 500 students, it was found that 220 students have failed, 170 have secured a third class, 90 have secured a second class and the rest, a first class. Do these figures support the general belief that above categories are in the ratio 4 : 3 : 2 : 1 respectively ? (The tabular value of χ^2 for d.f. 3 at 5% level of significance is 7.81).

[*U.P.T.U. (MBA) 2009*]

2. What is χ^2–test? [*G.B.T.U. 2010 ; G.B.T.U. MCA (C.O.) 2010*]

A die is thrown 90 times with the following results:

Face:	1	2	3	4	5	6	Total
Frequency:	10	12	16	14	18	20	90

Use χ^2-test to test whether these data are consistent with the hypothesis that die is unbiased.

Given $\chi^2_{0.05} = 11.07$ for 5 degrees of freedom. [*U.P.T.U. (MCA) 2007*]

3. A survey of 320 families with 5 children shows the following distribution:

No. of boys & girls:	5 boys & 0 girl	4 boys & 1 girl	3 boys & 2 girls	2 boys & 3 girls	1 boy & 4 girls	0 boy & 5 girls	Total
No. of families:	18	56	110	88	40	8	320

Given that values of χ^2 for 5 degrees of freedom are 11.1 and 15.1 at 0.05 and 0.01 significance level respectively, test the hypothesis that male and female births are equally probable.

(*G.B.T.U. 2010*)

4. A chemical extraction plant processes sea water to collect sodium chloride and magnesium. It is known that sea water contains sodium chloride, magnesium and other elements in the ratio 62 : 4 : 34. A sample of 200 tonnes of sea water has resulted in 130 tonnes of sodium chloride and 6 tonnes of magnesium. Are these data consistent with the known composition of sea water at 5% level of significance? (Given that the tabular value of χ^2 is 5.991 for 2 degree of freedom).

[*U.P.T.U. MCA (C.O.) 2008*]

5. The demand for a particular spare part in a factory was found to vary from day-to-day. In a sample study, the following information was obtained:

Days:	Mon	Tue	Wed	Thurs	Fri	Sat
No. of parts demanded:	1124	1125	1110	1120	1126	1115

Test the hypothesis that the number of parts demanded does not depend on the day of the week.

[**Given.** The values of chi-square significance at 5, 6, 7 d.f. are respectively 11.07, 12.59, 14.07 at 5% level of significance] (*G.B.T.U. 2011*)

6. The sales in a supermarket during a week are given below. Test the hypothesis that the sales donot depend on the day of the week using a significant level of 0.05.

Days	*Mon*	*Tue*	*Wed*	*Thurs*	*Fri*	*Sat*
Sales (in 1000 ₹)	65	54	60	56	71	84

7. 4 coins were tossed at a time and this operation is repeated 160 times. It is found that 4 heads occur 6 times, 3 heads occur 43 times, 2 heads occur 69 times, one head occur 34 times. Discuss whether the coin may be regarded as unbiased?

8. 200 digits are chosen at random from a set of tables. The frequencies of the digits were :

Digits:	0	1	2	3	4	5	6	7	8	9
Frequency:	18	19	23	21	16	25	22	20	21	15

Use χ^2-test to assess the correctness of the hypothesis that the digits were distributed in equal numbers in the table, given that the value of χ^2 are respectively 16.9, 18.3 and 19.7 for 9, 10 and 11 degrees of freedom at 5% level of significance.

9. A genetical law says that children having one parent of blood group M and the other parent of blood group N will always be one of the three blood groups M, MN, N and that the average no. of children in these groups will be in the ratio 1 : 2 : 1. The report on an experiment states as follows:

"Of 162 children having one M parent and one N parent, 28.4% were found to be of group M, 42% of group MN and the rest of the group N." Do the data in the report conform to the expected genetic ratio 1 : 2 : 1?

10. Every clinical thermometer is classified into one of the four categories A, B, C and D on the basis of inspection and test. From past experience, it is knwon that thermometers produced by a certain manufacturer are distributed among the four categories in the following proportions:

Category:	A	B	C	D
Proportion:	0.87	0.09	0.03	0.01

A new lot of 1336 thermometers is submitted by the manufacturer for inspection and test and the following distribution into four categories results :

Category:	A	B	C	D
No. of the rmometers reported:	1188	91	47	10

Does this new lot of thermometers differ from the previous experience with regards to proportion of thermometers in each category?

11. Test for goodness of fit of a poisson distribution at 5% level of significance to the following frequency distribution:

(*i*)

x:	0	1	2	3	4	5	6	7	8
f:	52	151	130	102	45	12	5	1	2

[**Hint.** Group the last three frequencies]

(*ii*)

x:	0	1	2	3	4	5	6	7	8	9	10	11	12	13
f:	3	15	47	76	68	74	46	39	15	9	5	2	0	1

[**Hint.** Group the first two and last four frequencies]

(*iii*)

x:	0	1	2	3	4	5
f:	275	138	75	7	4	1

[**Hint.** Club the last three frequencies]

(*iv*)

x:	0	1	2	3	4
f:	419	352	154	56	19

12. (*i*) Fit a binomial distribution to the data and test for goodness of fit at 5% level of significance.

x:	0	1	2	3	4	5
y:	38	144	342	287	164	25

(*ii*) A random number table of 250 digits showed the following distribution of digits 0, 1, 2, ..., 9.

Digit:	0	1	2	3	4	5	6	7	8	9
Observed: *Frequency*	17	31	29	18	14	20	35	30	20	36
Expected: *Frequency*	25	25	25	25	25	25	25	25	25	25

Does the observed distribution differ significantly from expected distributions using a significance level of 0.01? Given that $\chi^2_{0.99}$ for 9 degrees of freedom is 21.7. [*G.B.T.U. MCA (SUM) 2010*]

Answers

1. No **2.** Yes

3. H_0 accepted at 1% level of significance and rejected at 5% level of significance

4. $\chi^2 = 1.025$, Yes **5.** H_0 accepted at 5% level **6.** H_0 accepted

7. Coin is unbiased **8.** H_0 accepted at 5% level **9.** H_0 accepted at 5% level

10. H_0 rejected at 5% level

11. (*i*) H_0 accepted at 5% level, (*ii*) H_0 accepted at 5% level, (*iii*) H_0 rejected at 5% level, not a good fit. (*iv*) H_0 accepted at 5% level

12. (*i*) H_0 accepted at 5% level, provides a good fit (*ii*) Yes.

3.88 χ^2 TEST AS A TEST OF INDEPENDENCE

With the help of χ^2 test, we can find whether or not, two attributes are associated. We take the null hypothesis that there is no association between the attributes under study, *i.e.*, **we assume that the two attributes are independent. If the calculated value of χ^2 is less than the table value** at a specified level (generally 5%) of significance, the hypothesis holds good, *i.e.*, **the attributes are independent** and do not bear any association. On the other hand, if the calculated value of χ^2 is greater than the table value at a specified level of significance, we say that the results of the experiment do not support the hypothesis. In other words, the attributes are associated. Thus a very useful application of χ^2 test is to investigate the relationship between trials or attributes which can be classified into two or more categories.

The sample data set out into two-way table, called **contingency table.**

Let us consider two attributes A and B divided into r classes $A_1, A_2, A_3, \ldots\ldots, A_r$ and B divided into s classes $B_1, B_2, B_3, \ldots\ldots, B_s$. If (A_i), (B_j) represents the number of persons possessing the attributes A_i, B_j respectively, $(i = 1, 2, \ldots\ldots, r, j = 1, 2, \ldots\ldots, s)$ and $(A_i B_j)$ represent the number of persons possessing attributes A_i and B_j. Also we have $\sum_{i=1}^{r} A_i = \sum_{j=1}^{s} B_j = N$ where N is the total frequency. The contingency table for $r \times s$ is given as follows:

A / *B*	A_1	A_2	A_3	$\ldots A_r$	*Total*
B_1	(A_1B_1)	(A_2B_1)	(A_3B_1)	$\ldots\ldots(A_rB_1)$	B_1
B_2	(A_1B_2)	(A_2B_2)	(A_3B_2)	$\ldots\ldots(A_rB_2)$	B_2
B_3	(A_1B_3)	(A_2B_3)	(A_3B_3)	$\ldots\ldots(A_rB_3)$	B_3
......	......	......	......	......	......
......	......	......	......	......	......
B_s	(A_1B_s)	(A_2B_s)	(A_3B_s)	$\ldots\ldots(A_rB_s)$	(B_s)
Total	(A_1)	(A_2)	(A_3)	$\ldots\ldots(A_r)$	N

H_0 : Both the attributes are independent. *i.e.,* A and B are independent under the null hypothesis, we calculate the expected frequency as follows:

$$P(A_i) = \text{Probability that a person possesses the attribute } A_i = \frac{(A_i)}{N} \quad i = 1, 2, \ldots\ldots, r$$

$P(B_j)$ = Probability that a person possesses the attribute $B_j = \frac{(B_j)}{N}$

$P(A_iB_j)$ = Probability that a person possesses both attributes A_i and $B_j = \frac{(A_iB_j)}{N}$

If $(A_iB_j)_0$ is the expected number of persons possessing both the attributes A_i and B_j

$$(A_iB_j)_0 = NP(A_iB_j) = NP(A_i)(B_j)$$

$$= N\frac{(A_i)}{N}\frac{(B_j)}{N} = \frac{(A_i)(B_j)}{N} \qquad (\because \text{ A and B are independent})$$

Hence
$$\boxed{\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{s}\left[\frac{[(A_iB_j)-(A_iB_j)_0]^2}{(A_iB_j)_0}\right]}$$

which is distributed as a χ^2 variate with $(r-1)(s-1)$ degrees of freedom.

Note 1. For a 2 × 2 contingency table where the frequencies are $\frac{a/b}{c/d}$, χ^2 can be calculated from independent frequencies as $\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+b)(c+d)(b+d)(a+c)}$.

Note 2. If the contingency table is not 2 × 2, then the above formula for calculating χ^2 can't be used. Hence, we have another formula for calculating the expected frequency $(A_iB_j)_0 = \frac{(A_i)(B_j)}{N}$

i.e., expected frequency in each cell is = $\frac{\text{Product of column total and row total}}{\text{whole total}}$.

Note 3. If $\frac{a|b}{c|d}$ is the 2 × 2 contingency table with two attributes, $Q = \frac{ad-bc}{ad+bc}$ is called the coefficient of association. If the attributes are independent then $\frac{a}{b} = \frac{c}{d}$.

Note 4. Yates's Correction. In a 2 × 2 table, if the frequencies of a cell is small, we make Yates's correction to make χ^2 continuous. Decrease by $\frac{1}{2}$ those cell frequencies which are greater than expected frequencies, and increase by $\frac{1}{2}$ those which are less than expectation. This will not affect the marginal columns. This correction is known as Yates's correction to continuity. After Yates's correction

$$\chi^2 = \frac{N\left(bc-ad-\frac{1}{2}N\right)^2}{(a+c)(b+d)(c+d)(a+b)} \quad \text{when} \quad ad-bc<0$$

and
$$\chi^2 = \frac{N\left(ad-bc-\frac{1}{2}N\right)^2}{(a+c)(b+d)(c+d)(a+b)} \quad \text{when} \quad ad-bc>0.$$

EXAMPLES

Example 1. *What are the expected frequencies of 2 × 2 contingency tables given below:*

(*i*)

a	b
c	d

(*ii*)

2	10
6	6

Sol. Observed frequencies → Expected frequencies

(*i*)

a	b	$a+b$
c	d	$c+d$
$a+c$	$b+d$	$a+b+c+d = N$

→

$\frac{(a+c)(a+b)}{a+b+c+d}$	$\frac{(b+d)(a+b)}{a+b+c+d}$
$\frac{(a+c)(c+d)}{a+b+c+d}$	$\frac{(b+d)(c+d)}{a+b+c+d}$

Observed frequencies → Expected frequencies

(*ii*)

2	10	12
6	6	12
8	16	24

→

$\frac{8 \times 12}{24} = 4$	$\frac{16 \times 12}{24} = 8$
$\frac{8 \times 12}{24} = 4$	$\frac{16 \times 12}{24} = 8$

Example 2. *From the following table regarding the colour of eyes of father and son, test if the colour of son's eye is associated with that of the father.*

		Eye colour of son	
		Light	*Not light*
Eye colour of father	*Light*	*471*	*51*
	Not light	*148*	*230*

Sol. Null hypothesis H_0: The colour of son's eye is not associated with that of the father. *i.e.,* they are independent.

Under H_0, we calculate

the expected frequency in each cell $= \dfrac{\text{Product of column total and row total}}{\text{whole total}}$

Expected frequencies are:

Eye colour of son / *Eye colour of father*	*Light*	*Not light*	*Total*
Light	$\frac{619 \times 522}{900} = 359.02$	$\frac{289 \times 522}{900} = 167.62$	522
Not light	$\frac{619 \times 378}{900} = 259.98$	$\frac{289 \times 378}{900} = 121.38$	378
Total	619	289	900

$$\chi^2 = \frac{(471 - 359.02)^2}{359.02} + \frac{(51 - 167.62)^2}{167.62} + \frac{(148 - 259.98)^2}{259.98} + \frac{(230 - 121.38)^2}{121.38} = 261.498.$$

Tabulated value of χ^2 at 5% level for 1 d.f. is 3.841.

Conclusion. Since the calculated value of χ^2 > tabulated value of χ^2, H_0 is rejected. They are dependent *i.e.,* the colour of son's eye is associated with that of the father.

Example 3. *The following table gives the number of good and bad parts produced by each of the three shifts in a factory:*

	Good parts	*Bad parts*	*Total*
Day shift	*960*	*40*	*1000*
Evening shift	*940*	*50*	*990*
Night shift	*950*	*45*	*995*
Total	*2850*	*135*	*2985*

Test whether or not the production of bad parts is independent of the shift on which they were produced.

Sol. Null hypothesis H_0. The production of bad parts is independent of the shift on which they were produced. *i.e.,* the two attributes, production and shifts are independent.

Under H_0, $$\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{3} \left[\frac{[(A_iB_j)_0 - (A_iB_j)]^2}{(A_iB_j)_0} \right]$$

Calculation of expected frequencies

Let A and B be the two attributes namely production and shifts. A is divided into two classes A_1, A_2 and B is divided into three classes B_1, B_2, B_3.

$$(A_1B_1)_0 = \frac{(A_1)(B_1)}{N} = \frac{(2850) \times (1000)}{2985} = 954.77$$

$$(A_1B_2)_0 = \frac{(A_1)(B_2)}{N} = \frac{(2850) \times (990)}{2985} = 945.226$$

$$(A_1B_3)_0 = \frac{(A_1)(B_3)}{N} = \frac{(2850) \times (995)}{2985} = 950$$

$$(A_2B_1)_0 = \frac{(A_2)(B_1)}{N} = \frac{(135) \times (1000)}{2985} = 45.27$$

$$(A_2B_2)_0 = \frac{(A_2)(B_2)}{N} = \frac{(135) \times (990)}{2985} = 44.773$$

$$(A_2B_3)_0 = \frac{(A_2)(B_3)}{N} = \frac{(135) \times (995)}{2985} = 45.$$

To calculate the value of χ^2

Class	O_i	E_i	$(O_i - E_i)^2$	$(O_i - E_i)^2/E_i$
(A_1B_1)	960	954.77	27.3529	0.02864
(A_1B_2)	940	945.226	27.3110	0.02889
(A_1B_3)	950	950	0	0
(A_2B_1)	40	45.27	27.7729	0.61349
(A_2B_2)	50	44.773	27.3215	0.61022
(A_2B_3)	45	45	0	0
				1.28126

The tabulated value of χ^2 at 5% level of significance for 2 degrees of freedom $(r - 1)(s - 1)$ is 5.991.

Conclusion: Since the calculated value of χ^2 is less than the tabulated value, we accept H_0. *i.e.,* the production of bad parts is independent of the shift on which they were produced.

Example 4. *From the following data, find whether hair colour and sex are associated.*

Colour / *Sex*	*Fair*	*Red*	*Medium*	*Dark*	*Black*	*Total*
Boys	*592*	*849*	*504*	*119*	*36*	*2100*
Girls	*544*	*677*	*451*	*97*	*14*	*1783*
Total	*1136*	*1526*	*955*	*216*	*50*	*3883*

Sol. Null hypothesis H_0. The two attributes hair colour and sex are not associated. *i.e.,* they are independent.

Let A and B be the attributes hair colour and sex respectively. A is divides into 5 classes $(r = 5)$. B is divided into 2 classes $(s = 2)$.

$\therefore$ Degrees of freedom $= (r - 1)(s - 1) = (5 - 1)(2 - 1) = 4$

Under H_0, we calculate $\chi^2 = \displaystyle\sum_{i=1}^{5} \sum_{j=1}^{2} \frac{[(A_iB_j)_0 - (A_iB_j)]^2}{(A_iB_j)_0}$

To calculate the expected frequency $(A_i B_j)_0$ as follows:

$$(A_1B_1)_0 = \frac{(A_1)(B_1)}{N} = \frac{1136 \times 2100}{3883} = 614.37$$

$$(A_1B_2)_0 = \frac{(A_1)(B_2)}{N} = \frac{1136 \times 1783}{3883} = 521.629$$

$$(A_2B_1)_0 = \frac{(A_2)(B_1)}{N} = \frac{1526 \times 2100}{3883} = 852.289$$

$$(A_2B_2)_0 = \frac{(A_2)(B_2)}{N} = \frac{1526 \times 1783}{3883} = 700.71$$

$$(A_3B_1)_0 = \frac{(A_3)(B_1)}{N} = \frac{955 \times 2100}{3883} = 516.482$$

$$(A_3B_2)_0 = \frac{(A_3)(B_2)}{N} = \frac{955 \times 1783}{3883} = 483.517$$

$$(A_4B_1)_0 = \frac{(A_4)(B_1)}{N} = \frac{216 \times 2100}{3883} = 116.816$$

$$(A_4B_2)_0 = \frac{(A_4)(B_2)}{N} = \frac{216 \times 1783}{3883} = 99.183$$

$$(A_5B_1)_0 = \frac{(A_5)(B_1)}{N} = \frac{50 \times 2100}{3883} = 27.04$$

$$(A_5B_2)_0 = \frac{(A_5)(B_2)}{N} = \frac{50 \times 1783}{3883} = 22.959$$

Calculation of χ^2

Class	O_i	E_i	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
A_1B_1	592	614.37	500.416	0.8145
A_1B_2	544	521.629	500.462	0.959
A_2B_1	849	852.289	10.8175	0.0127
A_2B_2	677	700.71	562.1641	0.8023
A_3B_1	504	516.482	155.800	0.3016
A_3B_2	451	438.517	155.825	0.3553
A_4B_1	119	116.816	4.7698	0.0408
A_4B_2	97	99.183	4.7654	0.0480
A_5B_1	36	27.04	80.2816	2.9689
A_5B_2	14	22.959	80.2636	3.495
				9.79975

$$\chi^2_{\text{cal.}} = 9.799.$$

Tabular value of χ^2 at 5% level of significance for 4 d.f. is 9.488.

Conclusion: Since the calculated value of χ^2 < tabulated value, H_0 is rejected. *i.e.,* the two attributes are not independent. *i.e.,* the hair colour and sex are associated.

Example 5. *Can vaccination be regarded as preventive measure of small pox as evidenced by the following data of 1482 persons exposed to small pox in a locality. 368 in all were attacked of these 1482 persons and 343 were vaccinated and of these only 35 were attacked.*

Sol. For the given data we form the contingency table. Let the two attributes be vaccination and exposed to small pox. Each attributes is divided into two classes.

Vaccination A / *Disease small pox B*	*Vaccinated*	*Not*	*Total*
Attacked	35	333	368
Not	308	806	1114
Total	343	1139	1482

Null hypothesis H_0. The two attributes are independent *i.e.,* vaccination can't be regarded as preventive measure of small pox.

Degrees of freedom $\nu = (r - 1)(s - 1) = (2 - 1)(2 - 1) = 1$

Under H_0, $\quad \chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{[(A_iB_j)_0 - (A_iB_j)]^2}{(A_iB_j)_0}$

Calculation of expected frequency

$$(A_1B_1)_0 = \frac{(A_1)(B_1)}{N} = \frac{343 \times 368}{1482} = 85.1713$$

$$(A_1B_2)_0 = \frac{(A_1)(B_2)}{N} = \frac{343 \times 1114}{1482} = 257.828$$

$$(A_2B_1)_0 = \frac{(A_2)(B_1)}{N} = \frac{1139 \times 368}{1482} = 282.828$$

$$(A_2B_2)_0 = \frac{(A_2)(B_2)}{N} = \frac{1139 \times 1114}{1482} = 856.171$$

Calculation of χ^2

Class	O_i	E_i	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
(A_1B_1)	35	85.1713	2517.159	29.554
(A_1B_2)	308	257.828	2517.229	8.1728
(A_2B_1)	333	282.828	2517.2295	7.5592
(A_2B_2)	806	856.171	2517.1292	2.9399
				48.2261

Calculated value of χ^2 = 48.2261.

Tabulated value of χ^2 at 5% level of significance for 1 d.f. is 3.841.

Conclusion. Since the calculated value of χ^2 > tabulated value, H_0 is rejected.

i.e., the two attributes are not independent. *i.e.,* the vaccination can be regarded as preventive measure of small pox.

Example 6. *To test the effectiveness of inoculation against cholera, the following table was obtained:*

	Attacked	*Not attacked*	*Total*
Inoculated	*30*	*160*	*190*
Not inoculated	*140*	*460*	*600*
Total	*170*	*620*	*790*

(The figures represent the number of persons.)

Use χ^2-test to defend or refute the statement that the inoculation prevents attack from cholera. (U.P.T.U. 2009)

Sol. Null hypothesis H_0: The inoculation does not prevent attack from cholera.

Under H_0, we calculate the expected frequencies as:

	Attacked	*Not attacked*
Inoculated	$\frac{190 \times 170}{790} = 40.886$	$\frac{190 \times 620}{790} = 149.11$
Not inoculated	$\frac{600 \times 170}{790} = 129.11$	$\frac{600 \times 620}{790} = 470.89$

Calculation of χ^2

O_i	30	160	140	460
E_i	40.886	149.11	129.11	470.89
$\frac{(O_i - E_i)^2}{E_i}$	2.898	0.795	0.918	0.252

$$\chi^2_{\text{cal.}} = \sum \frac{(O_i - E_i)^2}{E_i} = 4.863$$

Tabulated value of χ^2 at 5% level of significance for 1 d.f. is 3.841.

Conclusion: Since $\chi^2_{\text{cal.}} > \chi^2_{\text{tab.}}$ at 5% level of significance, null hypothesis H_0 is rejected. Hence we defend the statement that inoculation prevents attack from cholera.

ASSIGNMENT

1. In a locality 100 persons were randomly selected and asked about their educational achievements. The results are given below:

Education				
Sex		*Middle*	*High school*	*College*
	Male	10	15	25
	Female	25	10	15

Based on this information can you say that the education depends on sex.

2. The following data is collected on two characters:

	Smokers	*Non smokers*
Literate	83	57
Illiterate	45	68

Based on this information can you say that there is no relation between habit of smoking and literacy.

3. 500 students at school were graded according to their intelligences and economic conditions of their homes. Examine whether there is any association between economic condition and intelligence, from the following data:

Economic conditions	*Intelligence*	
	Good	*Bad*
Rich	85	75
Poor	165	175

4. In an experiment on the immunisation of goats from anthrox, the following results were obtained. Derive your inferences on the efficiency of the vaccine.

	Died anthrox	*Survived*
Inoculated with vaccine	2	10
Not inoculated	6	6

5. By using χ^2-test, find out whether there is any association between income level and type of schooling:

Income	*Public School*	*Govt. School*
Low	200	400
High	1000	400

(Given for degree of freedom 1, $\chi^2_{0.05} = 3.84$) [*U.P.T.U. 2008, G.B.T.U. (MBA) 2011*]

6. Examine by any suitable method, whether the nature of area is related to voting preference in the election for which the data are tabulated below:

Votes for Area	*A*	*B*	*Total*
Rural	620	480	1100
Urban	380	520	900
Total	1000	1000	2000

(*U.P.T.U. 2006*)

7. The groups of 100 people each were taken for testing the use of a vaccine. 15 persons contracted the disease out of the inoculated persons, while 25 contracted the disease in the other group. Test the efficiency of the vaccine using Chi-square test. (The value of χ^2 for one degree of freedom at 5% level of significance is 3.84).

Answers

1. Yes **2.** No **3.** No **4.** Not effective
5. Yes **6.** Yes **7.** Not associated.

TEST YOUR KNOWLEDGE

1. The fourth moment about the mean of a frequency distribution is 24. What must be the value of standard deviation in order that the distribution be platykurtic? (*M.T.U. 2012*)

2. Two events A and B have probabilities 0.25 and 0.50 respectively. The probability that both events A and B occurs in 0.14. Find the probability that neither A nor B occurs. (*M.T.U. 2013*)

$\left[\textbf{Hint. } P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)\right]$

3. (*i*) Define the coefficients of skewness and kurtosis. (*U.P.T.U. 2014*)
(*ii*) Define skewness, coefficient of skewness, kurtosis and coefficient of kurtosis. (*M.T.U. 2013*)
(*iii*) Write a short note on skewness. [(*M.T.U.* (*B.Pharma*) *2011*]
(*iv*) What is meant by skewness? How is it measured? [(*M.T.U.* (*MBA*) *2012*]

4. What is the total probability theorem? (*U.P.T.U. 2014*)

5. The first four central moments of a distribution are 0, 2.5, 0.7 and 18.75. Comment on the kurtosis of the distribution. (*M.T.U. 2013*)

6. Find the moment generating function of Poisson distribution. (*M.T.U. 2013*)

7. Find the parameters p and q of the binomial distribution whose mean is 9 and variance is $\frac{9}{4}$. (*M.T.U. 2012*)

8. If the sum of the mean and variance of a binomial distribution of 5 trials is $\frac{9}{5}$, find $P(X \geq 1)$. (*M.T.U. 2013*)

9. It has been found that 2% of the tools produced by a certain machine are defective. What is the probability that in a shipment of 400 such tools, 3 or more will be defective? (*M.T.U. 2013*)

10. If $P(X = 0) = P(X = 1) = k$ in a Poisson distribution, then what is k?

11. For a Poisson variate X if $P(X = 1) = P(X = 2)$, then find $P(X = 4)$.

12. Find the total area under the curve of p.d.f. of a normal curve.

13. If for a Poisson distribution, $P(2) = P(3)$, then what is its probability function?

14. Find the parameters of a binomial distribution with mean = 8 and variance = 4.

15. If X is a normal variate with mean 30 and S.D. 5, find the probabilities that

(*i*) $26 \le X \le 40$ (*ii*) $X \ge 45$ and (*iii*) $|X - 30| > 5$

Answers

2. 0.39 **5.** $\beta_2 = 3$, Mesokurtic **6.** $M_x(t) = e^{\lambda(e^t - 1)}$ **7.** $q = \frac{1}{4}, p = \frac{3}{4}$

8. 0.67232 **9.** 0.9862 **10.** $\frac{1}{e}$ **11.** $\frac{2}{3e^2}$

12. 1 **13.** $\frac{e^{-3}(3)^x}{x\,!}$ **14.** $n = 16, p = \frac{1}{2}$

15. (*i*) 0.7653 (*ii*) 0.00135 (*iii*) 0.3174.

UNIT 4

Numerical Techniques–I

4.1 SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Consider the equation of the form $f(x) = 0$

If $f(x)$ is a quadratic, cubic or biquadratic expression then algebraic formulae are available for expressing the roots. But when $f(x)$ is a polynomial of higher degree or an expression involving transcendental functions *e.g.,* $1 + \cos x - 5x$, $x \tan x - \cosh x$, $e^{-x} - \sin x$ etc., algebraic methods are not available.

Here, we shall describe some numerical methods for the solution of $f(x) = 0$, where $f(x)$ is algebraic or transcendental.

4.2 ORDER OF CONVERGENCE OF ITERATIVE METHODS

Convergence of an iterative method is judged by the order at which the error between successive approximations to the root decreases.

An iterative method is said to be k^{th} order convergent if k is the largest positive real number such that

$$\lim_{i\to\infty} \left| \frac{e_{i+1}}{e_i^{\,k}} \right| \le \text{A}$$

where A is a non-zero finite number called asymptotic error constant and it depends on derivative of $f(x)$ at an approximate root x. e_i and e_{i+1} are the errors in successive approximations. k^{th} order convergence gives us idea in each iteration, the no. of significant digits in each approximation increases k times.

The error in any step is proportional to the k^{th} power of the error in the previous step.

4.3 CONVERGENCE OF A SEQUENCE

A sequence $< x_n >$ of successive approximations of a root $x = \alpha$ of the equation $f(x) = 0$ is said to converge to $x = \alpha$ with order $p \ge 1$ iff

$$| x_{n+1} - \alpha | \le c \, | x_n - \alpha |^p, \; n \ge 0$$

c being some constant greater than zero.

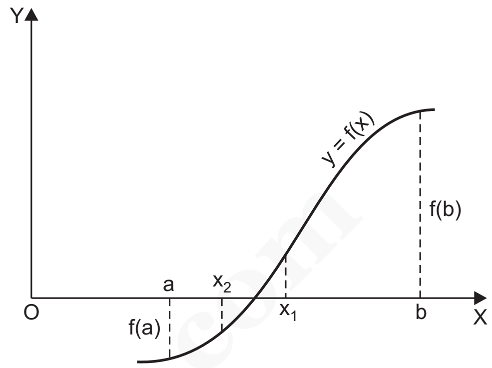
Particularly, if $| x_{n+1} - \alpha | = c \, | x_n - \alpha |$, $n \ge 0$, $0 < c < 1$ then convergence is called **geometric.** Also, if $p = 1$ and $0 < c < 1$ then convergence is called **linear** or of first order. Constant c is called the **rate of linear convergence.** Convergence is rapid or slow according as c is near 0 or 1.

Using induction, condition for linear convergence can be simplified to the form

$$| x_n - \alpha | \le c^n | x_0 - \alpha |, n \ge 0, 0 < c < 1.$$

4.4 BISECTION METHOD

This method is based on the repeated application of intermediate value property.



Let the function $f(x)$ be continuous between a and b. For definiteness, let $f(a)$ be (–)ve and $f(b)$ be (+)ve. Then the first approximation to the root is $x_1 = \frac{1}{2}(a + b)$.

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$ otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is (+) ve or (–) ve. Then, we bisect the interval as before and continue the process until the root is found to desired accuracy.

In the above figure, $f(x_1)$ is (+)ve so that the root lies between a and x_1. Then second approximation to the root is $x_2 = \frac{1}{2}(a + x_1)$. If $f(x_2)$ is (–)ve ; the root lies between x_1 and x_2. Then the third approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2)$ and so on.

We observe that this method uses only the end points of the interval $[a_n, b_n]$ for which $f(a_n) \cdot f(b_n) < 0$ and not the values of $f(x)$ at these end points to obtain the next approximation to the root.

The number of iterations required may be determined from the relation

$$\frac{|b - a|}{2^n} \le \varepsilon$$

or

$$n \ge \frac{\log_e |b - a| - \log_e \varepsilon}{\log_e 2}$$

where ε is the permissible error.

This method requires a large number of iterations to achieve a reasonable degree of accuracy for the root. It requires one function evaluation for each iteration. This method is also called as **Bolzano method** or **Interval halving method.**

4.5 PROVE THAT BISECTION METHOD ALWAYS CONVERGES

Let $[p_n, q_n]$ be the interval at n^{th} step of bisection, having a root of the equation $f(x) = 0$. Let x_n be the n^{th} approximation for the root. Then initially $p_1 = a$ and $q_1 = b$.

$$\Rightarrow \qquad x_1 = \text{first approximation} = \left(\frac{p_1 + q_1}{2}\right)$$

$$\Rightarrow \qquad p_1 < x_1 < q_1$$

Now either the root lies in $[a, x_1]$ or in $[x_1, b]$.

$\therefore$ either $[p_2, q_2] = [p_1, x_1]$ or $[p_2, q_2] = [x_1, q_1]$

$\Rightarrow$ either $p_2 = p_1, q_2 = x_1$ or $p_2 = x_1, q_2 = q_1$

$\Rightarrow$ $p_1 \le p_2, q_2 \le q_1$

Also, $x_2 = \dfrac{p_2 + q_2}{2}$ so that $p_2 < x_2 < q_2$

Continuing this way, we obtain that at n^{th} step,

$$x_n = \frac{p_n + q_n}{2}, p_n < x_n < q_n$$

and $p_1 \le p_2 \le \ldots\ldots \le p_n$ and $q_1 \ge q_2 \ge \ldots\ldots \ge q_n$

$\therefore$ $< p_1, p_2, \ldots\ldots, p_n, \ldots\ldots >$ is bounded non-decreasing sequence bounded by b and $< q_1, q_2, \ldots\ldots, q_n, \ldots\ldots >$ is a bounded non-increasing sequence of numbers bounded by a.

Hence, both these sequences converge.

Let, $\lim\limits_{n \to \infty} p_n = p$ and $\lim\limits_{n \to \infty} q_n = q$.

Now since length of the interval is decreasing at every step, we get that

$$\lim_{n \to \infty} (q_n - p_n) = 0 \quad \Rightarrow \quad q = p$$

Also, $p_n < x_n < q_n$

$\Rightarrow$ $\lim p_n \le \lim x_n \le \lim q_n$

$\Rightarrow$ $p \le \lim x_n \le q$

$\Rightarrow$ $\lim x_n = p = q$ ...(1)

Further since a root lies in $[p_n, q_n]$, we shall have

$$f(p_n) \cdot f(q_n) < 0$$

$\Rightarrow$ $0 \ge \lim\limits_{n \to \infty} [f(p_n)] \cdot f(q_n)]$

$\Rightarrow$ $0 \ge f(p) \cdot f(q)$

$\Rightarrow$ $0 \ge [f(p)]^2$

But $[f(p)]^2 \ge 0$ being a square

$\therefore$ we get $f(p) = 0$

$\therefore$ p is a root of $f(x) = 0$ ...(2)

From (1) and (2), we see that $<x_n>$ converges *necessarily* to a root of equation $f(x) = 0$

The method is not rapidly converging but it is useful in the sense that it converges surely.

EXAMPLES

Example 1. *Find the real root of the equation $x \log_{10} x = 1.2$ by bisection method correct to four decimal places.*

Sol. $f(x) = x \log_{10} x - 1.2$ ...(1)

Since, $f(2.74) = -.000563$ *i.e.,* (–)ve

and $f(2.75) = .0081649$ *i.e.,* (+)ve

Hence, a root lies between 2.74 and 2.75.

$\therefore$ First approximation to the root is

$$x_1 = \frac{2.74 + 2.75}{2} = 2.745$$

Now, $f(x_1) = f(2.745) = .003798$ *i.e.,* (+)ve

Hence, root lies between 2.74 and 2.745.

$\therefore$ Second approximation to the root is

$$x_2 = \frac{2.74 + 2.745}{2} = 2.7425$$

Now, $f(x_2) = f(2.7425) = .001617$ *i.e.,* (+)ve

Hence, root lies between 2.74 and 2.7425.

$\therefore$ Third approximation to the root is

$$x_3 = \frac{2.74 + 2.7425}{2} = 2.74125$$

Now, $f(x_3) = f(2.74125) = .0005267$ *i.e.,* (+)ve

Hence, root lies between 2.74 and 2.74125.

$\therefore$ Fourth approximation to the root is

$$x_4 = \frac{2.74 + 2.74125}{2} = 2.740625$$

Now, $f(x_4) = f(2.740625) = -\ .00001839$ *i.e.,* (–)ve.

Hence, root lies between 2.740625 and 2.74125.

$\therefore$ Fifth approximation to the root is

$$x_5 = \frac{2.740625 + 2.74125}{2} = 2.7409375$$

Now, $f(x_5) = f(2.7409375) = .000254$ *i.e.,* (+)ve

Hence, root lies between 2.740625 and 2.7409375.

$\therefore$ Sixth approximation to the root is

$$x_6 = \frac{2.740625 + 2.7409375}{2} = 2.74078125$$

Now, $f(x_6) = f(2.74078125) = .0001178$ *i.e.,* (+)ve

Hence, root lies between 2.740625 and 2.74078125.

$\therefore$ Seventh approximation to the root is

$$x_7 = \frac{2.740625 + 2.74078125}{2} = 2.740703125$$

Since, x_6 and x_7 are same up to four decimal places hence the approximate real root is **2.7407.**

Example 2. *Find a positive real root of $x - \cos x = 0$ by bisection method, correct up to 4 decimal places between 0 and 1.*

Sol. Let $f(x) = x - \cos x$

$f(0.73) = (–)$ve and $f(0.74) = (+)$ve

Hence, the root lies between 0.73 and 0.74.

First approximation to the root is

$$x_1 = \frac{0.73 + 0.74}{2} = 0.735$$

Now, $f(0.735) = (–)$ve

Hence, the root lies between 0.735 and 0.74.

Second approximation to the root is

$$x_2 = \frac{0.73 + 0.74}{2} = 0.7375$$

Now, $f(0.7375) = (-)$ve

Hence, the root lies between 0.7375 and 0.74.

Third approximation to the root is

$$x_3 = \frac{0.7375 + 0.74}{2} = 0.73875$$

Now, $f(0.73875) = (-)$ve

Hence, the root lies between 0.73875 and 0.74.

Fourth approximation to the root is

$$x_4 = \frac{1}{2}(0.73875 + 0.74) = 0.739375$$

Now, $f(x_4) = f(0.739375) = (+)$ve

Hence, the root lies between 0.73875 and 0.739375.

Fifth approximation to the root is

$$x_5 = \frac{1}{2}(0.73875 + 0.739375) = 0.7390625$$

Now, $f(0.7390625) = (-)$ve

Hence, the root lies between 0.7390625 and 0.739375

Sixth approximation to the root is

$$x_6 = \frac{1}{2}(0.7390625 + 0.739375) = 0.73921875$$

Now, $f(0.73921875) = (+)$ve

Hence, the root lies between 0.7390625 and 0.73921875

Seventh approximation to the root is

$$x_7 = \frac{1}{2}(0.7390625 + 0.73921875) = 0.73914$$

Now, $f(0.73914) = (+)$ve

Hence, the root lies between 0.7390625 and 0.73914

Eighth approximate to the root is

$$x_8 = \frac{1}{2}(0.7390625 + 0.73914) = 0.73910$$

Since x_7 and x_8 are same up to four decimal places hence the approximate real root is 0.7391.

Example 3. *Perform five iterations of bisection method to obtain the smallest positive root of equation $f(x) \equiv x^3 - 5x + 1 = 0$.* [G.B.T.U 2011; G.B.T.U. (C.O.) 2011]

Sol. $f(x) = x^3 - 5x + 1$ ...(1)

Since, $f(0.2016) = 0.0001935$ *i.e.,* (+)ve

and $f(0.2017) = -0.0002943$ *i.e.,* (–)ve

Hence, root lies between 0.2016 and 0.2017.

First approximation to the root is

$$x_1 = \frac{0.2016 + .2017}{2} = 0.20165$$

Now, $f(x_1) = -\ 0.00005036$ *i.e.,* (–)ve

Hence, root lies between 0.2016 and 0.20165.

Second approximation to the root is

$$x_2 = \frac{0.2016 + 0.20165}{2} = 0.201625$$

Now, $f(x_2) = 0.00007159$ *i.e.,* (+)ve

Hence, root lies between 0.201625 and .20165.

Third approximation to the root is

$$x_3 = \frac{0.201625 + 0.20165}{2} = 0.2016375$$

Now, $f(x_3) = 0.00001061$ *i.e.,* (+)ve

Hence, root lies between 0.2016375 and 0.20165.

Fourth approximation to the root is

$$x_4 = \frac{0.2016375 + 0.20165}{2} = 0.20164375$$

Now, $f(x_4) = -\ 0.00001987$ *i.e.,* (–)ve

Hence, root lies between 0.2016375 and 0.20164375.

∴ Fifth approximation to the root is

$$x_5 = \frac{0.2016375 + 0.20164375}{2} = 0.201640625$$

Hence, after performing five iterations, the **smallest positive root** of the given equation is **0.20164** correct to **five decimal places.**

Example 4. *Find a real root of $x^3 - x = 1$ between 1 and 2 by bisection method. Compute five iterations.*

Sol. Here, $f(x) = x^3 - x - 1$ ...(1)

Since, $f(1.324) = -\ 0.00306$ *i.e.,* (–)ve

and $f(1.325) = 0.00120$ *i.e.,* (+)ve

Hence, root lies between 1.324 and 1.325.

∴ First approximation to the root is

$$x_1 = \frac{1.324 + 1.325}{2} = 1.3245$$

Now, $f(x_1) = -\ 0.000929$ *i.e.,* (–)ve

Hence, root lies between 1.3245 and 1.325

∴ Second approximation to the root is

$$x_2 = \frac{1.3245 + 1.325}{2} = 1.32475$$

Now, $f(x_2) = 0.000136$ *i.e.,* (+)ve

Hence, root lies between 1.3245 and 1.32475.

Third approximation to the root is

$$x_3 = \frac{1.3245 + 1.32475}{2} = 1.324625$$

Now, $f(x_3) = -\ 0.000396$ *i.e.,* (–)ve

Hence, root lies between 1.324625 and 1.32475.

∴ Fourth approximation to the root is

$$x_4 = \frac{1.324625 + 1.32475}{2} = 1.3246875$$

Now, $f(x_4) = -\ .0001298$ *i.e.,* (–)ve

Hence, root lies between 1.3246875 and 1.32475

∴ Fifth approximation to the root is

$$x_5 = \frac{1.3246875 + 1.32475}{2} = 1.32471875$$

Hence, the real root of the given equation is **1.324** correct to three decimal places after computing five iterations.

Example 5. *Use bisection method to find out the positive square root of 30 correct to four decimal places.*

Sol. Let $f(x) = x^2 - 30$ ...(1)

Since, $f(5.477) = -\ .00247$ *i.e.,* (–)ve

and $f(5.478) = .00848$ *i.e.,* (+)ve

Hence, root lies between 5.477 and 5.478

∴ First approximation to the root is

$$x_1 = \frac{5.477 + 5.478}{2} = 5.4775$$

Now, $f(x_1) = .003$ *i.e.,* (+)ve

Hence, root lies between 5.477 and 5.4775

∴ Second approximation to the root is

$$x_2 = \frac{5.477 + 5.4775}{2} = 5.47725$$

Now, $f(x_2) = .00026$ *i.e.,* (+)ve

Hence, root lies between 5.477 and 5.47725

∴ Third approximation to the root is

$$x_3 = \frac{5.477 + 5.47725}{2} = 5.477125$$

Now, $f(x_3) = -\ .0011$ *i.e.,* (–)ve

Hence, root lies between 5.477125 and 5.47725

∴ Fourth approximation to the root is

$$x_4 = \frac{5.477125 + 5.47725}{2} = 5.4771875$$

Since, x_3 and x_4 are same up to four decimal places hence the positive square root of 30 correct to 4 decimal places is **5.4771.**

ASSIGNMENT

1. (*i*) Transcendental equation is given as

$$f(x) = 2^x - x - 3$$

Calculate $f(x)$ for $x = -4, -3, -2, -1, 0, 1, 2, 3, 4$ and compute, between which integers values roots are lying.

(*ii*) The equation $x^2 - 2x - 3\cos x = 0$ is given. Locate the smallest root in magnitude in an interval of length one unit.

2. Find a real root of $e^x = 3x$ by Bisection method.

3. The smallest positive root of the equation $f(x) = x^4 - 3x^2 + x - 10 = 0$ is to be obtained.

(*i*) Find an interval of unit length which contains this root.

(*ii*) Perform two iterations of the Bisection method.

4. Find real root lying in interval (1, 2) up to four decimal places for the equation $x^6 - x^4 - x^3 - 1 = 0$ by Bisection method.

5. Find a real root of $\cos x - xe^x = 0$ correct to three decimal places by Bisection method. (*U.P.T.U. 2008*)

6. The negative root of the smallest magnitude of the equation $f(x) = 3x^3 + 10x^2 + 10x + 7 = 0$ is to be obtained.

(*i*) Find an interval of unit length which contains this root.

(*ii*) Perform two iterations of Bisection method.

7. Compute the root of $f(x) = \sin 10x + \cos 3x$ using Bisection method. The initial approximations are 4 and 5.

8. Find the real root correct to three decimal places for the following equations:

(*i*) $x^3 - x - 4 = 0$ (*ii*) $x^3 - x^2 - 1 = 0$

(*iii*) $x^3 + x^2 - 1 = 0$ (*iv*) $x^3 - 3x - 5 = 0$.

9. Solve $x^3 - 9x + 1 = 0$ for the root between $x = 2$ and $x = 4$ by the method of Bisection. Also find the smallest positive root.

10. (*i*) Find a real root of the equation $x^3 - 2x - 5 = 0$ using Bisection method.

(*ii*) Find a positive root of the equation $xe^x = 1$ which lies between 0 and 1.

11. If a root of $f(x) = 0$ lies in the interval (a, b), then find the minimum number of iterations required when the permissible error is E.

12. Apply Bisection method to find a root of the equation $x^4 + 2x^3 - x - 1 = 0$ in the interval [0, 1].

13. Find the approximate value of the root of the equation $3x - \sqrt{1 + \sin x} = 0$ by Bisection method. (five iterations) [*U.P.T.U. (MCA) 2008*]

14. Find a real root of $x^3 - 2x - 1 = 0$ which lies between 1 and 2 by using Bisection method correct to two places of decimals.

15. Find a positive root of the equation $x^3 + 3x - 1 = 0$ by Bisection method.

Answers

1. (*i*)

x:	– 4	– 3	– 2	– 1	0	1	2	3	4
$f(x)$:	1.0625	.125	– .75	– 1.5	– 2	– 2	– 1	2	9

Roots lie in (– 3, – 2) and (2, 3).

(*ii*) 1.7281 in interval (1, 2).

2. 1.5121375 **3.** (*i*) (2, 3) (*ii*) Root lies in the interval (2, 2.25).

4. 1.4036 **5.** 0.517

6. (*i*) (– 3, – 2) (*ii*) Root lies in the interval (– 2.5, – 2.25) **7.** 4.712389

8. (*i*) 1.796 (*ii*) 1.466 (*iii*) 0.755 (*iv*) 2.279

9. 2.94282, 0.111 **10.** (*i*) 2.094551482 (*ii*) 0.56714333

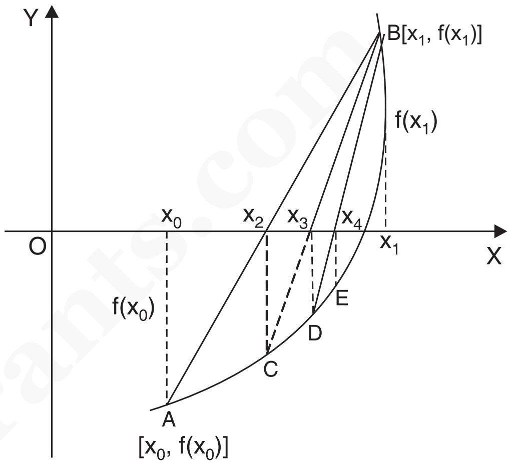
11. $n \geq \dfrac{\log_e \left(\dfrac{|b-a|}{E}\right)}{\log_e 2}$ **12.** 0.8667605 **13.** 0.39188 **14.** 1.618 **15.** 0.322

4.6 METHOD OF FALSE POSITION *OR* REGULA-FALSI METHOD

This is the oldest method for finding the real roots of a numerical equation.

It is sometimes known as **method of linear interpolation.**

In this method, we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Since the graph of $y = f(x)$ crosses the X-axis between these two points, a root must lie in between these points.



Consequently, $f(x_0)\, f(x_1) < 0$

Equation of the chord joining points $\{x_0, f(x_0)\}$ and $\{x_1, f(x_1)\}$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with X-axis as an approximation to the root.

So the abscissa of the point where chord cuts $y = 0$ is given by

$$x_2 = x_0 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_0) \quad \ldots(1)$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So, replacing x_1 by x_2 in (1), we obtain the next approximation x_3. However, the root could as well lie between x_1 and x_2 then we find x_3 accordingly.

This procedure is repeated till the root is found to the desired accuracy.

EXAMPLES

Example 1. *Find a real root of the equation $3x + \sin x - e^x = 0$ by the method of false position correct to four decimal places. Choose suitable initial approximations.*

[U.P.T.U. 2010 , U.P.T.U. (MCA) 2009]

Sol. Let $f(x) \equiv 3x + \sin x - e^x = 0$

$f(0.3) = -\,0.154$ *i.e.,* (–)ve

and $f(0.4) = 0.0975$ *i.e.,* (+)ve

$\therefore$ The root lies between 0.3 and 0.4.

Using Regula-Falsi method,

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= (0.3) - \frac{(0.4) - (0.3)}{(0.0975) - (-\,0.154)}(-\,0.154) \quad \Big| \because \; x_0 = 0.3 \text{ and } x_1 = 0.4 \text{ (let)}$$

$$= (0.3) + \left(\frac{0.1 \times 0.154}{0.2515}\right) = 0.3612$$

Now, $f(x_2) = f(0.3612) = 0.0019 = (+)\text{ve}$

Hence, the root lies between 0.3 and 0.3612.

Now, again, $x_3 = x_0 - \dfrac{(x_2 - x_0)}{f(x_2) - f(x_0)} f(x_0)$ | Replacing x_1 by x_2

$$= (0.3) - \left\{\frac{(0.3612) - (0.3)}{(0.0019) - (-0.154)}\right\}(-0.154)$$

$$= (0.3) + \left(\frac{0.0612}{0.1559}\right)(0.154) = 0.3604$$

Now, $f(x_3) = f(0.3604) = -0.00005 = (-)\text{ve}$

$\therefore$ The root lies between 0.3604 and 0.3612.

Now, again, $x_4 = x_3 - \left\{\dfrac{x_2 - x_3}{f(x_2) - f(x_3)}\right\} f(x_3)$ | Replacing x_0 by x_3

$$= (0.3604) - \left[\frac{(0.3612 - 0.3604)}{(0.0019) - (-0.00005)}\right](-0.00005)$$

$$= 0.3604 + \left(\frac{0.0008}{0.00195}\right)(0.00005) = 0.36042$$

Since, x_3 and x_4 are approximately the same, hence, the required real root is **0.3604** correct to four decimal places.

Example 2. *Find the root of the equation $xe^x = \cos x$ in the interval (0, 1) using Regula-Falsi method correct to four decimal places.* [U.P.T.U. (MCA) 2006]

Sol. Let $f(x) = xe^x - \cos x$

Since $f(0.51) = -0.02344$ and $f(0.52) = 0.00683$

$\therefore$ The root lies between 0.51 and 0.52.

Let $x_0 = 0.51$ and $x_1 = 0.52$

By Regula-Falsi method, first approximation to the root is

$$x_2 = x_0 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_0)$$

$$= 0.51 - \left[\frac{0.52 - 0.51}{0.00683 - (-0.02344)}\right](-0.02344) = 0.517744$$

$$f(x_2) = -0.000041$$

Hence, the root lies between x_2 and x_1.

Second approximation to the root is

$$x_3 = x_2 - \left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right] f(x_2) \qquad \text{| Replacing } x_0 \text{ by } x_2$$

$$= 0.517744 - \left[\frac{0.52 - 0.517744}{0.00683 - (-0.000041)}\right](-0.000041) = 0.517757$$

Since x_2 and x_3 are same up to four decimal places hence the approximate real root is 0.5177.

Example 3. *Find the root of the equation tan x + tanh x = 0 which lies in the interval (1.6, 3.0) correct to four significant digits using method of false position.*

[G.B.T.U. 2013, M.T.U. (MCA) 2012]

Sol. Let $f(x) \equiv \tan x + \tanh x = 0$

Since, $f(2.35) = -0.03$ and $f(2.37) = 0.009$

Hence, the root lies between 2.35 and 2.37. Let $x_0 = 2.35$ and $x_1 = 2.37$

Using Regula-Falsi method, first approximation to the root is

$$x_2 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0)$$

$$= 2.35 - \left(\frac{2.37 - 2.35}{0.009 + 0.03}\right)(-0.03) = 2.36538$$

Now, $f(x_2) = .000719$ (+)ve

Hence, the root lies between 2.35 and 2.36538.

Second approximation to the root is

$$x_3 = x_0 - \left\{\frac{x_2 - x_0}{f(x_2) - f(x_0)}\right\} f(x_0) \qquad | \text{ Replacing } x_1 \text{ by } x_2$$

$$= 2.35 - \left(\frac{2.36538 - 2.35}{0.000719 + .03}\right)(-.03)$$

$$= 2.36502$$

Since x_2 and x_3 are same up to four significant digits, hence the required root is **2.365.**

Example 4. *Using the method of false position, find the root of equation $x^6 - x^4 - x^3 - 1 = 0$ upto four decimal places.*

Sol. Let $f(x) = x^6 - x^4 - x^3 - 1$

Since, $f(1.4) = -0.056$ and $f(1.41) = 0.102$

Hence, the root lies between 1.4 and 1.41. Let $x_0 = 1.4$ and $x_1 = 1.41$.

Using method of false position, first approximation to the root is

$$x_2 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0)$$

$$= 1.4 - \left(\frac{1.41 - 1.4}{0.102 + 0.056}\right)(-0.056) = 1.4035$$

Now, $f(x_2) = -0.0016$ (–)ve

Hence, the root lies between x_2 and x_1.

Second approximation to the root is

$$x_3 = x_2 - \left\{\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right\} f(x_2) \qquad | \text{ Replacing } x_0 \text{ by } x_2$$

$$= 1.4035 - \left(\frac{1.41 - 1.4035}{0.102 + 0.0016}\right)(-0.0016)$$

$$= 1.4035 + \left(\frac{0.0065}{0.1036}\right)(0.0016) = 1.4036$$

Now, $f(x_3) = -\ 0.00003$ (–)ve

Hence, the root lies between x_3 and x_1.

Again, the third approximation to the root is

$$x_4 = x_3 - \left\{\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right\} f(x_3) \qquad | \text{ Replacing } x_2 \text{ by } x_3$$

$$= 1.4036 + \left(\frac{1.41 - 1.4036}{0.102 + 0.00003}\right)(0.00003)$$

$$= 1.4036 + \left(\frac{0.0064}{0.10203}\right)(0.00003) = 1.4036$$

Since, x_3 and x_4 are approximately the same up to four places of decimal, hence the required root of the given equation is **1.4036.** It is clear that –1 is also a root of the given equation.

Example 5. *Find a real root of the equation $x \log_{10} x = 1.2$ by Regula-Falsi method correct to four decimal places.*

Sol. Let $f(x) = x \log_{10} x - 1.2$ ...(1)

Since, $f(2.74) = -\ .0005634$ and $f(2.741) = .0003087$

Hence, the root lies between 2.74 and 2.741. Let $x_0 = 2.74$ and $x_1 = 2.741$.

Using method of False position, first approximation to the root is

$$x_2 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0)$$

$$= 2.74 - \left\{\frac{2.741 - 2.74}{.0003087 - (-\ .0005634)}\right\}(-\ .0005634)$$

$$= 2.74 + \left(\frac{.001}{.0008721}\right)(.0005634) = 2.740646027$$

Now, $f(x_2) = -\ .00000006016$ *i.e.,* (–)ve

Hence, the root lies between x_2 and x_1.

Second approximation to the root is

$$x_3 = x_2 - \left\{\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right\} f(x_2) \qquad | \text{ Replacing } x_0 \text{ by } x_2$$

$$= 2.740646027 - \left(\frac{2.741 - 2.740646027}{.0003087 + .00000006016}\right)(-\ .00000006016)$$

$$= 2.740646096$$

Since, x_2 and x_3 agree up to seven decimal places, the required root correct to four decimal places is **2.7406.**

Example 6. (*i*) *Apply False position method to find smallest positive root of the equation $x - e^{-x} = 0$, correct to three decimal places.*

(*ii*) *Using Regula-Falsi method, compute the smallest positive root of the equation $xe^x - 2 = 0$, correct up to four decimal places.* (U.P.T.U. 2006)

Sol. (*i*) Let $f(x) = x - e^{-x}$

Since $f(.56) = -.01121$ and $f(.58) = .0201$

Hence, root lies between .56 and .58. Let $x_0 = .56$ and $x_1 = .58$

Using method of False position, first approximation to the root is

$$x_2 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0)$$

$$= .56 - \left(\frac{.58 - .56}{.0201 + .01121}\right)(-.01121) = .56716$$

Now, $f(x_2) = .00002619$ *i.e.,* (+)ve

Hence the root lies between x_0 and x_2.

Second approximation to the root is

$$x_3 = x_0 - \left\{\frac{x_2 - x_0}{f(x_2) - f(x_0)}\right\} f(x_0) \qquad | \text{ Replacing } x_1 \text{ by } x_2$$

$$= .56 - \left(\frac{.56716 - .56}{.00002619 + .01121}\right)(-.01121) = .567143$$

Since, x_2 and x_3 agree up to four decimal places, the required root correct to three decimal places is **0.567.**

(*ii*) Let $f(x) = xe^x - 2$

Since, $f(.852) = -.00263$ and $f(.853) = .001715$

Hence, root lies between .852 and .853. Let $x_0 = .852$ and $x_1 = .853$

Using method of False position, first approximation to the root is

$$x_2 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0)$$

$$= .852 - \left\{\frac{.853 - .852}{.001715 - (-.00263)}\right\}(-.00263) = .852605293$$

Now $f(x_2) = -.00000090833$

Hence, root lies between x_2 and x_1.

Second approximation to the root is

$$x_3 = x_2 - \left\{\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right\} f(x_2) \qquad | \text{ Replacing } x_0 \text{ by } x_2$$

$$= (.852605293) - \left\{\frac{.853 - .852605293}{.001715 - (-.00000090833)}\right\}(-.00000090833)$$

$$= 0.852605501$$

Since, x_2 and x_3 agree upto 6 decimal places, hence the required root correct to 4 decimal places is **0.8526.**

Example 7. (*i*) *Solve $x^3 - 5x + 3 = 0$ by using Regula-Falsi method.*

(*ii*) *Use the method of false position to solve $x^3 - x - 4 = 0$.*

Sol. (*i*) Let $f(x) = x^3 - 5x + 3$

Since, $f(.65) = .024625$ and $f(.66) = -.012504$

Hence, root lies between .65 and .66. Let $x_0 = .65$ and $x_1 = .66$

Using method of False position, first approximation to the root is

$$x_2 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0)$$

$$= .65 - \left(\frac{.66 - .65}{-.012504 - .024625}\right)(.024625) = .656632282$$

Now, $f(x_2) = -.00004392$

Hence, root lies between x_0 and x_2.

Second approximation to the root is

$$x_3 = x_0 - \left\{\frac{x_2 - x_0}{f(x_2) - f(x_0)}\right\} f(x_0) \quad \text{| Replacing } x_1 \text{ by } x_2$$

$$= .65 - \left(\frac{.656632282 - .65}{-.00004392 - .024625}\right)(.024625)$$

$$= .656620474$$

Since, x_2 and x_3 agree up to four decimal places hence the required root is **.6566** correct up to four decimal places. Similarly the other roots of this equation are 1.8342 and – 2.4909.

(*ii*) Let $f(x) = x^3 - x - 4$

Since, $f(1.79) = -.054661$ and $f(1.80) = .032$

Hence, root lies between 1.79 and 1.80. Let $x_0 = 1.79$ and $x_1 = 1.80$.

Using method of False position, first approximation to the root is

$$x_2 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0)$$

$$= 1.79 - \left\{\frac{1.80 - 1.79}{.032 - (-.054661)}\right\}(-.054661) = 1.796307$$

Now, $f(x_2) = -.00012936$

Hence, root lies between x_2 and x_1.

Second approximation to the root is

$$x_3 = x_2 - \left\{\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right\} f(x_2) \quad \text{| Replacing } x_0 \text{ by } x_2$$

$$= 1.796307 - \left\{\frac{1.8 - 1.796307}{.032 - (-.00012936)}\right\}(-.00012936)$$

$$= 1.796321.$$

Since, x_2 and x_3 are same up to four decimal places hence the required root is **1.7963** correct up to four decimal places.

4.7 CONVERGENCE OF REGULA-FALSI METHOD (*M.T.U. 2013*)

If $< x_n >$ be the sequence of approximations obtained from

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n) \quad ...(1)$$

and α be the exact value of the root of equation $f(x) = 0$, then

Let $x_n = \alpha + e_n$ and $x_{n+1} = \alpha + e_{n+1}$

where e_n, e_{n+1} being errors involved in n^{th} and $(n + 1)^{\text{th}}$ approximations respectively.

Clearly, $f(\alpha) = 0$. Hence, (1) gives,

$$\alpha + e_{n+1} = \alpha + e_n - \frac{(e_n - e_{n-1})}{f(\alpha + e_n) - f(\alpha + e_{n-1})} \cdot f(\alpha + e_n)$$

or

$$e_{n+1} = \frac{e_{n-1} f(\alpha + e_n) - e_n f(\alpha + e_{n-1})}{f(\alpha + e_n) - f(\alpha + e_{n-1})}$$

$$= \frac{e_{n-1}\left[f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + ...\right] - e_n\left[f(\alpha) + e_{n-1} f'(\alpha) + \frac{e_{n-1}^2}{2!} f''(\alpha) + ...\right]}{\left[f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + ...\right] - \left[f(\alpha) + e_{n-1} f'(\alpha) + \frac{e_{n-1}^2}{2!} f''(\alpha) + ...\right]}$$

$$= \frac{(e_{n-1} - e_n) f(\alpha) + \frac{e_{n-1} e_n}{2!}(e_n - e_{n-1}) f''(\alpha) + ...}{(e_n - e_{n-1}) f'(\alpha) + \frac{(e_n - e_{n-1})(e_n + e_{n-1})}{2!} f''(\alpha) + ...}$$

$$= \frac{\frac{e_{n-1} e_n}{2} f''(\alpha) + ...}{f'(\alpha) + \left(\frac{e_n + e_{n-1}}{2}\right) f''(\alpha) + ...} \quad \Big| \because \ f(\alpha) = 0$$

or $$e_{n+1} \approx \frac{e_n e_{n-1}}{2!} \frac{f''(\alpha)}{f'(\alpha)} \quad ...(2) \quad \text{(neglecting high powers of } e_n, e_{n-1})$$

or $$e_{n+1} = c\, e_n\, e_{n-1} \quad ...(3)$$

where $c = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}$. Relation of the form (3) is called the error equation.

If the function $f(x)$ in the equation $f(x) = 0$ is convex in the interval (x_0, x_1) that contains the root, then one of the points x_0 or x_1 is always fixed and the other point varies with n. If the point x_0 is fixed then the function $f(x)$ is approximated by the straight line passing through the points (x_0, f_0) and (x_n, f_n), $k = 1, 2,...$. The error equation (3) becomes

$$e_{n+1} = c\, e_0\, e_n$$

where $e_0 = x_0 - \alpha$ is independent of n. Therefore, we can write

$$e_{n+1} = c^* e_n$$

where $c^* = c\, e_0$ is the asymptotic error constant. Hence, **the Regula-Falsi method has linear rate of convergence.**

ASSIGNMENT

1. (*i*) Solve $\cos x = 3x - 1$ correct to three decimal places using the method of False position. (*G.B.T.U. 2011*)

(*ii*) Solve $x^3 - 9x + 1 = 0$ for the root lying between 2 and 4 by the method of False position.

2. (*i*) Find real cube root of 18 by Regula-Falsi method.

(*ii*) Find the root of the following equation in the interval [0, 1] by Regula-Falsi method:

$$2x(1 - x^2 + x) \ln x = x^2 - 1$$ [*U.P.T.U. 2014, M.T.U.* (*MCA*) *2010*]

3. (*i*) Find the smallest positive root correct to three decimal places of $\cosh x \ \cos x = -1$.

(*ii*) Use Regula-Falsi method to find the smallest positive root of the following equation correct to four significant digits: $x^3 - 5x + 1 = 0$. (*M.T.U. 2012*)

4. Find all the roots of $\cos x - x^2 - x = 0$ up to 5 decimal places.

5. Find the real root of the equations by using method of false position.

(*i*) $x^4 - x^3 - 2x^2 - 6x - 4 = 0$ (*ii*) $x^6 - x^4 - x^3 - 3 = 0$

(*iii*) $xe^x = 3$ (*iv*) $x^2 - \log_{10} x - 12 = 0$

6. Solve the following equations by Regula-Falsi method:

(*i*) $(5 - x) e^x = 5$ near $x = 5$ (*ii*) $x^3 + x - 1 = 0$ near $x = 1$

(*iii*) $2x - \log_{10} x = 7$ lying between 3.5 and 4 (*iv*) $x^3 + x^2 - 3x - 3 = 0$ lying between 1 and 2

(*v*) $x^3 - 3x + 4 = 0$ between -2 and -3 (*vi*) $x^4 + x^3 - 7x^2 - x + 5 = 0$ lying between 2 and 3.

7. Find the real root of the equations by using method of False position.

(*i*) $x^3 - 4x + 1 = 0$ (*ii*) $x^3 - x^2 - 2 = 0$

(*iii*) $x^3 + x - 3 = 0$ (*iv*) $x^4 - x - 10 = 0$ (*G.B.T.U. 2012*)

8. (*i*) Explain Regula-Falsi method by stating at least one advantage over Bisection method.

(*ii*) Discuss method of False position. [*G.B.T.U. 2011; U.P.T.U. 2009*]

(*iii*) Illustrate False position method by plotting the function on graph and discuss the speed of convergence to the root.

(*iv*) Find the rate of convergence for Regula-Falsi method. (*M.T.U. 2013*)

(*v*) Determine the order of convergence of the iterative method

$$x_{k+1} = \frac{x_0\, f(x_k) - x_k\, f(x_0)}{f(x_k) - f(x_0)}$$

for finding a simple root of the equation $f(x) = 0$.

Answers

1. (*i*) 0.607 (*ii*) 2.942821 **2.** (*i*) 2.620741394 (*ii*) 1, 0.3279

3. (*i*) 1.875 (*ii*) 0.2016 **4.** $-$ 1.25115 and 0.55000

5. (*i*) 2.7320506 (*ii*) 1.501844 (*iii*) 1.04991 (*iv*) 3.5425

6. (*i*) 4.9651142 (*ii*) .682327803 (*iii*) 3.7892782 (*iv*) 1.73205 (*v*) $-$ 2.195823345 (*vi*) 2.0608526

7. (*i*) 1.860, .2541 (*ii*) 1.69562 (*iii*) 1.2134 (*iv*) 1.855

8. (*v*) linear : $e_{k+1} = ce_0e_k$.

4.8 NEWTON-RAPHSON METHOD (*M.T.U. 2013, U.P.T.U. 2009, 2015*)

This method is generally used to improve the result obtained by one of the previous methods. Let x_0 be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$.

Expanding $f(x_0 + h)$ by Taylor's series, we get

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots = 0$$

Since, h is small, neglecting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0 \quad \text{or} \quad h = -\frac{f(x_0)}{f'(x_0)} \qquad \ldots(1)$$

A better approximation than x_0 is therefore given by x_1, where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximations are given by $x_2, x_3, \ldots\ldots, x_{n+1}$, where

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad \ldots(2) \qquad (n = 0, 1, \ldots\ldots)$$

which is the Newton-Raphson formula.

Remarks. (1) This method is useful in cases of large values of $f'(x)$ *i.e.*, when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

(2) If $f'(x)$ is zero or nearly 0, the method fails.

(3) Newton's formula converges provided the initial approximation x_0 is chosen sufficiently close to the root. This method is also called the method of tangents.

(4) This method is also used to obtain complex roots.

4.9 CONVERGENCE

Comparing equation (2) with $x_{n+1} = \phi(x_n)$ of the iteration method, we get

$$\phi(x_n) = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In general, $\phi(x) = x - \dfrac{f(x)}{f'(x)}$ which gives $\phi'(x) = \dfrac{f(x)\,f''(x)}{[f'(x)]^2}$

Since, iteration method converges if $|\phi'(x)| < 1$

$\therefore$ Newton's method converges if

$$|f(x)\,f''(x)| < [f'(x)]^2$$

in the interval considered.

Assuming $f(x)$, $f'(x)$ and $f''(x)$ to be continuous, we can select a small interval in the vicinity of the root α in which above condition is satisfied.

The rate at which the iteration method converges if the initial approximation to the root is sufficiently close to the desired root is called the **rate of convergence.**

[*M.T.U. 2014, G.B.T.U. 2011, 2012, 2013*]

4.10 ORDER OF CONVERGENCE

[*M.T.U. 2012, G.B.T.U. (MCA) 2011; U.P.T.U. 2014, 2015; G.B.T.U. (M.Tech.) 2011*]

Suppose, x_n differs from the root α by a small quantity e_n so that

$$x_n = \alpha + e_n \quad \text{and} \quad x_{n+1} = \alpha + e_{n+1}$$

Then equation (2) becomes,

$$e_{n+1} = e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)}$$

$$= e_n - \frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \ldots\ldots}{f'(\alpha) + e_n f''(\alpha) + \ldots\ldots} \qquad \text{(By Taylor's expansion)}$$

$$= e_n - \frac{e_n f'(\alpha) + \frac{e_n^2}{2} f''(\alpha) + \ldots\ldots.}{f'(\alpha) + e_n f''(\alpha) + \ldots\ldots} \qquad | \because \quad f(\alpha) = 0$$

$$= \frac{e_n^2 f''(\alpha)}{2[f'(\alpha) + e_n f''(\alpha)]} \qquad | \text{ Neglect higher powers of } e_n$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)\left\{1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\right\}}$$

$$= \frac{e_n^2}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)} \left\{1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\right\}^{-1}$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \left\{1 - e_n \frac{f''(\alpha)}{f'(\alpha)} + \ldots\ldots.\right\}$$

or
$$\frac{e_{n+1}}{e_n^2} = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)} - \frac{e_n}{2} \left\{\frac{f''(\alpha)}{f'(\alpha)}\right\}^2 + \ldots\ldots.$$

$$\approx \frac{f''(\alpha)}{2f'(\alpha)} \qquad \text{(Neglecting terms containing powers of } e_n)$$

Hence, by definition, the order of convergence of Newton-Raphson method is 2 *i.e.,* Newton-Raphson method is **quadratic convergent.**

This also shows that subsequent error at each step is proportional to the square of the previous error and as such the *convergence is quadratic.*

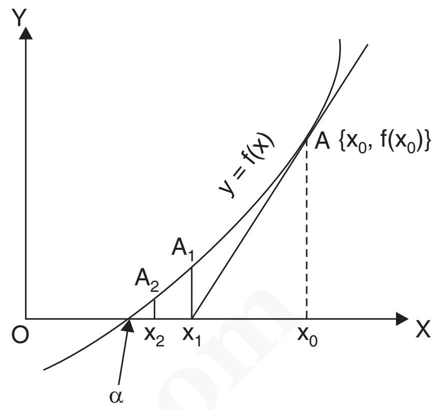
Hence, if at the first iteration, we have an answer correct to one decimal place then it should be correct up to two places at the II iteration, up to four places at III iteration.

It means that the number of correct decimal places at each iteration is almost doubled.

$\therefore$ Method converges very rapidly.

Due to its quadratic convergence, the formula (2) is also termed as **second order formula.**

4.11 GEOMETRICAL INTERPRETATION



Let x_0 be a point near the root α of equation $f(x) = 0$, then tangent at A $\{x_0, f(x_0)\}$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

It cuts x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

which is I approximation to root α. If A_1 corresponds to x_1 on the curve, then tangent at A_1 will cut x-axis at x_2, nearer to α and is therefore II approximation to root α.

Repeating this process, we approach the root α quite rapidly. Hence, the method consists in replacing the part of the curve between A and x-axis by the means of the tangent to the curve at A_0.

4.12 NEWTON'S ITERATIVE FORMULAE FOR FINDING INVERSE, SQUARE ROOT ETC.

1. Inverse. The reciprocal or inverse of a number 'a' can be considered as a root of the equation $\dfrac{1}{x} - a = 0$, which can be solved by Newton's method.

Since, $$f(x) = \frac{1}{x} - a,\ f'(x) = -\frac{1}{x^2}$$

∴ Newton's formula gives,

$$x_{n+1} = x_n + \frac{\left(\frac{1}{x_n} - a\right)}{\left(\frac{1}{x_n^2}\right)}$$

$$\boxed{x_{n+1} = x_n(2 - ax_n)}$$

2. Square root. The square root of 'a' can be considered as a root of equation $x^2 - a = 0$ solvable by Newton's method.

Since, $$f(x) = x^2 - a, \quad f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n}$$

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)}$$

3. Inverse square root. Equation is $\dfrac{1}{x^2} - a = 0$

Iterative formula is

$$\boxed{x_{n+1} = \frac{1}{2} x_n (3 - a x_n^2)}$$

4. General formula for p^th^ root. The p^{th} root of a can be considered as a root of the equation $x^p - a = 0$. To solve this by Newton's method, we have

$$f(x) = x^p - a \quad \text{and hence} \quad f'(x) = px^{p-1}$$

$\therefore$ Iterative formula is, $x_{n+1} = x_n - \dfrac{(x_n{}^p - a)}{px_n{}^{p-1}}$

$$\boxed{x_{n+1} = \frac{(p-1)\, x_n{}^p + a}{px_n{}^{p-1}}} \qquad \text{[U.P.T.U. MCA (SUM) 2008]}$$

Also, the general formula for reciprocal of p^{th} root of a is

$$x_{n+1} = x_n \left(\frac{p+1-ax_n{}^p}{p}\right).$$

4.13 ORDER OF CONVERGENCE OF NEWTON'S SQUARE ROOT FORMULA

Let $\sqrt{a} = \alpha$ so that $a = \alpha^2$. If we write

$$x_n = \alpha\left(\frac{1+e_n}{1-e_n}\right)$$

then,
$$x_{n+1} = \alpha\left(\frac{1+e_{n+1}}{1-e_{n+1}}\right) \qquad ...(1)$$

Also, by formula, $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$, we get

$$x_{n+1} = \frac{1}{2}\left[\alpha\left(\frac{1+e_n}{1-e_n}\right) + \frac{a}{\alpha}\left(\frac{1-e_n}{1+e_n}\right)\right]$$

$$= \alpha\left(\frac{1+e_n^2}{1-e_n^2}\right) \qquad ...(2) \quad (\because \ a = \alpha^2)$$

Comparing (1) and (2), we get $e_{n+1} = e_n^2$ confirming quadratic convergence of Newton's method.

4.14 ORDER OF CONVERGENCE OF NEWTON'S INVERSE FORMULA

Let $\alpha = \dfrac{1}{a}$ *i.e.,* $a = \dfrac{1}{\alpha}$. If we write $x_n = \alpha(1 - e_n)$

then, $x_{n+1} = \alpha\,(1 - e_{n+1})$

By formula, $x_{n+1} = x_n\,(2 - ax_n)$, we get

$$x_{n+1} = \alpha(1-e_n)\,[2 - a\alpha\,(1-e_n)] = \alpha(1-e_n{}^2) \qquad | \because \ a\alpha = 1$$

Comparing, we get $e_{n+1} = e_n{}^2$, hence, convergence is quadratic.

EXAMPLES

Example 1. *Using Newton-Raphson method, find the real root of the equation $3x = \cos x + 1$ correct to four decimal places.* [G.B.T.U. (M. Tech.) 2010, 2011]

Sol. Let $f(x) = 3x - \cos x - 1$

Since, $f(0) = -2 = (-)\text{ve}$;

$f(1) = 1.4597 = (+)\text{ve}$

$\therefore$ A root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Let us take $x_0 = 0.6$.

Also, $f'(x) = 3 + \sin x$

Newton's iteration formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \quad ...(1)$$

Put $n = 0$, the first approximation x_1 is given by,

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{0.6 \sin 0.6 + \cos 0.6 + 1}{3 + \sin 0.6} = .6071$$

Put $n = 1$ then second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1}$$

$$= \frac{.6071 \sin(.6071) + \cos(.6071) + 1}{3 + \sin(.6071)} = 0.6071$$

Clearly, $x_1 = x_2$. Hence, desired root is **0.6071** correct to 4 decimal places.

Example 2. *Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to six decimal places.*

Sol. $f(x) = x \log_{10} x - 1.2$

$\because$ $f(2.7) = -.0353 = (-)\text{ve}$

$f(2.8) = .05204 = (+)\text{ve}$

Hence, a root of $f(x) = 0$ lies between 2.7 and 2.8.

Since $|f(2.7)| < |f(2.8)|$

Hence, root is nearer to 2.7.

Let us take $x_0 = 2.71$

Also, $f'(x) = \log_{10} x + \log_{10} e = \log_{10} x + 0.43429$

Newton's iteration formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + .43429}\right) = \frac{.43429 x_n + 1.2}{\log_{10} x_n + .43429} \quad ...(1)$$

Put $n = 0$, the first approximation is

$$x_1 = \frac{.43429\, x_0 + 1.2}{\log_{10} x_0 + .43429} = 2.74073 \qquad |\text{ Taking } x_0 = 2.71$$

Similarly, Putting $n = 1, 2$, in (1), we get

$$x_2 = 2.740646$$

$$x_3 = 2.740646$$

Since x_2 and x_3 are same up to 6 decimal places, hence the required root is **2.740646.**

Example 3. *Evaluate $\sqrt{12}$ to five decimal places by Newton's iterative method.*

Sol. Let $x = \sqrt{12}$ so that $x^2 - 12 = 0$ ...(1)

Take $f(x) = x^2 - 12$, Newton's iteration formula gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 12}{2x_n} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right) \quad \text{...(2)}$$

Now since, $f(3) = -3$ (–)ve

$f(4) = 4$ (+)ve

∴ The root of (1) lies between 3 and 4.

Since $|f(3)| < |f(4)|$, hence, root is nearer to 3.

Take $x_0 = 3.4$, equation (2) gives,

$$x_1 = \frac{1}{2}\left(x_0 + \frac{12}{x_0}\right) = \frac{1}{2}\left(3.4 + \frac{12}{3.5}\right) = 3.464706$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{12}{x_1}\right) = 3.464102$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{12}{x_2}\right) = 3.4641016$$

Since, x_2 and x_3 are same up to 5 decimal places,

$$x = \sqrt{12} = \mathbf{3.46410.}$$

Example 4. *Using Newton's iterative method, find the real root of $x \sin x + \cos x = 0$ which is near $x = \pi$ correct to 3 decimal places.* [G.B.T.U. (C.O.) 2010]

Sol. We have $f(x) = x \sin x + \cos x$ and $f'(x) = x \cos x$

The iteration formula is,

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$$

with $x_0 = \pi$, $x_1 = x_0 - \dfrac{x_0 \sin x_0 + \cos x_0}{x_0 \cos x_0} = \pi - \dfrac{\pi \sin \pi + \cos \pi}{\pi \cos \pi} = 2.8233$

Successive iteratives are,

$$x_2 = 2.7986, \quad x_3 = 2.7984$$

Since, x_2 and x_3 are same up to three decimal places hence required root is **2.798.**

Example 5. *Find a real root of the equation $x = e^{-x}$ using the Newton-Raphson method.*

[U.P.T.U. (MCA) 2008]

Sol. We have $f(x) = x - e^{-x}$

∴ $f'(x) = 1 + e^{-x}$

Since, $f(.5) = -.1065$ (–)ve

$f(.6) = .05118$ (+)ve

Hence, the root lies between .5 and .6.

Since, $|f(.6)| < |f(.5)|$ $\therefore$ Root is nearer to 0.6.

Let us take $x_0 = 0.58$.

Newton's iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \left(\frac{x_n - e^{-x_n}}{1 + e^{-x_n}}\right)$$

Putting $n = 0$, we get

$$x_1 = x_0 - \frac{(x_0 - e^{-x_0})}{(1 + e^{-x_0})} = 0.567113$$

Putting $n = 1, 2$, we get successively

$$x_2 = 0.567143$$
$$x_3 = 0.56714329$$

Since x_2 and x_3 are same up to 6 decimal places, hence, the required real root is **0.567143.**

Example 6. *Find a positive value of $(17)^{1/3}$ correct to six decimal places by Newton-Raphson method.* (M.T.U. 2013, U.P.T.U. 2007, 2014)

Sol. Let $x = (17)^{1/3}$ so that $x^3 - 17 = 0$ ...(1)

Let $f(x) = x^3 - 17$ $\therefore$ $f'(x) = 3x^2$

Since $f(2.5) = -1.375$ (–)ve

and $f(2.6) = 0.576$ (+)ve

$\therefore$ The root lies between 2.5 and 2.6.

Since, $|f(2.6)| < |f(2.5)|$

Therefore, root is nearer to 2.6. Let us take $x_0 = 2.58$.

Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \left(\frac{x_n^3 - 17}{3x_n^2}\right)$$

Putting $n = 0, 1, 2, \ldots$, successively, we get

$$x_1 = x_0 - \left(\frac{x_0^3 - 17}{3x_0^2}\right) = 2.571311 \qquad | \because \quad x_0 = 2.58$$

$$x_2 = 2.571281$$
$$x_3 = 2.5712815$$

Since x_2 and x_3 are same up to 6 decimal places, hence, the required positive root is **2.571281.**

Example 7. *Show that the following two sequences, both have convergence of the second order with the same limit $\sqrt{a}$.*

$$x_{n+1} = \frac{1}{2}x_n\left(1 + \frac{a}{x_n^2}\right) \text{ and, } x_{n+1} = \frac{1}{2}x_n\left(3 - \frac{x_n^2}{a}\right).$$

(U.P.T.U. 2006, 2009)

Sol. Since, $$x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right), \text{ we have}$$

$$x_{n+1} - \sqrt{a} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right) - \sqrt{a} = \frac{1}{2}\left(x_n + \frac{a}{x_n} - 2\sqrt{a}\right)$$

$$= \frac{1}{2}\left(\sqrt{x_n} - \frac{\sqrt{a}}{\sqrt{x_n}}\right)^2 = \frac{1}{2x_n}(x_n - \sqrt{a})^2$$

Thus, $$e_{n+1} = \frac{1}{2x_n} e_n^2 \qquad ...(1)$$

which shows the quadratic convergence. Similarly for the second,

$$x_{n+1} - \sqrt{a} = \frac{1}{2} x_n \left(3 - \frac{x_n^2}{a}\right) - \sqrt{a} = \frac{1}{2} x_n \left(1 - \frac{x_n^2}{a}\right) + (x_n - \sqrt{a})$$

$$= \frac{x_n}{2a}(a - x_n^2) + (x_n - \sqrt{a}) = (x_n - \sqrt{a})\left[1 - \frac{x_n}{2a}\left(x_n + \sqrt{a}\right)\right]$$

$$e_{n+1} = \frac{x_n - \sqrt{a}}{2a}[2a - x_n^2 - x_n\sqrt{a}] = \frac{x_n - \sqrt{a}}{2a}[(a - x_n^2) + (a - x_n\sqrt{a})]$$

$$= -\left(\frac{x_n - \sqrt{a}}{2a}\right)(x_n - \sqrt{a})(x_n + 2\sqrt{a})$$

$$e_{n+1} = -\frac{(x_n - \sqrt{a})^2}{2a}(x_n + 2\sqrt{a}) = -\frac{(x_n + 2\sqrt{a})}{2a} \cdot e_n^2 \qquad ...(2)$$

which shows the quadratic convergence.

Example 8. *If x_n is suitable close approximation to $\sqrt{a}$, show that error in the formula $x_{n+1} = \frac{1}{2} x_n\left(1 + \frac{a}{x_n^2}\right)$ is about $\frac{1}{3}$rd that in the formula, $x_{n+1} = \frac{1}{2} x_n \left(3 - \frac{x_n^2}{a}\right)$ and deduce that the formula $x_{n+1} = \frac{x_n}{8}\left(6 + \frac{3a}{x_n^2} - \frac{x_n^2}{a}\right)$ gives a sequence with third order convergence.*

Sol. Since x_n is very close to $\sqrt{a}$

$$e_{n+1} \simeq -\left(\frac{x_n + 2x_n}{2x_n^2}\right) e_n^2 \qquad | \text{ From (2)}$$

$$= 3 \cdot \frac{1}{2x_n} e_n^2 \qquad ...(3)$$

A simple observation shows that from (1) (see example 7) and (3), error in first formula for e_{n+1} is about $\frac{1}{3}$rd of that in second formula.

To find the rate of convergence of given formula, we have

$$x_{n+1} - \sqrt{a} = \frac{x_n}{8}\left(6 + \frac{3a}{x_n^2} - \frac{x_n^2}{a}\right) - \sqrt{a} = \frac{x_n(6x_n^2 a + 3a^2 - x_n^4)}{8\,a x_n^2} - \sqrt{a}$$

$$= \frac{6x_n{}^2 a + 3a^2 - x_n{}^4 - 8x_n\, a\sqrt{a}}{8x_n a} = \frac{-(x_n + 3\sqrt{a})(x_n - \sqrt{a})^3}{8x_n a}$$

$$\therefore \qquad e_{n+1} = -\left(\frac{x_n + 3\sqrt{a}}{8x_n a}\right) e_n{}^3$$

It shows that above formula has a convergence of third order.

Example 9. *Show that the square root of $N = AB$ is given by*

$$\sqrt{N} \simeq \frac{S}{4} + \frac{N}{S}, \text{ where } S = A + B.$$

Sol. Let $\quad x = \sqrt{\text{N}} \quad \Rightarrow \quad x^2 - \text{N} = 0$

Now, let $\quad f(x) = x^2 - \text{N} \quad \therefore \quad f'(x) = 2x$

By Newton-Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - \text{N}}{2x_n} = \frac{x_n}{2} + \frac{\text{N}}{2x_n}$$

Let $\quad x_n = \dfrac{\text{A} + \text{B}}{2}$

then, $\quad x_{n+1} = \dfrac{\text{A}+\text{B}}{4} + \dfrac{\text{N}}{\text{A}+\text{B}} \simeq \dfrac{\text{S}}{4} + \dfrac{\text{N}}{\text{S}} \qquad | \text{ Since S} = \text{A} + \text{B}$

Example 10. *Using the starting value $2(1 + i)$, solve $x^4 - 5x^3 + 20x^2 - 40x + 60 = 0$ by Newton-Raphson method given that all the roots of the given equation are complex.*

Sol. Let $\quad f(x) = x^4 - 5x^3 + 20x^2 - 40x + 60$

so that, $\quad f'(x) = 4x^3 - 15x^2 + 40x - 40$

$\therefore$ Newton-Raphson method gives,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 5x_n^3 + 20x_n^2 - 40x_n + 60}{4x_n^3 - 15x_n^2 + 40x_n - 40}$$

$$= \frac{3x_n^4 - 10x_n^3 + 20x_n^2 - 60}{4x_n^3 - 15x_n^2 + 40x_n - 40}$$

Put $n = 0$, take $x_0 = 2(1 + i)$ by trial, we get

$$x_1 = 1.92\,(1 + i)$$

Again, $\quad x_2 = 1.915 + 1.908\, i$

Since, imaginary roots occur in conjugate pairs, roots are $1.915 \pm 1.908\, i$ up to three places of decimal. Assuming other pair of roots to be $\alpha \pm i\beta$, then

$$\text{Sum} = \begin{pmatrix} \alpha + i\beta + \alpha - i\beta \\ + 1.915 + 1.908\, i \\ + 1.915 - 1.908\, i \end{pmatrix} = 2\alpha + 3.83 = 5$$

$\Rightarrow \quad \alpha = 0.585$

Also, product of roots $= (\alpha^2 + \beta^2)\ [(1.915)^2 + (1.908)^2] = 60$

$\Rightarrow \quad \beta = 2.805$

Hence, other two roots are $0.585 \pm 2.805\, i$.

Example 11. *Determine the value of p and q so that rate of convergence of the iterative method* $x_{n+1} = px_n + q\dfrac{N}{x_n^2}$ *for computing* $N^{1/3}$ *becomes as high as possible.*

Sol. We have $\quad x^3 = \text{N}$

$$\therefore \qquad f(x) = x^3 - \text{N} \qquad ...(1)$$

Let α be the exact root, we have

$$\alpha^3 = \text{N} \qquad ...(2)$$

Substituting $x_n = \alpha + e_n$, $x_{n+1} = \alpha + e_{n+1}$, $\text{N} = \alpha^3$ in

$$x_{n+1} = px_n + q\frac{\text{N}}{x_n^2}, \text{ we get}$$

$$\alpha + e_{n+1} = p(\alpha + e_n) + q\,\frac{\alpha^3}{(\alpha + e_n)^2}$$

$$= p(\alpha + e_n) + q\,\frac{\alpha^3}{\alpha^2\left(1 + \dfrac{e_n}{\alpha}\right)^2} = p(\alpha + e_n) + q\alpha\left(1 + \frac{e_n}{\alpha}\right)^{-2}$$

$$= p(\alpha + e_n) + q\alpha\left\{1 - 2\frac{e_n}{\alpha} + 3\left(\frac{e_n}{\alpha}\right)^2 - \ldots\ldots\ldots\right\}$$

$$= p(\alpha + e_n) + q\alpha - 2qe_n + 3q\,\frac{e_n^2}{\alpha} - \ldots\ldots\ldots$$

$$\Rightarrow \qquad e_{n+1} = (p + q - 1)\alpha + (p - 2q)e_n + 0(e_n^2) + \ldots\ldots.$$

Now, for the method to become of order as high as possible *i.e.,* of order 2, we must have

$$p + q = 1 \quad \text{and} \quad p - 2q = 0$$

so that, $$p = \frac{2}{3} \quad \text{and} \quad q = \frac{1}{3}.$$

Example 12. *The graph of y = 2 sin x and y = log x + c touch each other in the nbd. of point x = 8. Find c and the coordinates of point of contact.*

Sol. The graphs will touch each other if values of dy/dx at their point of contact is same.

For $\qquad y = 2 \sin x, \qquad \dfrac{dy}{dx} = 2 \cos x$

For $\qquad y = \log x + c \qquad \dfrac{dy}{dx} = \dfrac{1}{x}$

$$\therefore \qquad 2 \cos x = \frac{1}{x} \quad \Rightarrow \quad x \cos x - .5 = 0$$

Let $\qquad f(x) = x \cos x - .5$

$$\therefore \qquad f'(x) = \cos x - x \sin x$$

$\therefore$ Newton's iterative formula is $x_{n+1} = x_n - \dfrac{x_n \cos x_n - 0.5}{\cos x_n - x_n \sin x_n}$

For $n = 0$, $x_0 = 8$, first app. $x_1 = 7.793$

Second approximation, $x_2 = 7.789 \approx 7.79$

Now, $y = 2 \sin 7.79 = 1.9960$

$\therefore$ Point of contact $\rightarrow (7.79, 1.996)$

Now, $y = \log x + c$

$\Rightarrow$ $1.996 = \log 7.79 + c \Rightarrow c = -0.054.$

Example 13. *How should the constant* α *be chosen to ensure the fastest possible convergence with the iteration formula* $x_{n+1} = \dfrac{\alpha x_n + x_n^{-2} + 1}{\alpha + 1}$?

Sol. Since, $\lim\limits_{n \to \infty} x_n = \lim\limits_{n \to \infty} x_{n+1} = \xi$, we have

$$\xi = \left(\frac{\alpha \xi + \dfrac{1}{\xi^2} + 1}{\alpha + 1} \right)$$

$$\Rightarrow \quad (\alpha + 1)\xi^3 = \alpha \xi^3 + \xi^2 + 1$$

$$\Rightarrow \quad \xi^3 - \xi^2 - 1 = 0$$

ξ can be obtained by finding a root of the equation $x^3 - x^2 - 1 = 0$.

We have $f(x) = x^3 - x^2 - 1$

$f'(x) = 3x^2 - 2x$

Since, $f(1.45) = (-)$ve and $f(1.47) = (+)$ve

$\therefore$ Root lies between 1.45 and 1.47. Let $x_0 = 1.46$.

By Newton-Raphson method,

First approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \left(\frac{x_0^3 - x_0^2 - 1}{3x_0^2 - 2x_0} \right) = 1.465601.$$

Second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \left(\frac{x_1^3 - x_1^2 - 1}{3x_1^2 - 2x_1} \right) = 1.46557$$

Hence, $\xi = 1.465$ correct to three decimal places.

Now, we have

$$x_{n+1} = \frac{\alpha x_n + x_n^{-2} + 1}{\alpha + 1} \qquad \text{...(1)}$$

Putting $x_n = \xi + e_n$ and $x_{n+1} = \xi + e_{n+1}$ in (1), we get

$$(\alpha + 1)(\xi + e_{n+1}) = \alpha(\xi + e_n) + \frac{1}{(\xi + e_n)^2} + 1 = \alpha(\xi + e_n) + \frac{1}{\xi^2}\left(1 + \frac{e_n}{\xi}\right)^{-2} + 1$$

which gives,

$$(1 + \alpha)e_{n+1} = \left(\alpha - \frac{2}{\xi^3}\right) e_n + O(e_n{}^2)$$

For fastest convergence, we must have $\alpha = \frac{2}{\xi^3} = \frac{2}{(1.465)^3} = 0.636$.

Example 14. *Use the Newton-Raphson method to find a solution accurate to within 10^{-5} for the problem, $e^{6x} + 1.441\, e^{2x} - 2.079\, e^{4x} - 0.3330 = 0; -1 \le x \le 0$. Use the initial point as $x_0 = -0.5$.* [U.P.T.U. MCA (SUM) 2008]

Sol. $f(x) = e^{6x} + 1.441\, e^{2x} - 2.079\, e^{4x} - 0.3330$

$x_0 = -0.5$ | Given

Also $f'(x) = 6e^{6x} + 2.882e^{2x} - 8.316e^{4x}$

By Newton-Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left[\frac{e^{6x_n} + 1.441e^{2x_n} - 2.079\,e^{4x_n} - 0.3330}{6e^{6x_n} + 2.882\,e^{2x_n} - 8.316\,e^{4x_n}}\right] \quad ...(1)$$

Putting $n = 0$, the first approximation is, $x_1 = -0.35241839$

Putting $n = 1$, the second approximation is, $x_2 = -0.284980263$

Putting $n = 2$, the third approximation is, $x_3 = -0.246813527$

Putting $n = 3$, the fourth approximation is, $x_4 = -0.223232852$

Putting $n = 4$, the fifth approximation is, $x_5 = -0.207509457$

Putting $n = 5$, the sixth approximation is, $x_6 = -0.195400823$

Similarly, putting $n = 6, 7, 8, 9, ...$ in (1), we get at last,

$$x_{12} = -0.169602323 \text{ and } x_{13} = -0.16960655$$

Since x_{12} and x_{13} are same up to five decimal places, hence, the root of the given equation is -0.16960 correct up to five decimal places.

ASSIGNMENT

1. (*i*) By using Newton-Raphsons method, find the root of $x^4 - x - 10 = 0$ which is near to $x = 2$ correct to three places of decimal. (*U.P.T.U. 2014, G.B.T.U. 2012*)

(*ii*) Find the root of $2 \sin x - 2x + 1 = 0$ correct to five significant digits with initial approximation $x_0 = 1.0$. (*M.T.U. 2012*)

2. (*i*) Compute one positive root of $2x - \log_{10} x = 7$ by Newton-Raphson method correct to four decimal places. [*G.B.T.U. (C.O.) 2011*]

(*ii*) Find the real root of the equation $\log_{10} x - x + 3 = 0$ correct to four demical places using Newton-Raphson method. (*G.B.T.U. 2011*)

(*iii*) Use Newton-Raphson method to find the root of $3x - \log_{10} x = 6$ correct to four decimal places. (*G.B.T.U. 2012*)

3. (*i*) Use the Newton-Raphson method to find a root of the equation $x^3 - 2x - 5 = 0$.

(*ii*) Use Newton-Raphson method to find a root of the equation $x^3 - 3x - 5 = 0$.

4. Find the real root of the equations:

(*i*) $\log x = \cos x$ (*ii*) $x^2 + 4 \sin x = 0$ (*iii*) $x^3 - 4x + 1 = 0$

(*M.T.U. 2012*)

by Newton-Raphson method correct to three decimal places.

5. Use Newton-Raphson method to obtain a root, correct to three decimal places of following equations :

(*i*) $\sin x = 1 - x$ (*ii*) $x^3 - 5x + 3 = 0$ (*iii*) $x^4 + x^2 - 80 = 0$

(*iv*) $x^3 + 3x^2 - 3 = 0$ (*v*) $4(x - \sin x) = 1$ (*vi*) $x - \cos x = 0$

(*vii*) $\sin x = \dfrac{x}{2}$ (*viii*) $x \log_{10} x = 4.77$ (*G.B.T.U. 2011*)

6. (*i*) Explain the method of Newton-Raphson for computing roots.

[*U.P.T.U. 2009, U.P.T.U. MCA (SUM) 2009, G.B.T.U. (MCA) 2010*]

(*ii*) Explain the limitations of Newton-Raphson method for finding out the root of an equation.

(*U.P.T.U. 2009*)

(*iii*) Explain the order of convergence and prove that Newton-Raphson method is second order convergent. (*M.T.U. 2012, G.B.T.U. 2010, 2011*)

7. If an approximate root of the equation $x(1 - \log_e x) = 0.5$ lies between 0.1 and 0.2, find the value of the root correct to three decimal places by Newton-Raphson method. (*U.P.T.U. 2015*)

8. Find all the roots of $\cos x - x^2 - x = 0$ to five decimal places by Newton-Raphson method.

(*U.P.T.U. 2006*)

9. Find all positive roots of the equation $10 \int_0^x e^{-x^2} dt - 1 = 0$ with six correct decimals. Use Newton-Raphson method. (*U.P.T.U. 2008*)

10. Use Newton-Raphson method to find the smallest positive root of the equation $\tan x = x$.

$\left[\textbf{Hint: } x \text{ lies in } \left(\pi, \dfrac{3\pi}{2}\right)\right]$ (*G.B.T.U. 2011, U.P.T.U. 2007*)

11. Find the positive root of the equation $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} e^{0.3x}$ correct to 6 decimal places.

12. Show that the equation $f(x) = \cos\left\{\dfrac{\pi(x+1)}{8}\right\} + 0.148x - 0.9062 = 0$ has one root in the interval $(-1, 0)$ and one in $(0, 1)$. Calculate the negative root correct to 4 decimals. (*U.P.T.U. 2009*)

13. (*i*) Use Newton's formula to prove that square root of N can be obtained by the recursion formula,

$$x_{i+1} = x_i\left(1 - \frac{x_i^2 - \text{N}}{2\text{N}}\right)$$

Hence, find the square root of

(*a*) 13 (*U.P.T.U. 2008*) (*b*) 21 [*U.P.T.U. (MCA) 2009*] (*c*) 35.

correct to 4 decimal places.

(*ii*) Find the positive value of $\left(\dfrac{1}{17}\right)^{1/3}$ correct up to 4 decimal places using Newton-Raphson method. (*U.P.T.U. 2009*)

14. Find the iterative method based on Newton-Raphson method to find cube root of N where N is a positive real number. Apply the method for N = 18 and obtain result correct to two decimal places. [*G.B.T.U. MCA (SUM) 2010*]

15. (*i*) The equation $2e^{-x} = \frac{1}{x+2} + \frac{1}{x+1}$ has two roots greater than – 1. Calculate these roots correct to five decimal places. (*M.T.U. 2012, G.B.T.U. 2013*)

(*ii*) Use Newton-Raphson method to find a root of non-linear equation
$f(x) = x^3 + 2x^2 + 10x - 20 = 0$ up to 10 iterations. (*U.P.T.U. 2009*)

16. Determine p, q and r so that the order of the iterative method

$$x_{n+1} = px_n + \frac{qa}{x_n^{\ 2}} + \frac{ra^2}{x_n^{\ 5}}$$

for $a^{1/3}$ becomes as high as possible. [*U.P.T.U.* (*MCA*) *2007*]

[**Hint:** $p + q + r = 1$, $p - 2q - 5r = 0$, $3q + 15r = 0$.]

Answers

1. (*i*) 1.855	(*ii*) 1.4973			
2. (*i*) 3.7892	(*ii*) 3.5502	(*iii*) 2.1079		
3. (*i*) 2.094568	(*ii*) 2.279			
4. (*i*) 1.303	(*ii*) – 1.934	(*iii*) 1.860, 0.2541, – 2.1147		
5. (*i*) 0.511	(*ii*) 0.657	(*iii*) 2.908	(*iv*) – 2.533	
(*v*) 1.171	(*vi*) .739	(*vii*) 1.896	(*viii*) 6.083	

7. 0.186 **8.** 0.55000 ; – 1.25115

9. Roots lie in (0, 1) and (1, 2) ; 0.100336, 1.679631 **10.** 4.4934 **11.** 2.363376

12. – 0.5081. **13.** (*i*) (*a*) 3.6055 (*b*) 4.5825 (*c*) 5.9160 (*ii*) 0.3889

14. 2.62 **15.** (*i*) Roots lie in (– 0.8, 0) and (0, 1) ; – 0.689752, 0.770091 (*ii*) 1.3688

16. $p = \frac{5}{9}$, $q = \frac{5}{9}$, $r = -\frac{1}{9}$; Third order.

4.15 DEFINITIONS

1. A number α is a solution of $f(x) = 0$ if $f(\alpha) = 0$. Such a solution α is a root or a zero of $f(x) = 0$. Geometrically, a root of the equation $f(x) = 0$ is the value of x at which the graph of $y = f(x)$ intersects x-axis.

2. If we can write $f(x) = 0$ as

$$f(x) = (x - \alpha)^m \, g(x) = 0$$

where $g(x)$ is bounded and $g(\alpha) \neq 0$ then α is called a **multiple root** of multiplicity m. In this case,

$$\mathbf{f(\alpha) = f'(\alpha) = \ldots\ldots\ldots = f^{(m-1)}(\alpha) = 0, \; f^{(m)}(\alpha) \neq 0}$$

For $m = 1$, the number α is said to be a simple root.

4.16 METHOD FOR MULTIPLE ROOTS

If α is a multiple root of multiplicity m of the eqn. $f(x) = 0$ then, we have

$$f(\alpha) = f'(\alpha) = \ldots\ldots = f^{(m-1)}(\alpha) = 0 \quad \text{and} \quad f^{(m)}(\alpha) \neq 0$$

It can easily be verified that all the iteration methods discussed so far have only linear rate of convergence when $m > 1$.

For example, in Newton-Raphson method, we have

$$f(x_k) = f(\alpha + e_k) = \frac{e_k^m}{m!} f^{(m)}(\alpha) + \frac{e_k^{m+1}}{(m+1)!} f^{(m+1)}(\alpha) + \frac{e_k^{m+2}}{(m+2)!} f^{(m+2)}(\alpha) + \ldots\ldots.$$

$$f'(x_k) = f'(\alpha + e_k) = \frac{e_k^{m-1}}{(m-1)!} f^{(m)}(\alpha) + \frac{e_k^m}{m!} f^{(m+1)}(\alpha) + \ldots\ldots.$$

The error equation for N–R method becomes,

$$e_{k+1} = \left(1 - \frac{1}{m}\right) e_k + \frac{1}{m^2 (m+1)} \frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)} e_k^2 + O(e_k^3)$$

If $m \neq 1$, we obtain,

$$e_{k+1} = \left(1 - \frac{1}{m}\right) e_k + O(e_k^2) \qquad \ldots(1)$$

which shows that the method has **only linear rate of convergence.**

However, if the multiplicity of the root is known in advance, we can modify the methods by introducing parameters dependent on the multiplicity of the root to increase their order of convergence.

For example, consider Newton-Raphson method in the form

$$x_{k+1} = x_k - \beta \frac{f_k}{f_k'} \qquad \ldots(2)$$

where β is an arbitrary parameter to be determined.

If α is a multiple root of multiplicity m, we obtain from (2), the error equation

$$e_{k+1} = \left(1 - \frac{\beta}{m}\right) e_k + \frac{\beta}{m^2 (m+1)} \frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)} {e_k}^2 + O(e_k^3)$$

If the method (2) is to have the quadratic rate of convergence then the coefficient of e_k must vanish which gives

$$1 - \frac{\beta}{m} = 0 \quad \text{or} \quad \beta = m$$

Thus, the method

$$\boxed{x_{k+1} = x_k - m \frac{f_k}{f_k'}}$$

has quadratic rate of convergence for determining a multiple root of multiplicity m.

If the multiplicity of the root is not known in advance, then we use the following procedure.

It is known that if $f(x) = 0$ has a root α of multiplicity m, then $f'(x) = 0$ has the same root α of multiplicity $m - 1$.

Hence, $g(x) = \dfrac{f(x)}{f'(x)}$ has a simple root α, we can now use Newton-Raphson method

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}$$

to find the approximate value of the multiple root α.

Simplifying, we have

$$x_{k+1} = x_k - \frac{f_k\, f_k'}{f_k'^{2} - f_k\, f_k''}$$

which has quadratic rate of convergence for multiple roots.

Note. If initial approximation x_0 is sufficiently close to the root, then the expressions,

$$x_0 - m\,\frac{f\,(x_0)}{f'\,(x_0)},\; x_0 - (m-1)\,\frac{f'\,(x_0)}{f''\,(x_0)},\; x_0 - (m-2)\,\frac{f''\,(x_0)}{f'''\,(x_0)} \text{ will have same value.}$$

EXAMPLES

Example 1. *Show that the modified Newton-Raphson method* $x_{n+1} = x_n - \dfrac{2f(x_n)}{f'(x_n)}$ *gives a quadratic convergence when f(x) = 0, has a pair of double roots in neighbourhood of* $x = x_n$.

Sol. $e_{n+1} = e_n - \dfrac{2f(a+e_n)}{f'(a+e_n)}$, where a, e_n and e_{n+1} have their usual meanings. Expanding in powers of e_n and using $f(a) = 0, f'(a) = 0$ since, $x = a$ is a double root near $x = x_n$, we get

$$e_{n+1} = e_n - \frac{2\left[\dfrac{e_n^2}{2!} f''(a) + \ldots\ldots\right]}{\left[e_n\, f''(a) + \dfrac{e_n^2}{2!} f'''(a) + \ldots\ldots\right]}$$

$$= e_n - \frac{2\,e_n^2\left[\dfrac{1}{2!} f''(a) + \dfrac{1}{3!} f'''(a) + \ldots\ldots\right]}{e_n\left[f''(a) + \dfrac{e_n}{2!} f'''(a) + \ldots\ldots\right]}$$

$$\simeq e_n - \frac{2e_n\left[\dfrac{1}{2!} f''(a) + \dfrac{1}{3!} f'''(a)\right]}{f''(a) + \dfrac{e_n}{2!} f'''(a)}$$

$$e_{n+1} \simeq \frac{1}{6}\, e_n^2 \cdot \frac{f'''(a)}{\left[f''(a) + \dfrac{e_n}{2!} f'''(a)\right]}$$

$$\therefore \qquad e_{n+1} \approx \frac{1}{6}\, e_n^2\, \frac{f'''(a)}{f''(a)}$$

$$\Rightarrow \qquad e_{n+1} \propto e_n^2$$

and hence convergence is quadratic.

Example 2. *Find the double root of the equation* $x^3 - x^2 - x + 1 = 0$.

Sol. Let $\qquad f(x) = x^3 - x^2 - x + 1$

so that $\qquad f'(x) = 3x^2 - 2x - 1,\; f''(x) = 6x - 2$

Starting with $x_0 = 0.9$, we have

$$x_0 - 2\,\frac{f(x_0)}{f'(x_0)} = .9 - \frac{2 \times .019}{(-.37)} = 1.003$$

and $$x_0 - (2-1)\,\frac{f'(x_0)}{f''(x_0)} = .9 - \frac{(-.37)}{3.4} = 1.009$$

The closeness of these values implies that there is a double root near $x = 1$.

Choosing $x_1 = 1.01$ for next approximation, we get

$$x_1 - 2\,\frac{f(x_1)}{f'(x_1)} = 1.01 - 2 \times \frac{0.0002}{0.0403} = 1.0001$$

and $$x_1 - (2-1)\,\frac{f'(x_1)}{f''(x_1)} = 1.01 - \frac{.0403}{4.06} = 1.0001$$

This shows that there is a double root at $x = 1.0001$ which is quite near the actual root $x = 1$.

4.17 NEARLY EQUAL ROOTS

So far, Newton's method is applicable when $f'(x) \neq 0$ in the neighbourhood of actual root $x = a$, *i.e.,* in the interval $(a - h, a + h)$.

If the quantity h is very very small, it will not satisfy the above restriction. The application of Newton's method will not be practical in that case. This condition arrives when roots are very close to each other.

We know that in case of double root $x = a$, $f(x)$ and $f'(x)$ both vanish at $x = a$. Thus, while applying Newton's method, if x_i is simultaneously near zeros of $f(x)$ and $f'(x)$ *i.e.,* $f(x_i)$ and $f'(x_i)$ both are very small, then it is usually practical to depart from the standard sequence and proceed to obtain two new starting values for the two nearly equal roots.

To obtain these values, we first apply Newton's method to the equation $f'(x) = 0$ *i.e.,* we use the iteration formula

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)} \qquad ...(1)$$

with last available iterate as the initial value x_0 for (1).

Suppose $x = c$ is the solution obtained by (1).

Now by Taylor's series, we have

$$f(x) = f(c) + (x-c)\,f'(c) + \frac{1}{2}\,(x-c)^2 f''(c) + ...$$

$$= f(c) + \frac{1}{2}\,(x-c)^2 f''(c) + \text{R} \qquad \Big|\ \because\ f'(c) = 0$$

Assuming R to be small, we conclude that the zeros of $f(x)$ near $x = c$ are approximately given by

$$f(c) + \frac{1}{2}\,(x-c)^2 f''(c) = 0 \quad \Rightarrow \quad x = c \pm \sqrt{\frac{-2f(c)}{f''(c)}} \qquad ...(2)$$

Using these values as starting values, we can use the original iteration formula to get two close roots of $f(x) = 0$.

Example. *Use synthetic division to solve* $f(x) \equiv x^3 - x^2 - 1.0001\,x + 0.9999 = 0$ *in the neighbourhood of* $x = 1$.

Sol. To find $f(1)$ and $f'(1)$,

1	–1	–1.0001	0.9999	1
	1	0	– 1.0001	
1	0	– 1.0001	– 0.0002 = $f(1)$	
	1	1		
1	1	– .0001 = $f'(1)$		
	1			
1	$2 = \frac{1}{2} f''(1)$			

From the above synthetic division, we observe that $f(1)$ and $f'(1)$ are small. Hence, there exists two nearly equal roots. Taking $x_0 = 1$, we will use $x_{i+1} = x_i - \dfrac{f'(x_i)}{f''(x_i)}$ to modify the root. For this, we require $f''(1)$.

From the above synthetic division, we have

$$\frac{1}{2} f''(1) = 2 \quad \Rightarrow \quad f''(1) = 4$$

$\therefore$ First approximation $x_1 = 1 - \dfrac{f'(1)}{f''(1)} = 1 - \dfrac{(-.0001)}{4} = 1.000025$

Now, we again calculate $f(x_1)$ and $f''(x_1)$ by synthetic division.

1	–1	– 1.000100	0.999900	1.000025
	1.000025	0.000025	– 1.000075	
1	0.000025	– 1.000075	– 0.000175 = $f(x_1)$	
	1.000025	1.000050		
1	1.000050	– 0.000025 = $f'(x_1)$		
	1.000025			
1	$2.000075 = \frac{1}{2} f''(x_1)$			

$\therefore$ $f(1.000025) = -0.000175$ and $f''(1.000025) = 4.000150$

Now, for nearly equal roots,

$$x = c \pm \sqrt{\frac{-2\,f(c)}{f''(c)}}, \quad \text{where } c = 1.000025$$

$$= 1.000025 \pm \sqrt{\frac{-2\,(-.000175)}{4.000150}} = 1.009378,\ 0.990671$$

4.18 COMPARISON OF NEWTON'S METHOD WITH REGULA-FALSI METHOD

Regula-Falsi is surely convergent while Newton's method is conditionally convergent. But once, Newton's method converges, it converges faster.

In Falsi method, we calculate only one more value of function at each step *i.e.,* $f(x^{(n)})$ while in Newton's method, we require two calculations $f(x_n)$ and $f'(x_n)$ at each step.

$\therefore$ Newton's method generally requires less number of iterations but requires more time for computation at each iteration.

When $f'(x)$ is large near the root, correction to be applied is smaller in case of Newton's method and then this method is preferred while if $f'(x)$ is small near the root, correction to be applied is large and curve becomes parallel to x-axis. In this case Regula-Falsi method should be applied.

Newton's method has the fastest rate of convergence. This method is quite sensitive to starting value. It may diverge if $f'(x) \approx 0$ during iterative cycle.

ASSIGNMENT

1. The equation $f(x) = (x-1)^2 (x-3)^2$ has roots at $x = 1$ and $x = 3$. Which of the following methods can be applied to find all the roots?

(*i*) Bisection method (*ii*) False position method (*iii*) Newton-Raphson method

Justify your answer. (*G.B.T.U. 2010*)

2. A sphere of wood, 2 m in diameter, floating in water sinks to a depth d given by

$$d^3 - 3d^2 + 2.5 = 0$$

Find d correct to two decimal places.

Answers

1. (*iii*) Newton-Raphson method since it deals with multiple roots as well.

2. 1.16

4.19 INTERPOLATION [*G.B.T.U.* (*MCA*) *2010, G.B.T.U. MCA* (*SUM*) *2010*]

According to Theile, '*Interpolation is the art of reading between the lines of the table*'.

It also means insertion or filling up intermediate terms of the series.

Suppose, we are given the following values of $y = f(x)$ for a set of values of x:

x :	x_0	x_1	x_2	......	x_n
y :	y_0	y_1	y_2	......	y_n

Thus, the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called **interpolation.**

Hence, interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called **extrapolation.**

4.20 ASSUMPTIONS FOR INTERPOLATION

1. There are no sudden jumps or falls in the values during the period under consideration.

2. The rise and fall in the values should be uniform.

e.g., if we are given data regarding deaths in various years in a particular town and some of the observations are for the years in which epidemic or war overtook the town then interpolation methods are not applicable in such cases.

3. When we apply calculus of finite differences, we assume that the given set of observations are capable of being expressed in a polynomial form.

If the function $f(x)$ is known explicitly, the value of y corresponding to any value of x can easily be found.

If the function $f(x)$ is not known, it is required to find a simpler function, say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. Such a process is called interpolation. If $\phi(x)$ is a polynomial, then the process is called polynomial interpolation and $\phi(x)$ is called the interpolating polynomial.

4.21 ERRORS IN POLYNOMIAL INTERPOLATION

Let the function $y(x)$ defined by $(n + 1)$ points (x_i, y_i) $i = 0, 1, 2, ..., n$ be continuous and differentiable $(n + 1)$ times and let $y(x)$ be approximated by a polynomial $\phi_n(x)$ of degree not exceeding n such that

$$\phi_n(x_i) = y_i \; ; i = 0, 1, 2, ..., n \qquad ...(1)$$

Now problem lies in finding the accuracy of this approximation if we use $\phi_n(x)$ to obtain approximate values of $y(x)$ at some points other than those defined above.

Since the expression $y(x) - \phi_n(x)$ vanishes for $x = x_0, x_1, ..., x_n$, we put

$$y(x) - \phi_n(x) = \text{L}\, \Pi_{n+1}(x) \qquad ...(2)$$

where $$\Pi_{n+1}(x) = (x - x_0)(x - x_1) ...... (x - x_n) \qquad ...(3)$$

and L is to be determined such that equation (2) holds for any intermediate value of x say x' where $x_0 < x' < x_n$.

Clearly, $$\text{L} = \frac{y(x') - \phi_n(x')}{\Pi_{n+1}(x')} \qquad ...(4)$$

Construct a function, $\text{F}(x) = y(x) - \phi_n(x) - \text{L}\, \Pi_{n+1}(x)$ ...(5)

where L is given by (4).

It is clear that, $\text{F}(x_0) = \text{F}(x_1) = ... = \text{F}(x_n) = \text{F}(x') = 0$

i.e., $\text{F}(x)$ vanishes $(n + 2)$ times in interval $[x_0, x_n]$ consequently, by repeated application of Rolle's theorem, $\text{F}'(x)$ must vanish $(n + 1)$ times, $\text{F}''(x)$ must vanish n times in the interval $[x_0, x_n]$.

Particularly, $\text{F}^{(n+1)}(x)$ must vanish once in $[x_0, x_n]$.

Let this point be $x = \xi \; ; x_0 < \xi < x_n$.

Differentiating (5) $(n + 1)$ times w.r.t. x and put $x = \xi$, we get

$$0 = (y)^{(n+1)}(\xi) - \text{L}\,(n+1)! \qquad \left|\; \frac{d^{n+1}}{dx^{n+1}}(x^{n+1}) = (n+1)! \right.$$

so that, $$\text{L} = \frac{y^{(n+1)}(\xi)}{(n+1)!} \qquad ...(6)$$

Comparison of (4) and (6) gives

$$y(x') - \phi_n(x') = \frac{y^{(n+1)}(\xi)}{(n+1)!}\ \Pi_{n+1}(x')$$

Hence, required expression for error is

$$y(x) - \phi_n(x) = \frac{\Pi_{n+1}(x)}{(n+1)!}\ y^{n+1}(\xi),\ x_0 < \xi < x_n \qquad ...(7)$$

Since $y(x)$ is generally unknown, hence we do not have any information concerning $y^{(n+1)}(x)$, equation (7) is useless in practical computations.

Particularly, we will use it to determine errors in Newton's interpolating formulae.

The various methods of interpolation are as follows:

(1) The method of graph (2) The method of curve fitting

(3) Use of calculus of finite difference formulae.

The merits of last method over others are

(*i*) It does not assume the form of function to be known.

(*ii*) It is less approximate than method of graphs.

(*iii*) Calculations remain simple even if some additional observations are included in the given data.

The demerit is that there is no definite way to verify whether the assumptions for the application of finite difference calculus are valid for the given set of observations.

4.22 FINITE DIFFERENCES

The calculus of finite differences deals with the changes that take place in the value of the function (dependent variable) due to finite changes in the independent variable.

Suppose, we are given a set of values (x_i, y_i) ; $i = 1, 2, 3, ..., n$ of any function $y = f(x)$. A value of independent variable x is called **argument** and the corresponding value of dependent variable y is called **entry.**

Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0$, $x_0 + h, x_0 + 2h, ..., x_0 + nh$ giving $y = y_0, y_1, y_2, ..., y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x, given three types of differences are useful:

1. Forward differences. The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, ..., y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \Delta y_2, ..., \Delta y_{n-1}$ respectively, are called the first forward differences where Δ is the forward difference operator. (*U.P.T.U. 2009*)

Thus, the first forward differences are

$$\Delta y_r = y_{r+1} - y_r$$

Similarly, the second forward differences are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

Particularly, $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$

Similarly, $\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$

$$\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.$$

Clearly, any higher order difference can easily be expressed in terms of ordinates since the coefficients occurring on R.H.S. are the binomial coefficients*. In general, $\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$ defines the p^{th} forward differences.

Following table shows how the forward differences of all orders can be formed.

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
$(= x_0 + h)$		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
$(= x_0 + 2h)$		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
$= (x_0 + 3h)$		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
$= (x_0 + 4h)$		Δy_4				
x_5	y_5					
$= (x_0 + 5h)$						

Here, the first entry y_0 is called leading term and Δy_0, $\Delta^2 y_0$, ... are called leading differences.

Remark. Δ obeys distributive, commutative and index laws :

1. $\Delta\,[f(x) \pm \phi(x)] = \Delta f(x) \pm \Delta\phi\,(x)$ 2. $\Delta\,[c\,f(x)] = c\,\Delta\,f(x)$; c is constant

3. $\Delta^m\,\Delta^n\,f(x) = \Delta^{m+n}\,f(x)$, m, n being (+)ve integers.

But, $\Delta[f(x)\,.\,\phi(x)] \neq f(x)\,.\,\Delta\,\phi(x)$.

2. **Backward differences.** The differences $y_1 - y_0, y_2 - y_1, ..., y_n - y_{n-1}$ when denoted by ∇y_1, ∇y_2, ..., ∇y_n respectively are called first backward differences where ∇ is the backward difference operator. (*U.P.T.U. 2009*)

Similarly, we define higher order backward differences as,

$$\nabla y_r = y_r - y_{r-1}$$
$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$
$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}$$

Particularly,
$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0 \text{ etc.}$$

* $\Delta^n(y_0) = y_n - {}^nC_1\,y_{n-1} + {}^nC_2 y_{n-2} + ... + (-1)^n\,y_0$

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
$(= x_0 + h)$		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
$(= x_0 + 2h)$		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
$(= x_0 + 3h)$		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
$(= x_0 + 4h)$		∇y_5				
x_5	y_5					
$(= x_0 + 5h)$						

3. **Central differences.** The central difference operator δ is defined by the relations

$$y_1 - y_0 = \delta y_{1/2}, y_2 - y_1 = \delta y_{3/2}, \ldots\ldots, y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}.$$

[*G.B.T.U. 2012, U.P.T.U. 2009*]

Similarly, high order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1, \quad \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2 \text{ and so on.}$$

These differences are shown as follows:

Central difference table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

Note **1.** The central differences on the same horizontal line have same suffix.

2. It is only the notation which changes, not the differences *e.g.*,

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}.$$

4.23 OTHER DIFFERENCE OPERATORS

1. Shift operator E. [*G.B.T.U. 2012, U.P.T.U. 2009*]

It is the operation of increasing the argument x by h so that

$$Ef(x) = f(x + h)$$

$$E^2 f(x) = f(x + 2h) \text{ and so on.}$$

The inverse operator E^{-1} is defined by

$$E^{-1} f(x) = f(x - h).$$

Also $$E^n y_x = y_{x+nh}.$$

2. Averaging operator μ. [*G.B.T.U. 2012, U.P.T.U. 2009*]

It is defined by

$$\mu y_x = \frac{1}{2}\left[y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h}\right]$$

In difference calculus, E is fundamental operator and ∇, Δ, δ, μ can be expressed in terms of E.

4.24 RELATION BETWEEN OPERATORS

1. $$\boxed{\Delta = E - 1 \quad \text{or} \quad E = 1 + \Delta.}$$

[*M.T.U. (MCA) 2012, U.P.T.U. (MCA) 2009*]

Proof. We know that,

$$\Delta y_x = y_{x+h} - y_x = E y_x - y_x = (E - 1) y_x$$

$\Rightarrow$ $$\Delta = E - 1$$

or $$E = 1 + \Delta$$

2. $$\boxed{\nabla = 1 - E^{-1}}$$

Proof. $$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x$$

$\therefore$ $$\nabla = 1 - E^{-1}$$

3. $$\boxed{\delta = E^{1/2} - E^{-1/2}}$$ (*U.P.T.U. 2014*)

Proof. $$\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}} = E^{1/2} y_x - E^{-1/2} y_x = (E^{1/2} - E^{-1/2})\, y_x$$

$\therefore$ $$\delta = E^{1/2} - E^{-1/2}$$

4. $$\boxed{\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})}$$

Proof. $$\mu y_x = \frac{1}{2}\left(y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right) = \frac{1}{2}(E^{1/2} + E^{-1/2})\, y_x$$

$\Rightarrow$ $$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

5. $$\boxed{\Delta = E\nabla = \nabla E = \delta E^{1/2}}$$

[*G.B.T.U. 2011, U.P.T.U. (MCA) 2009, M.T.U. 2012*]

Proof. $\quad \mathrm{E}(\nabla y_x) = \mathrm{E}(y_x - y_{x-h}) = y_{x+h} - y_x = \Delta y_x$

$\Rightarrow \quad \mathrm{E}\nabla = \Delta$

$\nabla(\mathrm{E}\, y_x) = \nabla\, y_{x+h} = y_{x+h} - y_x = \Delta y_x$

$\Rightarrow \quad \nabla \mathrm{E} = \Delta$

$\delta \mathrm{E}^{1/2}\, y_x = \delta y_{x+\frac{h}{2}} = y_{x+h} - y_x = \Delta y_x$

$\Rightarrow \quad \delta \mathrm{E}^{1/2} = \Delta$

6. $\boxed{\mathrm{E} = e^{h\mathrm{D}}}$ [*U.P.T.U. (MCA) 2009*]

Proof. $\quad \mathrm{E}f(x) = f(x+h)$

$$= f(x) + h\, f'(x) + \frac{h^2}{2!}\, f''(x) + \ldots\ldots \qquad \text{(By Taylor series)}$$

$$= f(x) + h\mathrm{D}f(x) + \frac{h^2}{2!}\, \mathrm{D}^2 f(x) + \ldots\ldots$$

$$= \left[1 + h\mathrm{D} + \frac{(h\,\mathrm{D})^2}{2!} + \ldots\ldots\right] f(x) = e^{h\mathrm{D}}\, f(x)$$

$\therefore \quad \mathrm{E} = e^{h\mathrm{D}} \quad \text{or} \quad \Delta = e^{h\mathrm{D}} - 1.$

4.25 DIFFERENCES OF A POLYNOMIAL

The n^{th} differences of a polynomial of n^{th} degree are constant and all higher order differences are zero when the values of the independent variable are at equal interval.

Let $\quad f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots + kx + l$

$\therefore \quad \Delta f(x) = f(x+h) - f(x)$

$$= a[(x+h)^n - x^n] + b\,[(x+h)^{n-1} - x^{n-1}] + \ldots\ldots + kh$$

$$= anhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \ldots\ldots + k'x + l' \qquad \ldots(1)$$

where b', c', ..., l' are new constant coefficients.

$\therefore$ First differences of a polynomial of n^{th} degree is a polynomial of degree $(n-1)$.

Similarly, $\quad \Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$

$$= anh\,[(x+h)^{n-1} - x^{n-1}] + b'[(x+h)^{n-2} - x^{n-2}] + \ldots\ldots + k'h$$

$$= an(n-1)\, h^2 x^{n-2} + b''x^{n-3} + \ldots\ldots + k'' \qquad \ldots(2)$$

$\therefore$ Second differences represent a polynomial of degree $(n-2)$.

Continuing this process, for n^{th} differences, we get a polynomial of degree zero *i.e.*,

$$\Delta^n f(x) = an(n-1)(n-2) \ldots\ldots 1\, h^n = a\, n!\, h^n$$

which is a constant. Hence, the $(n+1)^{\text{th}}$ and higher differences of a polynomial of n^{th} degree will be zero. Converse of this theorem is also true.

EXAMPLES

Example 1. *Construct the forward difference table, given that*

x:	*5*	*10*	*15*	*20*	*25*	*30*
y:	*9962*	*9848*	*9659*	*9397*	*9063*	*8660*

and point out the values of $\Delta^2 y_{10}$, $\Delta^4 y_5$.

Sol. Forward difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	9962				
		– 114			
10	9848		– 75		
		– 189		2	
15	9659		$\boxed{-73}$		$\boxed{-1}$
		– 262		1	
20	9397		– 72		2
		– 334		3	
25	9063		– 69		
		– 403			
30	8660				

From the table, $\Delta^2 y_{10} = -73$ and $\Delta^4 y_5 = -1$.

Example 2. *Construct a backward difference table for y = log x given that*

x:	*10*	*20*	*30*	*40*	*50*
y:	*1*	*1.3010*	*1.4771*	*1.6021*	*1.6990*

and find values of ∇^3 *log 40 and* ∇^4 *log 50.*

Sol. Backward difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	1				
		0.3010			
20	1.3010		– 0.1249		
		0.1761		$\boxed{0.0738}$	
30	1.4771		– 0.0511		$\boxed{-0.0508}$
		0.1250		0.0230	
40	1.6021		– 0.0281		
		0.0969			
50	1.6990				

From the table, $\nabla^3 \log 40 = 0.0738$ and $\nabla^4 \log 50 = -0.0508$.

Example 3. (*i*) *Find f(6) given f(0) = – 3, f(1) = 6, f(2) = 8, f(3) = 12 ; third difference being constant.*

(*ii*) *Find* $\Delta^{10}(1 - ax)(1 - bx^2)(1 - cx^3)(1 - dx^4)$.

Sol. (*i*) The difference table is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	– 3			
		9		
1	6		– 7	
		2		9
2	8		2	
		4		
3	12			

$$f(0 + 6) = E^6 f(0) = (1 + \Delta)^6 f(0) = (1 + 6\Delta + 15\Delta^2 + 20\Delta^3)\, f(0)$$
$$= -3 + 6\,(9) + 15\,(-7) + 20\,(9) = 126.$$

(*ii*) Maximum power of x in polynomial will be 10 and co-efficient of x^{10} will be $abcd$.

Here, $k = abcd,\ h = 1,\ n = 10$

$\therefore$ Expression $= k\, h^n\, n\,! = abcd\ 10\,!$

Example 4. *Evaluate:*

(*i*) $\Delta \tan^{-1} x$ (*ii*) $\Delta^2 \cos 2x$

where h is the interval of differencing.

Sol. (*i*) $\Delta \tan^{-1} x = \tan^{-1}(x + h) - \tan^{-1} x$

$$= \tan^{-1}\left\{\frac{x + h - x}{1 + x\,(x + h)}\right\} = \tan^{-1}\left(\frac{h}{1 + hx + x^2}\right)$$

(*ii*)
$$\Delta^2 \cos 2x = \Delta[\cos 2(x + h) - \cos 2x]$$
$$= [\cos 2(x + 2h) - \cos 2(x + h)] - [\cos 2(x + h) - \cos 2x]$$
$$= -2 \sin(2x + 3h) \sin h + 2 \sin(2x + h) \sin h$$
$$= -2 \sin h\, [2 \cos(2x + 2h) \sin h] = -4 \sin^2 h \cos 2(x + h).$$

Example 5. *If $f(x) = \exp(ax)$, evaluate $\Delta^n f(x)$.*

Sol.
$$\Delta e^{ax} = e^{a(x + h)} - e^{ax} = (e^{ah} - 1)e^{ax}$$
$$\Delta^2 e^{ax} = \Delta(\Delta e^{ax}) = \Delta[(e^{ah} - 1)e^{ax}] = (e^{ah} - 1)(e^{ah} - 1)e^{ax} = (e^{ah} - 1)^2\, e^{ax}$$

Similarly,
$$\Delta^3 e^{ax} = (e^{ah} - 1)^3\, e^{ax}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\Delta^n e^{ax} = (e^{ah} - 1)^n\, e^{ax}.$$

Example 6. *With usual notations, prove that,* $\Delta^n\left(\dfrac{1}{x}\right) = (-1)^n \cdot \dfrac{n\,!\,h^n}{x\,(x + h) \ldots\ldots (x + nh)}$.

Sol.
$$\Delta^n\left(\frac{1}{x}\right) = \Delta^{n-1}\,\Delta\left(\frac{1}{x}\right) = \Delta^{n-1}\left[\frac{1}{x + h} - \frac{1}{x}\right] = (-1)\,\Delta^{n-2}\left[\Delta\left(\frac{1}{x} - \frac{1}{x + h}\right)\right]$$
$$= (-1)\,\Delta^{n-2}\left[\left(\frac{1}{x + h} - \frac{1}{x}\right) - \left(\frac{1}{x + 2h} - \frac{1}{x + h}\right)\right]$$

$$= (-1)\,\Delta^{n-2}\left[\frac{2}{x+h}-\frac{1}{x}-\frac{1}{x+2h}\right] = (-1)\,\Delta^{n-2}\left[\frac{-2h^2}{x(x+h)(x+2h)}\right]$$

$$= (-1)^2\,\Delta^{n-2}\left[\frac{2!\,h^2}{x(x+h)(x+2h)}\right] = (-1)^3\,\Delta^{n-3}\left[\frac{3!\,h^3}{x(x+h)(x+2h)(x+3h)}\right]$$

$$\vdots$$

$$= (-1)^n\,\frac{n!\,h^n}{x(x+h)\,\ldots\ldots\,(x+nh)}.$$

Example 7. *Evaluate:* (*i*) $\Delta^n\,[\sin\,(ax+b)]$ (*ii*) $\Delta^n\,[\cos\,(ax+b)]$.

Sol. (*i*) $\Delta \sin\,(ax+b) = \sin\,[a\,(x+h)+b] - \sin\,(ax+b)$

$$= 2\sin\frac{ah}{2}\cos\left[a\left(x+\frac{h}{2}\right)+b\right] = 2\sin\frac{ah}{2}\sin\left(ax+b+\frac{ah+\pi}{2}\right)$$

$$\therefore \quad \Delta^2 \sin\,(ax+b) = \Delta\left[2\sin\frac{ah}{2}\sin\left(ax+b+\frac{ah+\pi}{2}\right)\right]$$

$$= \left(2\sin\frac{ah}{2}\right)\left(2\sin\frac{ah}{2}\right)\sin\left[ax+b+\frac{ah+\pi}{2}+\frac{ah+\pi}{2}\right]$$

$$= \left(2\sin\frac{ah}{2}\right)^2\sin\left[ax+b+2\left(\frac{ah+\pi}{2}\right)\right]$$

Proceeding in the same manner, we get

$$\Delta^3 \sin\,(ax+b) = \left(2\sin\frac{ah}{2}\right)^3\sin\left[ax+b+\frac{3(ah+\pi)}{2}\right]$$

$$\vdots \qquad\qquad \vdots$$

$$\Delta^n \sin\,(ax+b) = \left(2\sin\frac{ah}{2}\right)^n\sin\left[ax+b+\frac{n(ah+\pi)}{2}\right]$$

Similarly,

(*ii*) $$\Delta^n \cos\,(ax+b) = \left(2\sin\frac{ah}{2}\right)^n\cos\left[ax+b+n\left(\frac{ah+\pi}{2}\right)\right].$$

Example 8. *Prove that*

(*i*) $\Delta \log f(x) = \log\left[1+\dfrac{\Delta f(x)}{f(x)}\right]$ (*ii*) $\mu = \sqrt{\left(1+\dfrac{\delta^2}{4}\right)}$.

[G.B.T.U. 2013, U.P.T.U. 2009]

Sol. (*i*) LHS $= \log f(x+h) - \log f(x)$

$$= \log\,[f(x)+\Delta f(x)] - \log f(x) \qquad \Big|\ \because\ \Delta f(x) = f(x+h)-f(x)$$

$$= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right] = \log\left[1+\frac{\Delta f(x)}{f(x)}\right] = \text{RHS}$$

(*ii*) $$\text{RHS} = \sqrt{1+\frac{\delta^2}{4}} = \sqrt{1+\frac{1}{4}(E^{1/2}-E^{-1/2})^2} \qquad | \because \quad \delta = \text{E}^{1/2} - \text{E}^{-1/2}$$

$$= \frac{1}{2}\sqrt{4+(\text{E}^{1/2}-\text{E}^{-1/2})^2}$$

$$= \frac{1}{2}\sqrt{(\text{E}^{1/2}+\text{E}^{-1/2})^2} = \frac{\text{E}^{1/2}+\text{E}^{-1/2}}{2} = \mu = \text{LHS}$$

Example 9. *Prove that* $e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$. (M.T.U. 2013)

Sol. $$\left(\frac{\Delta^2}{\text{E}}\right) e^x = \Delta^2\, \text{E}^{-1} e^x = \Delta^2 e^{x-h} = e^{-h}\, \Delta^2 e^x$$

$$\text{RHS} = e^{-h} \cdot \Delta^2 e^x \cdot \frac{\text{E}\, e^x}{\Delta^2\, e^x} = e^{-h} \cdot \text{E}\, e^x = e^{-h}\, e^{x+h} = e^x.$$

Example 10. *Prove that,*

(*i*) $(E^{1/2} + E^{-1/2})\,(1 + \Delta)^{1/2} = 2 + \Delta$ (U.P.T.U. 2009)

(*ii*) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+(\delta^2/4)}$ [G.B.T.U. 2013, U.P.T.U. 2009, G.B.T.U. (C.O.) 2011]

(*iii*) $\Delta^3 y_2 = \nabla^3 y_5$.

(*iv*) $hD = -\log(1-\nabla) = \sin h^{-1}(\mu\delta)$. (M.T.U. 2013, U.P.T.U. 2006)

Sol. (*i*) $(\text{E}^{1/2} + \text{E}^{-1/2})\, \text{E}^{1/2} = \text{E} + 1 = 1 + \Delta + 1 = \Delta + 2$

(*ii*) $$\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} = \frac{1}{2}(\text{E}^{1/2}-\text{E}^{-1/2})^2 + (\text{E}^{1/2}-\text{E}^{-1/2})\sqrt{1+\frac{1}{4}(\text{E}^{1/2}-\text{E}^{-1/2})^2}$$

$$= \frac{1}{2}(\text{E}+\text{E}^{-1}-2) + (\text{E}^{1/2}-\text{E}^{-1/2})\left(\frac{\text{E}^{1/2}+\text{E}^{-1/2}}{2}\right)$$

$$= \frac{1}{2}(2\text{E}-2) = \text{E}-1 = \Delta$$

(*iii*) $$\Delta^3 y_2 = (\text{E}-1)^3\, y_2$$
$$= (\text{E}^3 - 3\text{E}^2 + 3\text{E} - 1)\, y_2 = y_5 - 3y_4 + 3y_3 - y_2$$
$$\nabla^3 y_5 = (1-\text{E}^{-1})\, y_5$$
$$= (1 - 3\text{E}^{-1} + 3\text{E}^{-2} - \text{E}^{-3})\, y_5 = y_5 - 3y_4 + 3y_3 - y_2$$

(*iv*) $$h\text{D} = \log \text{E} = -\log(\text{E}^{-1}) = -\log(1-\nabla) \qquad | \because \quad \text{E}^{-1} = 1-\nabla$$

Also, $$\mu = \frac{1}{2}(\text{E}^{1/2}+\text{E}^{-1/2}) \quad \text{and} \quad \delta = \text{E}^{1/2}-\text{E}^{-1/2}$$

$\therefore$ $$\mu\delta = \frac{1}{2}(\text{E}-\text{E}^{-1}) = \frac{1}{2}(e^{h\text{D}} - e^{-h\text{D}}) = \sinh(h\text{D})$$

or $$h\text{D} = \sinh^{-1}(\mu\delta).$$

Example 11. *Prove that*

(*i*) $\Delta + \nabla = \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$, *where* Δ *and* ∇ *are forward difference and backward difference operators respectively.* [G.B.T.U. 2012, G.B.T.U. (MCA) 2010, 2011; U.P.T.U. 2008, 2009]

(*ii*) $\displaystyle\sum_{r=0}^{n-1} \Delta^2 y_r = \Delta y_n - \Delta y_0$ (*iii*) $\Delta^r y_k = \nabla^r y_{k+r}$.

Sol. (*i*) $$\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1} = \frac{E-1}{\left(\dfrac{E-1}{E}\right)} - \frac{\left(\dfrac{E-1}{E}\right)}{E-1} = E - \frac{1}{E} = E - E^{-1}$$

$$= (E-1) + (1-E^{-1}) = \Delta + \nabla$$

(*ii*) $$\sum_{r=0}^{n-1} \Delta^2 y_r = \sum_{r=0}^{n-1} (\Delta y_{r+1} - \Delta y_r) = \Delta y_1 - \Delta y_0 + \Delta y_2 - \Delta y_1 + \ldots\ldots + \Delta y_n - \Delta y_{n-1}$$

$$= \Delta y_n - \Delta y_0.$$

(*iii*) $$\nabla^r y_{k+r} = (1 - E^{-1})^r y_{k+r} = \left(\frac{E-1}{E}\right)^r y_{k+r} = (E-1)^r E^{-r} y_{k+r} = \Delta^r y_k.$$

Example 12. *Prove that*

(*i*) $\mu\delta = \dfrac{1}{2}(\Delta + \nabla) = \dfrac{\Delta E^{-1}}{2} + \dfrac{\Delta}{2}$ (*ii*) $1 + \left(\dfrac{\delta^2}{2}\right) = \sqrt{1 + \delta^2\mu^2}$

[G.B.T.U. 2011] [G.B.T.U. 2013, U.P.T.U. 2014]

(*iii*) $\nabla^2 = h^2D^2 - h^3D^3 + \dfrac{7}{12} h^4D^4 \ldots\ldots$ (*iv*) $\nabla - \Delta = -\nabla\Delta$

[G.B.T.U. (C.O.) 2010; G.B.T.U. (MCA) 2010, 2011]

Sol. (*i*) $$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$$

$$= \frac{1}{2}(E - E^{-1}) = \frac{1}{2}[(E-1) + (1-E^{-1})] \qquad \ldots(1)$$

$$\Rightarrow \qquad \mu\delta = \frac{1}{2}(\Delta + \nabla)$$

Also from (1),

$$\mu\delta = \frac{1}{2}[\Delta + (E-1)E^{-1}] = \frac{1}{2}(\Delta + \Delta E^{-1}) = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}$$

Hence, the results.

(*ii*) $$\text{LHS} = 1+\left(\frac{\delta^2}{2}\right) = 1+\frac{(E^{1/2}-E^{-1/2})^2}{2} = 1+\left(\frac{E+E^{-1}-2}{2}\right)$$

$$= \frac{1}{2}\,(E + E^{-1})$$

$$\text{RHS} = \sqrt{1+\delta^2\mu^2}$$

$$= \left[1+\left\{(E^{1/2}-E^{-1/2})^2 \cdot \frac{1}{4}(E^{1/2}+E^{-1/2})^2\right\}\right]^{1/2}$$

$$= \left\{1+\left(\frac{(E-E^{-1})^2}{4}\right)\right\}^{1/2} = \left(\frac{E^2+E^{-2}+2}{4}\right)^{1/2} = \left(\frac{E+E^{-1}}{2}\right)$$

Hence, LHS = RHS.

(*iii*) $E = e^{hD}$ and $\nabla = 1 - E^{-1}$

$\therefore$ $\nabla^2 = (1 - e^{-hD})^2$

$$= \left[1-\left\{1-hD+\frac{(hD)^2}{2!}-\frac{(hD)^3}{3!}+\frac{(hD)^4}{4!}-\ldots\right\}\right]^2$$

$$= \left\{hD-\frac{(hD)^2}{2!}+\frac{(hD)^3}{3!}-\frac{(hD)^4}{4!}+\ldots\right\}^2$$

$$= h^2D^2\left[1-\left\{\frac{hD}{2}-\frac{(hD)^2}{6}+\ldots\right\}\right]^2$$

$$= h^2D^2\left[1+\left\{\frac{hD}{2}-\frac{(hD)^2}{6}+\ldots\right\}^2-2\left\{\frac{hD}{2}-\frac{(hD)^2}{6}+\ldots\right\}\right]$$

$$= h^2D^2\left[1-hD+\left(\frac{1}{4}+\frac{1}{3}\right)(hD)^2-\ldots\right]$$

$$= h^2D^2\left(1-hD+\frac{7}{12}h^2D^2-\ldots\right) = h^2D^2 - h^3D^3 + \frac{7}{12}\,h^4D^4 - \ldots$$

(*iv*) $$\nabla - \Delta = (1 - E^{-1}) - (E - 1) = \left(\frac{E-1}{E}\right) - (E - 1) = (E - 1)(E^{-1} - 1)$$

$$= -(E - 1)\,(1 - E^{-1}) = -\nabla\Delta$$

Example 13. *Prove that if m is a (+)ve integer then* $\dfrac{(x+1)^{(m)}}{m!} = \dfrac{x^{(m)}}{m!} + \dfrac{x^{(m-1)}}{(m-1)!}$

Sol. RHS $= \dfrac{x(x-1)\ldots\ldots(x-m+1)}{m!}+\dfrac{x(x-1)\ldots\ldots(x-m+2)}{(m-1)!}$

$$= \frac{x(x-1)(x-2)\ldots\ldots(x-m+2)}{m!}\,[(x-m+1)+m]$$

$$= \frac{(x+1)\,x(x-1)(x-2)\ldots\ldots(x-m+2)}{m!} = \frac{(x+1)^{(m)}}{m!} = \text{LHS.}$$

Example 14. *Given* $u_0 + u_8 = 1.9243$, $u_1 + u_7 = 1.9590$
$u_2 + u_6 = 1.9823$, $u_3 + u_5 = 1.9956$. *Find* u_4.

Sol. Taking $\Delta^8 u_0 = 0$

$\Rightarrow$ $(E-1)^8 u_0 = 0$

$\Rightarrow$ $u_8 - 8c_1u_7 + 8c_2u_6 - 8c_3u_5 + 8c_4u_4 - 8c_5u_3 + 8c_6u_2 - 8c_7u_1 + 8c_8u_0 = 0$

$\Rightarrow$ $(u_0 + u_8) - 8(u_1 + u_7) + 28(u_2 + u_6) - 56(u_3 + u_5) + 70\,u_4 = 0$

$\Rightarrow$ $u_4 = 0.99996.$ (After putting the values)

ASSIGNMENT

1. (*i*) Form a table of differences for the function:
$f(x) = x^3 + 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$. Continue the table to obtain $f(6)$ and $f(7)$.

(*ii*) If $y = x^3 + x^2 - 2x + 1$, calculate values of y for $x = 0, 1, 2, 3, 4, 5$ and form the difference table. Find the value of y at $x = 6$ by extending the table and verify that the same value is obtained by substitution.

2. Prove that: D $\sin^{-1} x = \sin^{-1}\left[(x+1)\sqrt{1-x^2} - x\sqrt{1-(x+1)^2}\right]$.

3. (*i*) Write forward difference table for

x:	10	20	30	40
y:	1.1	2.0	4.4	7.9.

(*ii*) Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values:

x :	0	1	2	3	4	5	6	7
y :	1	– 1	1	– 1	1	—	—	—

4. Construct the table of differences for the data below.

x :	0	1	2	3	4
$f(x)$:	1.0	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 f(2)$.

5. Prove that:

(*i*) $\nabla = \Delta E^{-1} = E^{-1}\Delta = 1 - E^{-1}$ (*ii*) $E^{1/2} = \mu + \frac{1}{2}\delta$

(*iii*) $\delta = \Delta E^{-1/2} = \nabla E^{1/2}$ (*iv*) $E = (1-\nabla)^{-1}$

(*v*) $\Delta\nabla = \nabla\Delta = \delta^2$ (*G.B.T.U. 2011*) (*vi*) $\delta = \Delta(1+\Delta)^{-1/2} = \nabla(1-\nabla)^{-1/2}$ (*U.P.T.U. 2009*)

(*vii*) $\delta^2 E = \Delta^2$ (*G.B.T.U. 2012*) (*viii*) $\mu\delta = \frac{1}{2}(E - E^{-1})$ (*G.B.T.U. 2012*)

(*ix*) $\Delta \equiv \nabla(1-\nabla)^{-1}$ [*M.T.U. (MCA) 2012*]

6. u_x is a function of x for which fifth differences are constant and
$u_1 + u_7 = -786$, $u_2 + u_6 = 686$, $u_3 + u_5 = 1088$. Find u_4.

7. Prove that:

(*i*) $u_4 = u_3 + \Delta u_2 + \Delta^2 u_1 + \Delta^3 u_1$ (*ii*) $u_4 = u_0 + 4\Delta u_0 + 6\Delta^2 u_{-1} + 10\Delta^3 u_{-1}$.

8. Evaluate:

(*i*) $\Delta(e^{ax} \log bx)$ (*ii*) $\Delta\left(\dfrac{2^x}{(x+1)!}\right)$; $h = 1$.

Answers

1. (*i*) 239, 371 (*ii*) 1, 1, 9, 31, 73, 141; 241

3. (*ii*) 31, 129, 351 **4.** 0.4

6. 570.9.

8. (*i*) $e^{ax}\left[e^{ah}\log\left(1+\dfrac{h}{x}\right)+(e^{ah}-1)\log bx\right]$ (*ii*) $-\dfrac{x}{(x+2)!}\ 2^x$.

4.26 FACTORIAL NOTATION

A product of the form $x(x-1)(x-2)$ ...... $(x-r+1)$ is denoted by $[x]^r$ and is called a factorial.

Particularly, $[x] = x$; $[x]^2 = x(x-1)$; $[x]^3 = x\,(x-1)(x-2)$ etc.

In case the interval of differencing is h then

$$[x]^n = x(x-h)\,(x-2h)\ \ldots\ldots\ (x-\overline{n-1}\,h)$$

Factorial notation helps in finding the successive differences of a polynomial directly by simple rule of differentiation.

4.27 TO SHOW THAT (i) $\Delta^n[x]^n = n\,!$ (ii) $\Delta^{n+1}[x]^n = 0$

$$\begin{aligned}\Delta[x]^n &= [(x+h)]^n - [x]^n\\ &= (x+h)(x+h-h)\,(x+h-2h)\ \ldots\ldots\ (x+h-\overline{n-1}\,h)\\ &\qquad - x(x-h)\,(x-2h)\ \ldots\ldots\ (x-\overline{n-1}\,h)\\ &= x(x-h)\ \ldots\ldots\ (x-\overline{n-2}\ h)\ [x+h-(x-nh+h)] = nh\ [x]^{n-1}\end{aligned}$$

Similarly, $\Delta^2[x]^n = \Delta[nh\ [x]^{n-1}] = nh\ \Delta[x]^{n-1} = n(n-1)\,h^2\ [x]^{n-2}$

$$\vdots$$

$$\Delta^n[x]^n = n(n-1)\ \ldots..\ 2\,.\,1\,.\,h^{n-1}\,(x+h-x) = n\,!\,h^n$$

Also, $\Delta^{n+1}[x]^n = n\,!\,h^n - n\,!\,h^n = 0$

when $h = 1$, $\Delta[x]^n = n[x]^{n-1}$ and $\Delta^n[x]^n = n\,!$

Hence, the result of differencing $[x]^r$ is analogous to that of differencing x^r when $h = 1$.

4.28 TO SHOW THAT $x^{(-n)} = \dfrac{1}{(x+n)^{(n)}}$, THE INTERVAL OF DIFFERENCING BEING UNITY

By definition of $x^{(n)}$, we have

$$x^{(n)} = (x-\overline{n-1}\ h)\ x^{(n-1)} \qquad \ldots(1)$$

when interval of differencing is h.

$\therefore$ When $n = 0$, we have $x^{(0)} = (x+h)\ x^{(-1)}$ ...(2)

Since, $\quad \Delta x^{(n)} = nhx^{(n-1)}$ ...(3)

when $n = 1$, $\Delta x^{(1)} = hx^{(0)}$.

$$\Rightarrow \quad \Delta x = h\,x^{(0)} \quad \Rightarrow \quad h = hx^{(0)} \quad \Rightarrow \quad x^{(0)} = 1$$

From (2), $\quad x^{(-1)} = \dfrac{1}{(x+h)}$ ...(4)

when $n = -1$, from (1),

$$x^{(-1)} = (x + 2h)\,x^{(-2)}$$

$$\Rightarrow \quad \frac{1}{x+h} = (x + 2h)\,x^{(-2)} \quad \Rightarrow \quad x^{(-2)} = \frac{1}{(x+h)(x+2h)}$$

In general, $\quad x^{(-n)} = \dfrac{1}{(x+h)(x+2h)\ldots\ldots(x+nh)}$ ...(5)

$$x^{(-n)} = \frac{1}{(x+nh)^{(n)}}$$

Here $x^{(-n)}$ is called reciprocal factorial where n is a (+)ve integer.

Particular case. When $h = 1$, $\quad x^{(-n)} = \dfrac{1}{(x+n)^{(n)}}$.

4.29 MISSING TERM TECHNIQUE

Suppose n values out of $(n + 1)$ values of $y = f(x)$ are given, the values of x being equidistant.

Let the unknown value be N. We construct the difference table.

Since only n values of y are known, we can assume $y = f(x)$ to be a polynomial of degree $(n - 1)$ in x.

Equating to zero the n^{th} difference, we can get the value of N.

EXAMPLES

Example 1. *Express $f(x) = x^4 - 12x^3 + 24x^2 - 30x + 9$ and its successive differences in factorial notation. Hence show that $\Delta^5 f(x) = 0$.*

Sol. Let $\quad f(x) = A[x]^4 + B[x]^3 + C[x]^2 + D[x] + E$

Using method of synthetic division, we divide by x, $x - 1$, $x - 2$, $x - 3$ etc. successively, then

1	1	– 12	24	– 30	9 = E
		1	– 11	13	
2	1	– 11	13	– 17 = D	
		2	– 18		
3	1	– 9	– 5 = C		
		3			
4	1	– 6 = B			
	1 = A				

Hence, $f(x) = [x]^4 - 6[x]^3 - 5[x]^2 - 17[x] + 9$

$\therefore$ $\Delta f(x) = 4[x]^3 - 18[x]^2 - 10[x] - 17$

$\Delta^2 f(x) = 12[x]^2 - 36[x] - 10$

$\Delta^3 f(x) = 24[x] - 36$

$\Delta^4 f(x) = 24$ and $\Delta^5 f(x) = 0.$

Example 2. *Obtain the function whose first difference is* $9x^2 + 11x + 5$.

Sol. Let $f(x)$ be the required function so that $\Delta f(x) = 9x^2 + 11x + 5$

Let $9x^2 + 11x + 5 = 9[x]^2 + A[x] + B = 9x(x-1) + Ax + B$

Putting $x = 0$, $B = 5$ and $x = 1$, $A = 20$

$\therefore$ $\Delta f(x) = 9[x]^2 + 20[x] + 5$

Integrating, we get

$$f(x) = 9\,\frac{[x]^3}{3} + 20\,\frac{[x]^2}{2} + 5[x] + c$$

$$= 3x(x-1)(x-2) + 10x(x-1) + 5x + c = 3x^3 + x^2 + x + c$$

where c is the constant of integration.

Example 3. *Find the missing values in the table:*

x:	45	50	55	60	65
y:	3	—	2	—	– 2.4.

Sol. Difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	3			
		$y_1 - 3$		
50	y_1		$5 - 2y_1$	
		$2 - y_1$		$3y_1 + y_3 - 9$
55	2		$y_1 + y_3 - 4$	
		$y_3 - 2$		$3.6 - y_1 - 3y_3$
60	y_3		$-0.4 - 2y_3$	
		$-2.4 - y_3$		
65	– 2.4			

As only three entries y_0, y_2, y_4 are given, the function y can be represented by a second degree polynomial.

$\therefore$ $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

$\Rightarrow$ $3y_1 + y_3 = 9$ and $y_1 + 3y_3 = 3.6$

Solving these, we get

$$y_1 = 2.925, \quad y_2 = 0.225.$$

Example 4. *Find the relation between* α, β *and* γ *in order that* $\alpha + \beta x + \gamma x^2$ *may be expressible in one term in the factorial notation.*

Sol. Let $$f(x) = \alpha + \beta x + \gamma x^2 = (a + bx)^{(2)}$$

where a and b are certain unknown constants.

Now, $$(a + bx)^{(2)} = (a + bx)\,[a + b(x - 1)]$$
$$= (a + bx)\,(a - b + bx) = (a + bx)^2 - ab - b^2x$$
$$= (a^2 - ab) + (2ab - b^2)x + b^2x^2 = \alpha + \beta x + \gamma x^2$$

Comparing the co-efficients of various powers of x, we get
$$\alpha = a^2 - ab,\ \beta = 2ab - b^2,\ \gamma = b^2$$

Eliminating a and b from the above equations, we get $\gamma^2 + 4\alpha\gamma = \beta^2$

Example 5. (*i*) *Estimate the missing term in the following table:*

x:	0	1	2	3	4
y = f(x):	1	3	9	?	81.

[G.B.T.U. (C.O.) 2010]

(*ii*) *Find the missing term in the table:*

x:	2	3	4	5	6
f(x):	45	49.2	54.1	?	67.4

[M.T.U. 2013, U.P.T.U. 2008]

Sol. (*i*) We are given Four values

$\therefore$ $$\Delta^4 f(x) = 0 \quad \forall\ x$$
$\Rightarrow$ $$(\mathrm{E} - 1)^4 f(x) = 0 \quad \forall\ x$$
$\Rightarrow$ $$(\mathrm{E}^4 - 4\mathrm{E}^3 + 6\mathrm{E}^2 - 4\mathrm{E} + 1)\, f(x) = 0 \quad \forall\ x$$
$\Rightarrow$ $$f(x + 4) - 4f(x + 3) + 6f(x + 2) - 4f(x + 1) + f(x) = 0 \quad \forall\ x$$

where interval of differencing is 1.

Now putting $x = 0$, we obtain,
$$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0 \qquad \text{...(1)}$$
$\Rightarrow$ $$81 - 4f(3) + 54 - 12 + 1 = 0 \qquad \text{(From table)}$$
$\Rightarrow$ $$4f(3) = 124 \quad \Rightarrow \quad f(3) = 31.$$

(*ii*) We are given four values.

$\therefore$ $$\Delta^4 f(x) = 0 \quad \forall\ x$$
$\Rightarrow$ $$(\mathrm{E} - 1)^4 f(x) = 0 \quad \forall\ x$$
$\Rightarrow$ $$(\mathrm{E}^4 - 4\mathrm{E}^3 + 6\mathrm{E}^2 - 4\mathrm{E} + 1)\, f(x) = 0 \quad \forall x$$
$\Rightarrow$ $$f(x + 4) - 4f(x + 3) + 6f(x + 2) - 4f(x + 1) + f(x) = 0 \quad \forall\ x$$

where interval of differencing is 1.

Now, putting $x = 2$, we obtain,
$$f(6) - 4f(5) + 6f(4) - 4f(3) + f(2) = 0$$
$\Rightarrow$ $$67.4 - 4f(5) + (6 \times 54.1) - (4 \times 49.2) + 45 = 0$$
$\Rightarrow$ $$4f(5) = 240.2$$
$\Rightarrow$ $$f(5) = 60.05$$

Example 6. *Estimate the production for 1964 and 1966 from the following data:*

Year:	1961	1962	1963	1964	1965	1966	1967
Production:	200	220	260	—	350	—	430

Sol. Since five figures are known, assume all the fifth order differences as zero. Since two figures are unknown, we need two equations to determine them.

Hence, $\Delta^5 y_0 = 0$ and $\Delta^5 y_1 = 0$

$\Rightarrow$ $(E-1)^5 y_0 = 0$ and $(E-1)^5 y_1 = 0$

$\Rightarrow$ $y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$

and $y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0$

Substituting the known values, we get

$$y_5 - 1750 + 10y_3 - 2600 + 1100 - 200 = 0$$

and $430 - 5y_5 + 3500 - 10y_3 + 1300 - 220 = 0$

$\Rightarrow$ $y_5 + 10y_3 = 3450$ ...(1)

and $-5y_5 - 10y_3 = -5010$ ...(2)

Adding (1) and (2), we get

$$-4y_5 = -1560 \quad \Rightarrow \quad y_5 = 390$$

From (1), $390 + 10y_3 = 3450$

$\Rightarrow$ $10y_3 = 3060 \quad \Rightarrow \quad y_3 = 306$

Hence production for year 1964 = 306

and production for year 1966 = 390.

ASSIGNMENT

1. (*i*) Estimate the missing term in the following:

x :	1	2	3	4	5	6	7
y :	2	4	8	—	32	64	128

Explain why the result differs from 16?

(*ii*) Find the missing value of the following data:

x :	1	2	3	4	5
$f(x)$:	7	$\boxed{\times}$	13	21	37

2. (*i*) From the following data, find the value of U_{47}:

$U_{46} = 0.2884$, $U_{48} = 0.5356$, $U_{49} = 0.6513$, $U_{50} = 0.7620$.

[**Hint.** $\Delta^4 U_x = 0 \Rightarrow (E-1)^4 U_x = 0$.]

(*ii*) Given: log 100 = 2, log 101 = 2.0043, log 103 = 2.0128, log 104 = 2.0170. Find log 102.

3. (*i*) Determine the missing values in the following table:

x :	0	5	10	15	20	25
y :	6	10	—	17	—	31

(*G.B.T.U. 2011*)

(*ii*)

x :	1	1.5	2	2.5	3	3.5	4
$f(x)$:	6	?	10	20	?	15	5

[*G.B.T.U. 2010; G.B.T.U. (C.O.) 2011*]

(*iii*)

x :	1	2	3	4	5	6	7	8
$f(x)$:	1	8	?	64	?	216	343	512

(*iv*)

x :	2	2.1	2.2	2.3	2.4	2.5	2.6
y :	0.135	—	0.111	0.100	—	0.082	0.074

(*v*)

x :	10	15	20	25	30	35
$f(x)$:	43	—	29	32	—	77

[*M.T.U. 2012, G.B.T.U. 2011*]

(*vi*)

x :	1	2	3	4	5	6	7	8
$f(x)$:	2	4	8	—	32	—	128	256

Explain why the results differ from 16 and 64. [*G.B.T.U. 2013*]

4. Express $f(x) = \dfrac{x-1}{(x+1)(x+3)}$ in terms of negative factorial polynomials.

Answers

1. (*i*) 16.1, 2^x is not a polynomial (*ii*) 9.5

2. (*i*) 0.4147 (*ii*) 2.0086

3. (*i*) 13.25, 22.5 (*ii*) 0.222, 22.022

(*iii*) 27, 125 (*iv*) 0.123, 0.0904

(*v*) 33.933, 46.733 (*vi*) 16.257, 63.476; 2^x is not a polynomial.

4. $x^{(-1)} - 4x^{(-2)} + 4x^{(-3)}$

We now proceed to study the use of finite difference calculus for the purpose of interpolation. Thus we shall do in following cases which are as follows:

(*i*) The value of argument in given data varies by an equal interval. The technique is called an **interpolation with equal intervals.**

(*ii*) The values of argument are not at equal intervals. This is known as **interpolation with unequal intervals.**

4.30 NEWTON'S FORMULAE FOR INTERPOLATION

Newton's formula is used for constructing the interpolation polynomial. It makes use of divided differences. This result was first discovered by the Scottish mathematician James Gregory (1638–1675) a contemporary of Newton.

Gregory and Newton did extensive work on methods of interpolation but now the formula is referred to as Newton's interpolation formula. Newton has derived general forward and backward difference interpolation formulae.

4.31 NEWTON'S GREGORY FORWARD INTERPOLATION FORMULA

(*M.T.U. 2013*)

Let $y = f(x)$ be a function of x which assumes the values $f(a)$, $f(a + h)$, $f(a + 2h)$, ..., $f(a + nh)$ for $(n + 1)$ equidistant values $a, a + h, a + 2h, ..., a + nh$ of the independent variable x. Let $f(x)$ be a polynomial of n^{th} degree.

Let $f(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + A_3(x-a)(x-a-h)(x-a-2h) + \ldots\ldots + A_n(x-a)\ldots\ldots(x-a-\overline{n-1}h)$ ...(1)

where $A_0, A_1, A_2, \ldots\ldots, A_n$ are to be determined.

Put $x = a, a + h, a + 2h, \ldots\ldots, a + nh$ in (1) successively.

For $x = a$, $\quad f(a) = A_0$ ...(2)

For $x = a + h$, $\quad f(a+h) = A_0 + A_1 h$

$\Rightarrow \quad f(a+h) = f(a) + A_1 h$ | By (2)

$\Rightarrow \quad A_1 = \dfrac{\Delta f(a)}{h}$ ...(3)

For $x = a + 2h$,

$$f(a + 2h) = A_0 + A_1 (2h) + A_2 (2h) h$$

$$= f(a) + 2h \left\{ \frac{\Delta f(a)}{h} \right\} + 2h^2 A_2$$

$$\Rightarrow \quad 2h^2 A_2 = f(a + 2h) - 2f(a + h) + f(a) = \Delta^2 f(a)$$

$$\Rightarrow \quad A_2 = \frac{\Delta^2 f(a)}{2!\, h^2}$$

Similarly, $A_3 = \dfrac{\Delta^3 f(a)}{3!\, h^3}$ and so on.

Thus, $A_n = \dfrac{\Delta^n f(a)}{n!\, h^n}$.

From (1),

$$f(x) = f(a) + (x - a) \frac{\Delta f(a)}{h} + (x - a)(x - a - h) \frac{\Delta^2 f(a)}{2!\, h^2} + \ldots\ldots.$$
$$+ (x - a) \ldots\ldots (x - a - \overline{n-1}\, h) \frac{\Delta^n f(a)}{n!\, h^n}$$

Put $x = a + hu \Rightarrow u = \dfrac{x - a}{h}$, we have

$$f(a + hu) = f(a) + hu \frac{\Delta f(a)}{h} + \frac{(hu)(hu - h)}{2!\, h^2} \Delta^2 f(a) + \ldots\ldots$$
$$+ \frac{(hu)(hu - h)(hu - 2h) \ldots\ldots. (hu - \overline{n-1}\, h)}{n!\, h^n} \Delta^n f(a)$$

$$\Rightarrow \quad \boxed{f(a + hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!} \Delta^2 f(a) + \ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!} \Delta^n f(a)}$$

This formula is particularly useful for interpolating the values of $f(x)$ near the beginning of the set of values given. h is called interval of differencing while Δ is forward difference operator.

EXAMPLES

Example 1. *The population of a town in the decennial census was as given below. Estimate the population for the year 1895.* (G.B.T.U. 2011)

Year x:	*1891*	*1901*	*1911*	*1921*	*1931*
Population y: (in thousands)	*46*	*66*	*81*	*93*	*101*

Sol. Here $a = 1891, h = 10,$

$$a + hu = 1895 \Rightarrow 1891 + 10u = 1895 \Rightarrow u = 0.4$$

The difference table is as under:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	$\boxed{46}$				
		$\boxed{20}$			
1901	66		$\boxed{-5}$		
		15		$\boxed{2}$	
1911	81		-3		$\boxed{-3}$
		12		-1	
1921	93		-4		
		8			
1931	101				

Applying Newton's forward difference formula,

$$y(1895) = y(1891) + u\ \Delta y(1891) + \frac{u(u-1)}{2!}\ \Delta^2 y(1891) + \frac{u(u-1)(u-2)}{3!}\ \Delta^3 y(1891)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\ \Delta^4 y(1891)$$

$$\Rightarrow \quad y(1895) = 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{2}\ (-5)$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)}{6}(2) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}\ (-3)$$

$\Rightarrow \quad y(1895) = 54.8528$ thousands $= 54852.8 \approx 54853$

Hence the population for the year 1895 is **54853** approximately.

Example 2. *From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46.* (U.P.T.U. 2014)

Age	*45*	*50*	*55*	*60*	*65*
Premium (in rupees)	*114.84*	*96.16*	*83.32*	*74.48*	*68.48*

Sol. Here $h = 5,\ a = 45,\ a + hu = 46$

$\therefore \quad 45 + 5u = 46 \quad \Rightarrow \quad u = 0.2$

The difference table is:

Age *(x)*	*Premium* *(in rupees)* *(y)*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
45	$\boxed{114.84}$				
		$\boxed{-18.68}$			
50	96.16		$\boxed{5.84}$		
		– 12.84		$\boxed{-1.84}$	
55	83.32		4		$\boxed{0.68}$
		– 8.84		– 1.16	
60	74.48		2.84		
		– 6			
65	68.48				

By Newton's forward difference formula,

$$y_{46} = y_{45} + u\,\Delta y_{45} + \frac{u(u-1)}{2\,!}\Delta^2 y_{45} + \frac{u(u-1)(u-2)}{3\,!}\Delta^3 y_{45} + \frac{u(u-1)(u-2)(u-3)}{4\,!}\Delta^4 y_{45}$$

$$= 114.84 + (0.2)(-18.68) + \frac{(0.2)(0.2-1)}{2\,!}(5.84)$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)}{3\,!}(-1.84) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{4\,!}(0.68)$$

$$= 110.525632$$

Hence the premium for policies maturing at the age of 46 is **₹ 110.52.**

Example 3. *From the following table, find the value of* $e^{0.24}$*:*

x:	*0.1*	*0.2*	*0.3*	*0.4*	*0.5*
e^x*:*	*1.10517*	*1.22140*	*1.34986*	*1.49182*	*1.64872.*

Sol. The difference table is:

x	$10^5 y$	$10^5 \Delta y$	$10^5 \Delta^2 y$	$10^5\,\Delta^3 y$	$10^4 \Delta^4 y$
0.1	$\boxed{110517}$				
		$\boxed{11623}$			
0.2	122140		$\boxed{1223}$		
		12846		$\boxed{127}$	
0.3	134986		1350		$\boxed{17}$
		14196		144	
0.4	149182		1494		
		15690			
0.5	164872				

Here $h = 0.1$. $\therefore$ $0.24 = 0.1 + (0.1)\,u$ or $u = 1.4$

Newton-Gregory forward difference formula is

$$y(0.24) = y(0.1) + u\,\Delta y(0.1) + \frac{u(u-1)}{2!}\Delta^2 y(0.1) + \frac{u(u-1)(u-2)}{3!}\Delta^3 y(0.1) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y(0.1)$$

$$\Rightarrow \quad 10^5\, y(0.24) = 10^5\, y(0.1) + u\, 10^5\, \Delta y(0.1) + \frac{u(u-1)}{2!} 10^5\, \Delta^2 y(0.1) + \frac{u(u-1)(u-2)}{3!} 10^5\, \Delta^3 y(0.1) + \frac{u(u-1)(u-2)(u-3)}{4!} 10^5\, \Delta^4 y(0.1)$$

$$\Rightarrow \quad 10^5\, y(0.24) = 110517 + (1.4)(11623) + \frac{(1.4)(1.4-1)}{2}(1223) + \frac{(1.4)(1.4-1)(1.4-2)}{3!}(127) + \frac{(1.4)(1.4-1)(1.4-2)(1.4-3)}{4!}(17)$$

$$= 127124.9088$$

$$\therefore \quad y(0.24) = e^{.24} = 1.271249088$$

Example 4. *The following table gives the population of a town during the last six censuses. Estimate the population in 1913 by Newton's forward difference formula*

Year	*1911*	*1921*	*1931*	*1941*	*1951*	*1961*
Population (in thousands)	*12*	*15*	*20*	*27*	*39*	*52*

[U.P.T.U. (MCA) 2009]

Sol. Here, $a = 1911,\ h = 10,\ x = 1913$

$$\therefore \quad u = \frac{x-a}{h} = \frac{1913-1911}{10} = 0.2$$

Forward difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
1911	$\boxed{12}$					
		$\boxed{3}$				
1921	15		$\boxed{2}$			
		5		$\boxed{0}$		
1931	20		2		$\boxed{3}$	
		7		3		$\boxed{-10}$
1941	27		5		-7	
		12		-4		
1951	39		1			
		13				
1961	52					

Newton's forward difference formula is

$$f(1913) = f(1911) + u\Delta f(1911) + \frac{u(u-1)}{2!}\Delta^2 f(1911) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(1911)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(1911) + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 f(1911)$$

$$= 12 + (0.2)(3) + \frac{(0.2)(-0.8)}{2}(2) + \frac{(0.2)(-0.8)(-1.8)(-2.8)}{4!}(3)$$

$$+ \frac{(0.2)(-0.8)(-1.8)(-2.8)(-3.8)}{5!}(-10)$$

$f(1913) = 12.08384$ thousands

Hence the population of the town in the year 1913 $\simeq$ 12083.84 = 12084 (approximately).

Example 5. *From the table, estimate the number of students who obtained marks between 40 and 45.* (M.T.U. 2013)

Marks:	*30–40*	*40–50*	*50–60*	*60–70*	*70–80*
No. of students:	*31*	*42*	*51*	*35*	*31.*

Sol. Difference table is:

Marks less than (x)	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$\boxed{31}$				
		$\boxed{42}$			
50	73		$\boxed{9}$		
		51		$\boxed{-25}$	
60	124		– 16		$\boxed{37}$
		35		12	
70	159		– 4		
		31			
80	190				

We shall find y_{45}, number of students with marks less than 45.

$$a = 40,\ h = 10,\ a + hu = 45.$$

$$\therefore \quad 40 + 10u = 45 \quad \Rightarrow \quad u = 0.5$$

By Newton's forward difference formula,

$$y(45) = y(40) + u\,\Delta\, y(40) + \frac{u(u-1)}{2!}\Delta^2 y(40)$$

$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 y(40) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y(40)$$

$$= 31 + (0.5)(42) + \frac{(0.5)(0.5-1)}{2}(9) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(37)$$

$$= 47.8672 \approx 48$$

Hence number of students getting marks less than 45 = 48

By number of students getting marks less than 40 = 31

Hence number of students getting marks between 40 and 45 = 48 – 31 = 17.

Example 6. *Find the cubic polynomial which takes the following values:*

x:	*0*	*1*	*2*	*3*
f(x):	*1*	*2*	*1*	*10.*

Sol. Let us form the difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	$\boxed{1}$			
		$\boxed{1}$		
1	2		$\boxed{-2}$	
		– 1		$\boxed{12}$
2	1		10	
		9		
3	10			

Here, $h = 1$. Hence using the formula, $x = a + hu$

and choosing $a = 0$, we get $x = u$

∴ By Newton's forward difference formula,

$$y = y_0 + x\,\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0$$

$$= 1 + x(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1$$

Hence the required cubic polynomial is

$$y = f(x) = 2x^3 - 7x^2 + 6x + 1.$$

Example 7. *The following table gives the marks secured by100 students in the Numerical Analysis subject:*

Range of marks:	*30–40*	*40–50*	*50–60*	*60–70*	*70–80*
No. of students:	*25*	*35*	*22*	*11*	*7*

Use Newton's forward difference interpolation formula to find.

(*i*) *the number of students who got more than 55 marks.*

(*ii*) *the number of students who secured marks in the range from 36 to 45.*

Sol. The given table is re-arranged as follows:

Marks obtained	*No. of students*
Less than 40	25
Less than 50	60
Less than 60	82
Less than 70	93
Less than 80	100

(*i*) Here, $a = 40$, $h = 10$, $a + hu = 55$

$\therefore$ $40 + 10u = 55$ $\Rightarrow$ $u = 1.5$

First, we find the number of students who got less than 55 marks.

The difference table is as under:

Marks obtained less than	*No. of students* = *y*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$\boxed{25}$				
		$\boxed{35}$			
50	60		$\boxed{-13}$		
		22		$\boxed{2}$	
60	82		– 11		$\boxed{5}$
		11		7	
70	93		– 4		
		7			
80	100				

Applying Newton's forward difference formula,

$$y_{55} = y_{40} + u\Delta y_{40} + \frac{u(u-1)}{2!}\Delta^2 y_{40} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{40} + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_{40}$$

$$= 25 + (1.5)(35) + \frac{(1.5)(.5)}{2!}(-13) + \frac{(1.5)(.5)(-.5)}{3!}(2) + \frac{(1.5)(.5)(-.5)(-1.5)}{4!}(5)$$

$$= 71.6171875 \approx 72$$

There are 72 students who got less than 55 marks.

$\therefore$ No. of students who got more than 55 marks = 100 – 72 = 28

(*ii*) To calculate the number of students securing marks between 36 and 45, take the difference of y_{45} and y_{36}.

$$u = \frac{x-a}{h} = \frac{36-40}{10} = -.4$$

Also, $$u = \frac{45-40}{10} = .5$$

By Newton's forward difference formula.

$$y_{36} = y_{40} + u\,\Delta\, y_{40} + \frac{u(u-1)}{2!}\Delta^2\, y_{40} + \frac{u(u-1)(u-2)}{3!}\Delta^3\, y_{40} + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4\, y_{40}$$

$$= 25 + (-0.4)(35) + \frac{(-0.4)(-1.4)}{2!}(-13) + \frac{(-0.4)(-1.4)(-2.4)}{3!}(2)$$

$$+ \frac{(-0.4)(-1.4)(-2.4)(-3.4)}{4!}(5) = 7.864 \approx 8$$

Also,
$$y_{45} = y_{40} + u\,\Delta y_{40} + \frac{u(u-1)}{2!}\Delta^2 y_{40} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{40} + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_{40}$$

$$= 25 + (0.5)(35) + \frac{(0.5)(-0.5)}{2}(-13) + \frac{(0.5)(-0.5)(-1.5)}{6}(2) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}(5)$$

$$= 44.0546 \approx 44.$$

Hence, the number of students who secured marks in the range from 36 to 45 is $= y_{45} - y_{36} = 44 - 8 = 36$.

Example 8. *The following are the numbers of deaths in four successive ten year age groups. Find the number of deaths at 45–50 and 50–55.*

Age group:	*25–35*	*35–45*	*45–55*	*55–65*
Deaths:	*13229*	*18139*	*24225*	*31496.*

Sol. Difference table of cumulative frequencies:

Age up to x	*No. of deaths f(x)*	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
35	$\boxed{13229}$			
		$\boxed{18139}$		
45	31368		$\boxed{6086}$	
		24225		$\boxed{1185}$
55	55593		7271	
		31496		
65	87089			

Here, $h = 10, a = 35, a + hu = 50$

$\therefore \quad 35 + 10u = 50 \quad \Rightarrow \quad u = 1.5$

By Newton's forward difference formula,

$$y_{50} = y_{35} + u\,\Delta y_{35} + \frac{u(u-1)}{2!}\Delta^2 y_{35} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{35}$$

$$= 13229 + (1.5)(18139) + \frac{(1.5)(0.5)}{2}(6086) + \frac{(1.5)(0.5)(-0.5)}{6}(1185)$$

$$= 42645.6875 \approx 42646$$

$\therefore$ Deaths at the age between 45 – 50 is 42646 – 31368 = 11278

and Deaths at the age between 50 – 55 is 55593 – 42646 = 12947.

ASSIGNMENT

1. Estimate the value of $f(22)$ from the following available data:

x:	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

2. Given that:

x:	1	2	3	4	5	6
$y(x)$:	0	1	8	27	64	125

Find the value of $f(2.5)$. [*U.P.T.U. (MCA) 2008*]

3. (*i*) The table below gives value of tan x for $0.10 \le x \le 0.30$.

x:	0.10	0.15	0.20	0.25	0.30
tan x:	0.1003	0.1511	0.2027	0.2553	0.3093

Evaluate tan 0.12 using Newton's forward difference formula.

(*ii*) Find the value of sin 52° from the following table using Newton's forward difference formula:

θ:	45°	50°	55°	60°
sin θ:	0.7071	0.7660	0.8192	0.8660

4. (*i*) Fit a polynomial of degree 3 and hence determine $y(3.5)$ for the following data:

x:	3	4	5	6
y:	6	24	60	120

[*M.T.U. 2013, G.B.T.U. 2011, 2012*]

(*ii*) Find the interpolating polynomial to the following data and hence find the value of y for $x = 5$:

x:	4	6	8	10
$f(x)$:	1	3	8	16

(*G.B.T.U. 2013*)

(*iii*) Express the value of θ in terms of x using the following data:

x:	40	50	60	70	80	90
θ:	184	204	226	250	276	304

Also find θ at $x = 43$. (*M.T.U. 2012*)

5. (*i*) Obtain the value of $f(3.5)$ from the following data:

x:	3	4	5	6	7
$f(x)$:	3	6.6	15	22	35

(*G.B.T.U. 2010*)

(*ii*) Use Newton-Gregory formula to compute y at $x = 24$ from the following data:

x:	21	25	29	33	37
y:	18.4	17.8	17.1	16.3	15.5

[*G.B.T.U. (C.O.) 2011*]

6. (*i*) Find the cubic polynomial which takes the following values:

$$y(0) = 1, \quad y(1) = 0, \quad y(2) = 1 \text{ and } y(3) = 10$$

Hence or otherwise obtain $y(4)$.

(*ii*) Find the polynomial interpolating the data:

x:	0	1	2
y:	0	5	2

(*U.P.T.U. 2008*)

7. Ordinates $f(x)$ of a normal curve in terms of standard deviation x are given as

x:	1.00	1.02	1.04	1.06	1.08
$f(x)$:	0.2420	0.2371	0.2323	0.2275	0.2227

Find the ordinate for standard deviation $x = 1.025$.

8. Find the number of men getting wages between ₹ 10 and ₹ 15 from following table:

Wages (in ₹):	0–10	10–20	20–30	30–40
Frequency:	9	30	35	42

(*G.B.T.U. 2011*)

9. Following are the marks obtained by 492 candidates in a certain examination

Marks:	0–40	40–45	45–50	50–55	55–60	60–65
No. of candidates:	210	43	54	74	32	79

Find out the number of candidates who secured

(*a*) more than 48 but not more than 50 marks

(*b*) less than 48 but not less than 45 marks.

10. Find the number of students from the following data who secured marks not more than 45

Marks range:	30–40	40–50	50–60	60–70	70–80
No. of students :	35	48	70	40	22

11. Use Newton's forward difference formula to obtain the interpolating polynomial $f(x)$ satisfying the following data:

x:	1	2	3	4
$f(x)$:	26	18	4	1

If another point $x = 5$, $f(x) = 26$ is added to the above data, will the interpolating polynomial be the same as before or different. Explain why?

12. Find the polynomial of degree four which takes the following values:

x:	2	4	6	8	10
y:	0	0	1	0	0

(*U.P.T.U. 2007*)

13. Use Newton's method to find a polynomial $p(x)$ of lowest possible degree such that $p(n) = 2^n$ for $n = 0, 1, 2, 3, 4$.

14. Find the order of the polynomial which might be suitable for the following function:

x:	2	2.1	2.2	2.3	2.4	2.5	2.6	2.7
$f(x)$:	0.577	0.568	0.556	0.540	0.520	0.497	0.471	0.442

Also find the value of $f(2.15)$ using difference formulae. [*G.B.T.U. (MCA) 2010*]

Answers

1. 352.22304 **2.** 3.375

3. (*i*) 0.1205 (*ii*) 0.7880

4. (*i*) $x^3 - 3x^2 + 2x$, 13.125 (*ii*) $\frac{3}{8}x^2 - \frac{11}{4}x + 6$, 1.625 (*iii*) $\frac{1}{100}x^2 + \frac{11}{10}x + 124$, 189.79

5. (*i*) 3.28125 (*ii*) 17.9571

6. (*i*) $x^3 - 2x^2 + 1$; 33 (*ii*) $9x - 4x^2$

7. 0.23589625 **8.** 15

9. (*a*) 27 (*b*) 27 **10.** 51

11. $\frac{17}{6}x^3 - 20x^2 + \frac{193}{6}x + 11$; no change since third differences are constant.

12. $\frac{1}{64}(x^4 - 24x^3 + 196x^2 - 624x + 640)$ **13.** $\frac{x^4}{24} - \frac{x^3}{12} + \frac{11}{12}x^2 + \frac{7x}{12} + 1$

14. 7^{th}, 0.562425293.

4.32 NEWTON'S GREGORY BACKWARD INTERPOLATION FORMULA

[*U.P.T.U. MCA (SUM) 2008*]

Let $y = f(x)$ be a function of x which assumes the values $f(a)$, $f(a + h)$, $f(a + 2h)$, ..., $f(a + nh)$ for $(n + 1)$ equidistant values $a, a + h, a + 2h, ..., a + nh$ of the independent variable x.

Let $f(x)$ be a polynomial of n^{th} degree.

Let, $$f(x) = A_0 + A_1(x - a - nh) + A_2 (x - a - nh)(x - a - \overline{n-1}\, h) + ...$$
$$+ A_n (x - a - nh)(x - a - \overline{n-1}\, h) ...... (x - a - h)$$

where $A_0, A_1, A_2, A_3, ..., A_n$ are to be determined. ...(1)

Put $x = a + nh, a + \overline{n-1}\, h, ..., a$ in (1) respectively.

Put $x = a + nh$, then $f(a + nh) = A_0$ ...(2)

Put $x = a + (n - 1)h$, then

$$f(a + \overline{n-1}\, h) = A_0 - h A_1 = f(a + nh) - h A_1 \qquad | \text{ By (2)}$$

$$\Rightarrow \qquad A_1 = \frac{\nabla f(a + nh)}{h} \qquad ...(3)$$

Put $x = a + (n - 2)h$, then

$$f(a + \overline{n-2}\, h) = A_0 - 2hA_1 + (-2h)(-h) A_2$$

$$\Rightarrow \qquad 2!\, h^2 A_2 = f(a + \overline{n-2}\, h) - f(a + nh) + 2\nabla f(a + nh) = \nabla^2 f(a + nh)$$

$$A_2 = \frac{\nabla^2 f(a + nh)}{2!\, h^2} \qquad ...(4)$$

Proceeding, we get $$A_n = \frac{\nabla^n f(a + nh)}{n!\, h^n} \qquad ...(5)$$

Substituting the values in (1), we get

$$f(x) = f(a + nh) + (x - a - nh) \frac{\nabla f(a + nh)}{h} + ... + (x - a - nh)(x - a - \overline{n-1}\, h)$$
$$..... (x - a - h) \frac{\nabla^n f(a + nh)}{n!\, h^n} \qquad ...(6)$$

Put $x = a + nh + uh$, then

$$x - a - nh = uh$$

and $$x - a - (n - 1)h = (u + 1)h$$
$$\vdots$$
$$x - a - h = (u + \overline{n-1})\, h$$

$\therefore$ (6) becomes,

$$f(x) = f(a + nh) + u\, \nabla f(a + nh) + \frac{u(u + 1)}{2!} \nabla^2 f(a + nh)$$
$$+ ...... + uh\, .\, (u + 1)h\, ... (u + \overline{n-1})(h) \frac{\nabla^n f(a + nh)}{n!\, h^n}$$

or

$$\boxed{\begin{aligned} f(a + nh + uh) = f(a + nh) + u\, \nabla f(a + nh) + \frac{u(u + 1)}{2!} \nabla^2 f(a + nh) \\ + ... + \frac{u(u + 1) ...... (u + \overline{n-1})}{n!} \nabla^n f(a + nh) \end{aligned}}$$

This formula is useful when the value of $f(x)$ is required near the end of the table.

EXAMPLES

Example 1. *The population of a town was as given. Estimate the population for the year 1925.*

Years (x):	*1891*	*1901*	*1911*	*1921*	*1931*
Population (y): *(in thousands)*	*46*	*66*	*81*	*93*	*101*

Sol. Here, $a + nh = 1931,\ h = 10,\ a + nh + uh = 1925$

$$\therefore \qquad u = \frac{1925 - 1931}{10} = -\,0.6$$

Difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		– 5		
		15		2	
1911	81		– 3		$\boxed{-3}$
		12		$\boxed{-1}$	
1921	93		$\boxed{-4}$		
		$\boxed{8}$			
1931	$\boxed{101}$				

Applying Newton's Backward difference formula, we get

$$y_{1925} = y_{1931} + u\nabla y_{1931} + \frac{u(u+1)}{2!}\nabla^2 y_{1931} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_{1931} + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_{1931}$$

$$= 101 + (-\,0.6)(8) + \frac{(-\,0.6)(0.4)}{2!}(-4) + \frac{(-\,0.6)(0.4)(1.4)}{3!}(-1) + \frac{(-\,0.6)(0.4)(1.4)(2.4)}{4!}(-3)$$

$= 96.8368$ thousands.

Hence the population for the year 1925 = 96836.8 ≈ 96837.

Example 2. *The population of a town is as follows:*

Year:	*1921*	*1931*	*1941*	*1951*	*1961*	*1971*
Population: *(in Lakhs)*	*20*	*24*	*29*	*36*	*46*	*51*

Estimate the increase in population during the period 1955 to 1961.

Sol. Here, $a + nh = 1971,\ h = 10,\ a + nh + uh = 1955$

$\therefore \qquad 1971 + 10u = 1955 \quad \Rightarrow \quad u = -\,1.6$

Difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1921	20					
		4				
1931	24		1			
		5		1		
1941	29		2		0	
		7		1		$\boxed{-9}$
1951	36		3		$\boxed{-9}$	
		10		$\boxed{-8}$		
1961	46		$\boxed{-5}$			
		$\boxed{5}$				
1971	$\boxed{51}$					

Applying Newton's backward difference formula, we get

$$y_{1955} = y_{1971} + u\nabla y_{1971} + \frac{u(u+1)}{2!}\nabla^2 y_{1971} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_{1971}$$
$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_{1971} + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_{1971}$$
$$= 51 + (-1.6)(5) + \frac{(-1.6)(-0.6)}{2!}(-5) + \frac{(-1.6)(-0.6)(0.4)}{6}(-8)$$
$$+ \frac{(-1.6)(-0.6)(.4)(1.4)}{24}(-9) + \frac{(-1.6)(-0.6)(0.4)(1.4)(2.4)}{120}(-9)$$
$$= 39.789632$$

$\therefore$ Increase in population during period 1955 to 1961 is

$$= 46 - 39.789632 = 6.210368 \text{ Lakhs} = 621036.8 \approx 621037.$$

Example 3. *Evaluate from following table f(3.8) to three significant figures using Gregory-Newton backward interpolation formula*

x:	*0*	*1*	*2*	*3*	*4*	
f(x):	*1*	*1.5*	*2.2*	*3.1*	*4.6*	(U.P.T.U. 2009)

Sol. Here, $a + nh = 4$, $h = 1$, $a + nh + uh = 3.8$

$$\therefore \quad u = \frac{3.8 - 4}{1} = -0.2$$

Backward difference table is

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0	1				
		0.5			
1	1.5		0.2		
		0.7		0	
2	2.2		0.2		$\boxed{0.4}$
		0.9		$\boxed{0.4}$	
3	3.1		$\boxed{0.6}$		
		$\boxed{1.5}$			
4	$\boxed{4.6}$				

By Newton's backward difference formula,

$$f(3.8) = f(4) + u\nabla f(4) + \frac{u(u+1)}{2}\nabla^2 f(4) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(4) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(4)$$

$$= 4.6 + (-0.2)(1.5) + \frac{(-0.2)(0.8)}{2}(0.6) + \frac{(-0.2)(0.8)(1.8)}{3!}(0.4) + \frac{(-0.2)(0.8)(1.8)(2.8)}{4!}(0.4) = 4.21936.$$

Example 4. *Given log x for x = 40, 45, 50, 55, 60 and 65 according to the following table:*

x:	*40*	*45*	*50*	*55*	*60*	*65*
log x:	*1.60206*	*1.65321*	*1.69897*	*1.74036*	*1.77815*	*1.81291*

Find the value of log 5875. [M.T.U. (MCA) 2012]

Sol. The difference table is:

x	$10^5 \log x = 10^5 y_x$	$10^5 \nabla y_x$	$10^5 \nabla^2 y_x$	$10^5 \nabla^3 y_x$	$10^5 \nabla^4 y_x$	$10^5 \nabla^5 y_x$
40	160206					
		5115				
45	165321		– 539			
		4576		102		
50	169897		– 437		– 25	
		4139		77		$\boxed{5}$
55	174036		– 360		$\boxed{-20}$	
		3779		$\boxed{57}$		
60	177815		$\boxed{-303}$			
		$\boxed{3476}$				
65	$\boxed{181291}$					

Newton's Backward difference formula is

$$f(a+nh+uh) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a+nh) + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 f(a+nh) \qquad ...(1)$$

First we shall find the value of log(58.75).

Here, $a + nh = 65, h = 5, a + nh + uh = 58.75$

$\therefore$ $65 + 5u = 58.75 \Rightarrow u = -1.25$

From (1),

$$10^5 f(58.75) = 181291 + (-1.25)(3476) + \frac{(-1.25)(-0.25)}{2!}(-303)$$
$$+ \frac{(-1.25)(-0.25)(0.75)}{3!}(57) + \frac{(-1.25)(-0.25)(0.75)(1.75)}{4!}(-20)$$
$$+ \frac{(-1.25)(-0.25)(0.75)(1.75)(2.75)}{5!}(5)$$

$\Rightarrow \quad 10^5 f(58.75) = 176900.588$

$\therefore \quad f(58.75) = \log 58.75 = 176900.588 \times 10^{-5} = 1.76900588$

Hence, $\log 5875 = 3.76900588$ $\quad | \because$ Mantissa remain the same

Example 5. *From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policy maturing at the age of 63:*

Age:	*45*	*50*	*55*	*60*	*65*
Premium: (in rupees)	*114.84*	*96.16*	*83.32*	*74.48*	*68.48*

Sol. The difference table is:

Age *(x)*	*Premium* *(in rupees)* *(y)*	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
45	114.84				
		– 18.68			
50	96.16		5.84		
		– 12.84		– 1.84	
55	83.32		4		$\boxed{0.68}$
		– 8.84		$\boxed{-1.16}$	
60	74.48		$\boxed{2.84}$		
		$\boxed{-6}$			
65	$\boxed{68.48}$				

Here $\quad a + nh = 65, \quad h = 5, a + nh + uh = 63$

$\therefore \quad 65 + 5u = 63 \quad \Rightarrow \quad u = -0.4$

By Newton's backward difference formula,

$$y(63) = y(65) + u\nabla y(65) + \frac{u(u+1)}{2!}\nabla^2 y(65) + \frac{u(u+1)(u+2)}{3!}\nabla^3 y(65)$$
$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y(65)$$

$$= 68.48 + (-0.4)(-6) + \frac{(-0.4)(0.6)}{2}(2.84)$$
$$+ \frac{(-0.4)(0.6)(1.6)}{6}(-1.16) + \frac{(-0.4)(0.6)(1.6)(2.6)}{24}(.68)$$

$= 70.585152$

ASSIGNMENT

1. Find a polynomial of degree three using Newton-Gregory backward difference formula which takes the following values. Hence find $y(7)$:

x:	3	4	5	6
y:	6	24	60	120

(*G.B.T.U. 2012*)

2. The population of a town in decennial census is as under. Estimate the population for the year 1955:

Year:	1921	1931	1941	1951	1961
Population (in lacs):	46	66	81	93	101

(*M.T.U. 2012*)

3. Estimate the value of $f(42)$ from the following available data:

x:	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

4. The table below gives the value of tan x for $0.10 \le x \le 0.30$:

x:	0.10	0.15	0.20	0.25	0.30
$y = \tan x$:	0.1003	0.1511	0.2027	0.2553	0.3093

Find: (*i*) tan 0.50 (*ii*) tan 0.26 (*iii*) tan 0.40.

5. (*i*) Given:

x:	1	2	3	4	5	6	7	8
$f(x)$:	1	8	27	64	125	216	343	512

Find $f(7.5)$ using Newton's Backward difference formula.

(*ii*) Compute $f(8)$ from the following data:

x:	1	3	5	7	9
$f(x)$:	9	21	81	237	537

(*G.B.T.U. 2013*)

6. Using Newton's backward difference formula, find the value of $e^{-1.9}$ from the following table of values of e^{-x}:

x:	1	1.25	1.50	1.75	2.00
e^{-x}:	0.3679	0.2865	0.2231	0.1738	0.1353

7. If $y(10) = 35.3$, $y(15) = 32.4$, $y(20) = 29.2$, $y(25) = 26.1$, $y(30) = 23.2$ and $y(35) = 20.5$, find $y(12)$ using Newton's forward as well as backward interpolation formula. Also explain why the difference (if any) in the result occur. (*U.P.T.U. 2007*)

8. From the following table of values of x and $f(x)$, determine (*i*) $f(0.23)$ (*ii*) $f(0.29)$:

x:	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$:	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

9. From the following table, find the value of tan 17°

$\theta°$:	0	4	8	12	16	20	24
$\tan \theta°$:	0	0.0699	0.1405	0.2126	0.2867	0.3640	0.4402

10. From the following table:

x:	10°	20°	30°	40°	50°	60°	70°	80°
$\cos x$:	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Calculate cos 25° and cos 73° using Gregory Newton formula. (*U.P.T.U. 2006*)

11. Use Newton–Gregory formula to interpolate the value of y at $x = 36$ from the following data:

x:	21	25	29	33	37
y:	18.4	17.8	17.1	16.3	15.5

(*U.P.T.U. 2014*)

Answers

1. $x^3 - 3x^2 + 2x$, 210	**2.** 9683680	**3.** 219
4. (*i*) 0.5543	(*ii*) 0.2662	(*iii*) 0.4241
5. (*i*) 421.875	(*ii*) 366	**6.** 0.1496
7. 34.22007, 34.30866	**8.** (*i*) 1.6751	(*ii*) 1.7081
9. 0.3057	**10.** 0.9063, 0.2923	**11.** 15.698

4.33 INTERPOLATION BY UNEVENLY SPACED POINTS

The interpolation formulae derived so far possess the disadvantage of being applicable only to equally spaced values of the argument. It is then desirable to develop interpolation formulae for unequally spaced values of x. We shall study two such formulae:

(1) Lagrange's interpolation formula

(2) Newton's general interpolation formula with divided differences.

4.34 LAGRANGE'S INTERPOLATION FORMULA [*G.B.T.U. (MCA) 2010, 2011*]

Let $f(x_0), f(x_1), ..., f(x_n)$ be $(n + 1)$ entries of a function $y = f(x)$, where $f(x)$ is assumed to be a polynomial corresponding to the arguments $x_0, x_1, x_2, ..., x_n$.

The polynomial $f(x)$ may be written as

$$f(x) = A_0 (x - x_1)(x - x_2) ...... (x - x_n) + A_1(x - x_0)(x - x_2) ...... (x - x_n) + ... + A_n (x - x_0)(x - x_1) ...... (x - x_{n-1}) \quad ...(1)$$

where $A_0, A_1, ..., A_n$ are constants to be determined.

Putting $x = x_0, x_1, ..., x_n$ in (1), we get

$$f(x_0) = A_0 (x_0 - x_1)(x_0 - x_2) .... (x_0 - x_n)$$

$$\therefore \quad A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) ...... (x_0 - x_n)} \quad ...(2)$$

$$f(x_1) = A_1 (x_1 - x_0)(x_1 - x_2) ...... (x_1 - x_n)$$

$$\therefore \quad A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) ...... (x_1 - x_n)} \quad ...(3)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\text{Similarly,} \quad A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) ...... (x_n - x_{n-1})} \quad ...(4)$$

Substituting the values of $A_0, A_1, ..., A_n$ in equation (1), we get

$$\boxed{f(x) = \frac{(x - x_1)(x - x_2) ...... (x - x_n)}{(x_0 - x_1)(x_0 - x_2) ...... (x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2) ...... (x - x_n)}{(x_1 - x_0)(x_1 - x_2) ...... (x_1 - x_n)} f(x_1) + ... + \frac{(x - x_0)(x - x_1) ...... (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) ...... (x_n - x_{n-1})} f(x_n)}$$

This is called **Lagrange's interpolation formula**.

4.35 ANOTHER FORM OF LAGRANGE'S FORMULA

Lagrange's formula can also be put in the form

$$\boxed{P_n(x) = \sum_{r=0}^{n} \frac{\phi(x)\, f(x_r)}{(x - x_r)\, \phi'(x_r)}}$$

where $\phi(x) = \prod_{r=0}^{n} (x - x_r)$ and $\phi'(x_r) = \left[\frac{d}{dx}\{\phi(x)\}\right]_{x = x_r}$

We have the Lagrange's formula,

$$P_n(x) = \sum_{r=0}^{n} \frac{(x - x_0)(x - x_1)\ldots(x - x_{r-1})(x - x_{r+1})\ldots(x - x_n)}{(x_r - x_0)(x_r - x_1)\ldots(x_r - x_{r-1})(x_r - x_{r+1})\ldots(x_r - x_n)} f(x_r)$$

$$= \sum_{r=0}^{n} \left\{\frac{\phi(x)}{x - x_r}\right\}\left\{\frac{f(x_r)}{(x_r - x_0)(x_r - x_1)\ldots(x_r - x_{r-1})(x_r - x_{r+1})\ldots(x_r - x_n)}\right\} \quad \ldots(1)$$

Now, $\phi(x) = \prod_{r=0}^{n} (x - x_r)$

$= (x - x_0)(x - x_1) \ldots\ldots (x - x_{r-1})(x - x_r)(x - x_{r+1}) \ldots\ldots (x - x_n)$

$\therefore \quad \phi'(x) = (x - x_1)(x - x_2) \ldots\ldots (x - x_r) \ldots\ldots (x - x_n) + (x - x_0)(x - x_2) \ldots\ldots (x - x_r) \ldots\ldots (x - x_n)$
$+ \ldots\ldots + (x - x_0)(x - x_1) \ldots\ldots (x - x_{r-1})(x - x_{r+1}) \ldots\ldots (x - x_n) + \ldots\ldots$
$+ (x - x_0)(x - x_1) \ldots\ldots (x - x_r) \ldots\ldots (x - x_{n-1})$

$\Rightarrow \quad \phi'(x_r) = [\phi'(x)]_{x = x_r}$

$= (x_r - x_0)(x_r - x_1) \ldots\ldots (x_r - x_{r-1})(x_r - x_{r+1}) \ldots\ldots (x_r - x_n) \quad \ldots(2)$

Hence from (1), $P_n(x) = \sum_{r=0}^{n} \frac{\phi(x)\, f(x_r)}{(x - x_r)\, \phi'(x_r)}$ | Using (2)

EXAMPLES

Example 1. *Using Lagrange's interpolation formula, find y(10) from the following table:*

x	5	6	9	11
y	12	13	14	16

(U.P.T.U. 2009)

Sol. Here, $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$

$f(x_0) = 12$, $f(x_1) = 13$, $f(x_2) = 14$, $f(x_3) = 16$

Lagrange's formula is

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

$$f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$
$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$
$$= -\frac{1}{2}(x-6)(x-9)(x-11) + \frac{13}{15}(x-5)(x-9)(x-11) - \frac{7}{12}(x-5)(x-6)(x-11)$$
$$+ \frac{4}{15}(x-5)(x-6)(x-9)$$

Putting $x = 10$, we get

$$f(10) = -\frac{1}{2}(10-6)(10-9)(10-11) + \frac{13}{15}(10-5)(10-9)(10-11)$$
$$- \frac{7}{12}(10-5)(10-6)(10-11) + \frac{4}{15}(10-5)(10-6)(10-9)$$
$$= 14.66666667$$

Hence, $y(10) = 14.66666667$.

Example 2. *Compute the value of f(x) for x = 2.5 from the following table:*

x:	*1*	*2*	*3*	*4*
f(x):	*1*	*8*	*27*	*64*

using Lagrange's interpolation method.

Sol. Here $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$

$f(x_0) = 1$, $f(x_1) = 8$, $f(x_2) = 27$, $f(x_3) = 64$

Lagrange's formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$
$$= \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}(1) + \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}(8) + \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}(27)$$
$$+ \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}(64)$$

Putting $x = 2.5$, we get

$$f(2.5) = -\frac{1}{6}(2.5-2)(2.5-3)(2.5-4) + 4(2.5-1)(2.5-3)(2.5-4)$$
$$- \frac{27}{2}(2.5-1)(2.5-2)(2.5-4) + \frac{32}{3}(2.5-1)(2.5-2)(2.5-3)$$
$$= 15.625.$$

Example 3. *Find the cubic Lagrange's interpolating polynomial from the following data:*

x:	*0*	*1*	*2*	*5*
f(x):	*2*	*3*	*12*	*147.*

Sol. Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 5$

$f(x_0) = 2$, $f(x_1) = 3$, $f(x_2) = 12$, $f(x_3) = 147$

Lagrange's formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$
$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12)$$
$$+ \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$
$$= -\frac{1}{5}(x-1)(x-2)(x-5) + \frac{3}{4}x(x-2)(x-5) - 2x(x-1)(x-5)$$
$$+ \frac{49}{20}x(x-1)(x-2)$$

$\Rightarrow \quad f(x) = x^3 + x^2 - x + 2$

Example 4. *Find the unique polynomial P(x) of degree 2 such that:*

P(1) = 1, P(3) = 27, P(4) = 64

Use Lagrange method of interpolation.

Sol. Here, $x_0 = 1, \quad x_1 = 3, \quad x_2 = 4$

$f(x_0) = 1, \quad f(x_1) = 27, \quad f(x_2) = 64$

By Lagrange's interpolation formula

$$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$
$$= \frac{(x-3)(x-4)}{(1-3)(1-4)}(1) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(27) + \frac{(x-1)(x-3)}{(4-1)(4-3)}(64)$$
$$= \frac{1}{6}(x^2-7x+12) - \frac{27}{2}(x^2-5x+4) + \frac{64}{3}(x^2-4x+3) = 8x^2 - 19x + 12$$

Example 5. *The function y = f(x) is given at the points (7, 3), (8, 1), (9, 1) and (10, 9). Find the value of y for x = 9.5 using Lagrange's interpolation formula.*

Sol. We are given

x:	7	8	9	10
$f(x)$:	3	1	1	9

Here, $x_0 = 7, \quad x_1 = 8, \quad x_2 = 9, \quad x_3 = 10$

$f(x_0) = 3, \quad f(x_1) = 1, \quad f(x_2) = 1, \quad f(x_3) = 9$

Lagrange's interpolation formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(x-8)(x-9)(x-10)}{(7-8)(7-9)(7-10)}(3) + \frac{(x-7)(x-9)(x-10)}{(8-7)(8-9)(8-10)}(1) + \frac{(x-7)(x-8)(x-10)}{(9-7)(9-8)(9-10)}(1)$$
$$+ \frac{(x-7)(x-8)(x-9)}{(10-7)(10-8)(10-9)}(9) \quad ...(1)$$

Putting $x = 9.5$ in eqn. (1), we get

$$f(9.5) = -\frac{1}{2}(9.5-8)(9.5-9)(9.5-10) + \frac{1}{2}(9.5-7)(9.5-9)(9.5-10)$$
$$-\frac{1}{2}(9.5-7)(9.5-8)(9.5-10) + \frac{3}{2}(9.5-7)(9.5-8)(9.5-9) = 3.625.$$

Example 6. *Use Lagrange's interpolation formula to fit a polynomial to the data:*

x:	*– 1*	*0*	*2*	*3*
u_x:	*– 8*	*3*	*1*	*12*

Hence or otherwise find the value of u_1.

Sol. Here,

$x_0 = -1, \quad x_1 = 0, \quad x_2 = 2, \quad x_3 = 3$

$f(x_0) = -8, \quad f(x_1) = 3, \quad f(x_2) = 1, \quad f(x_3) = 12$

Lagrange's interpolation formula is

$$u_x = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$
$$= \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3) + \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1)$$
$$+ \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12)$$
$$= \frac{2}{3}x(x-2)(x-3) + \frac{1}{2}(x+1)(x-2)(x-3) - \frac{1}{6}(x+1)x(x-3) + (x+1)x(x-2)$$

$$\Rightarrow \quad u_x = 2x^3 - 6x^2 + 3x + 3 \quad ...(1)$$

Putting $x = 1$ in (1), we get

$$u_1 = 2(1)^3 - 6(1)^2 + 3(1) + 3 = 2.$$

Example 7. *By means of Lagrange's formula, prove that*

(i) $y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$

(ii) $y_3 = 0.05(y_0 + y_6) - 0.3(y_1 + y_5) + 0.75(y_2 + y_4)$

(iii) $y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$.

Sol. (*i*) For the arguments –3, –1, 1, 3, the Lagrange's formula is

$$y_x = \frac{(x+1)(x-1)(x-3)}{(-3+1)(-3-1)(-3-3)} y_{-3} + \frac{(x+3)(x-1)(x-3)}{(-1+3)(-1-1)(-1-3)} y_{-1} + \frac{(x+3)(x+1)(x-3)}{(1+3)(1+1)(1-3)} y_1$$

$$+ \frac{(x+3)(x+1)(x-1)}{(3+3)(3+1)(3-1)} y_3 \quad ...(1)$$

Putting $x = 0$ in (1), we get

$$y_0 = -\frac{1}{16} y_{-3} + \frac{9}{16} y_{-1} + \frac{9}{16} y_1 - \frac{1}{16} y_3$$

$$= \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

(*ii*) For the arguments 0, 1, 2, 4, 5, 6, the Lagrange's formula is

$$y_x = \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} y_0 + \frac{(x-0)(x-2)(x-4)(x-5)(x-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} y_1$$

$$+ \frac{(x-0)(x-1)(x-4)(x-5)(x-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} y_2 + \frac{(x-0)(x-1)(x-2)(x-5)(x-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} y_4$$

$$+ \frac{(x-0)(x-1)(x-2)(x-4)(x-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} y_5 + \frac{(x-0)(x-1)(x-2)(x-4)(x-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} y_6 \quad ...(1)$$

Putting $x = 3$ in (1), we get

$$y_3 = 0.05\, y_0 - 0.3\, y_1 + 0.75\, y_2 + 0.75\, y_4 - 0.3\, y_5 + 0.05\, y_6$$

$$= 0.05\,(y_0 + y_6) - 0.3(y_1 + y_5) + 0.75\,(y_2 + y_4).$$

(*iii*) For the arguments –5, –3, 3, 5, the Lagrange's formula is

$$y_x = \frac{(x+3)(x-3)(x-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} + \frac{(x+5)(x-3)(x-5)}{(-3+5)(-3-3)(-3-5)} y_{-3}$$

$$+ \frac{(x+5)(x+3)(x-5)}{(3+5)(3+3)(3-5)} y_3 + \frac{(x+5)(x+3)(x-3)}{(5+5)(5+3)(5-3)} y_5 \quad ...(1)$$

Putting $x = 1$ in eqn. (1), we get

$$y_1 = -0.2\, y_{-5} + 0.5 y_{-3} + y_3 - 0.3\, y_5 = y_3 - 0.3\,(y_5 - y_{-3}) + 0.2\,(y_{-3} - y_{-5}).$$

ASSIGNMENT

1. Using Lagrange's interpolation formula, find polynomial which takes the values 3, 12, 15, –21 when x has the values 3, 2, 1 and –1. (*U.P.T.U. 2014*)

2. Values of $f(x)$ for values of x are given as

$$f(1) = 4,\ f(2) = 5,\ f(7) = 5,\ f(8) = 4$$

Find $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum.

3. (*i*) Using Lagrange interpolation formula, calculate $f(3)$ from the following table:

x:	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

(*U.P.T.U. 2006*)

(*ii*) Use Lagrange's interpolation formula to compute $f(5.5)$ from the following data:

x:	0	1	4	5	6
$f(x)$:	1	14	15	6	3

[*G.B.T.U. (C.O.) 2011*]

4. Compute $f(27)$ from the following data using Lagrange's interpolation formula.

x:	14	17	31	35
$f(x)$:	68.7	64.0	44.0	39.1

(*U.P.T.U. 2007*)

5. (*i*) Find the value of tan 33° by Lagrange's formula if

tan 30° = 0.5774, tan 32° = 0.6249, tan 35° = 0.7002, tan 38° = 0.7813.

(*ii*) Find the value of $f(0.55)$ using Lagrange interpolation with the following table of values:

x:	0.4	0.5	0.7	0.8
$f(x)$:	– 0.916	– 0.693	– 0.357	– 0.223

[*U.P.T.U. MCA (SUM) 2009*]

6. (*i*) Use Lagrange's formula to find $f(6)$ from the following table:

x:	2	5	7	10	12
$f(x)$:	18	180	448	1210	2028.

(*ii*) Apply Lagrange's formula to find $f(15)$, if

x:	10	12	14	16	18	20
$f(x)$:	2420	1942	1497	1109	790	540.

7. Derive the Lagrange's interpolation formula for unequal intervals. Find the value of $f(2.6)$ from the given table (Use above formula). [*U.P.T.U. (MCA) 2006*]

x:	– 2	– 1	1	2	6
$f(x)$:	– 4	14	– 4	– 16	196

8. If $y_0, y_1, ..., y_9$ are consecutive terms of a series, prove that

$$y_5 = \frac{1}{70}[56(y_4 + y_6) - 28(y_3 + y_7) + 8(y_2 + y_8) - (y_1 + y_9)]$$

9. Using the following table, find $f(x)$ as a polynomial in x:

x:	– 1	0	3	6	7
$f(x)$:	3	– 6	39	822	1611.

(*U.P.T.U. 2009*)

10. (*i*) If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, and $y(6) = 132$, find the four-point Lagrange interpolation polynomial that takes the same values as the function y at the given points.

(*ii*) Apply Lagrange's formula to find a cubic polynomial which approximates the data:

x:	– 2	– 1	2	3
$y(x)$:	– 12	– 8	3	5

[*G.B.T.U. (C.O.) 2010*]

(*iii*) Find the unique polynomial $P(x)$ of degree 2 or less such that $P(2) = 8$, $P(4) = 64$, $P(5) = 125$ using the Lagrange interpolation formula. Hence evaluate $P(2.5)$.

[*G.B.T.U. MCA (SUM) 2010*]

11. Given the table of values

x:	150	152	154	156
$y = \sqrt{x}$:	12.247	12.329	12.410	12.490

Evaluate $\sqrt{155}$ using Lagrange's interpolation formula.

12. Use Lagrange's method to find a polynomial $p(x)$ of lowest possible degree such that $p(n) = 2^n$ for $n = 0, 1, 2, 3, 4$.

13. Values of $f(x)$ are given at a, b and c. Show that the maximum is obtained by

$$x = \frac{f(a)(b^2 - c^2) + f(b)(c^2 - a^2) + f(c)(a^2 - b^2)}{f(a)(b - c) + f(b)(c - a) + f(c)(a - b)}.$$

14. (*i*) Find a Lagrange's interpolating polynomial for the data given below:

$x_0 = 1$, $x_1 = 2.5$, $x_2 = 4$ and $x_3 = 5.5$

$f(x_0) = 4$, $f(x_1) = 7.5$, $f(x_2) = 13$ and $f(x_3) = 17.5$

Also, find the value of $f(5)$.

(*ii*) Use Lagrange's method to find interpolating polynomial of degree 3 to fit the following data:

x:	0	1	2	3
$f(x)$:	0	1.7183	6.3891	19.0855

Hence compute the value of $f(1.5)$. [*G.B.T.U. (MCA) 2011*]

15. Determine by Lagrange's formula, the percentage number of criminals under 35 years:

Age	*% number of criminals*
under 25 years	52
under 30 years	67.3
under 40 years	84.1
under 50 years	94.4

16. Obtain Lagrange's interpolatory for the following data:

x:	1	3	5	7	10
$f(x)$:	13	31	25	37	101

Find the values of $f(4)$ and $f(8.5)$ (*U.P.T.U. 2015*)

Answers

1. $x^3 - 9x^2 + 17x + 6$ **2.** 5.66; $x = 4.5$

3. (*i*) 10 (*ii*) 3.096875 **4.** 49.31046

5. (*i*) 0.64942084 (*ii*) – 0.59721875 **6.** (*i*) 293.99856 (*ii*) 1294.8437

7. – 17.248 **9.** $x^4 - 3x^3 + 5x^2 - 6$

10. (*i*) $x^3 - 3x^2 + 5x - 6$ (*ii*) $y(x) = -\frac{1}{15}x^3 - \frac{3}{20}x^2 + \frac{241}{60}x - 3.9$

(*iii*) $P(x) = 11x^2 - 38x + 40$, 13.75

11. 12.45 **12.** $\frac{x^4}{24} - \frac{x^3}{12} + \frac{11}{24}x^2 + \frac{7x}{12} + 1$

14. (*i*) $-\frac{4}{27}x^3 + \frac{14}{9}x^2 - \frac{5}{3}x + \frac{115}{27}$; 16.2962963 (*ii*) $0.84551x^3 - 1.06028x^2 + 1.93307x$; 3.367571

15. 77.405

16. $f(x) = -\frac{1}{1080}(91x^4 - 2401x^3 + 19571x^2 - 60431x + 29130)$; $f(4) = 27.616$, $f(8.5) = 64.83$.

4.36 ERROR IN LAGRANGE'S INTERPOLATION FORMULA

Remainder,

$$y(x) - L_n(x) = R_n(x) = \frac{\Pi_{n+1}(x)}{(n+1)!}\, y^{(n+1)}(\xi),\ a < \xi < b$$

where Lagrange's formula is for the class of functions having continuous derivatives of order up to $(n + 1)$ on $[a, b]$.

Quantity $E_L = \max\limits_{[a,b]} |R_n(x)|$ may be taken as an estimate of error.

Let us assume

$$|y^{(n+1)}(\xi)| \le M_{n+1},\ a \le \xi \le b$$

then,
$$E_L \le \frac{M_{n+1}}{(n+1)!} \max_{[a,b]} |\Pi_{n+1}(x)|.$$

EXAMPLES

Example 1. *Show that the truncation error of quadratic interpolation in an equidistant table is bounded by* $\frac{h^3}{9\sqrt{3}}$ *max* $|f'''(\xi)|$ *where h is the step size and f is the tabulated function.*

Sol. Let x_{i-1}, x_i, x_{i+1} denote three consecutive equi-spaced points with step size h.

The truncation error of the quadratic Lagrange interpolation is bounded by

$$|E_2(f;x)| \le \frac{M_3}{6} \max |(x-x_{i-1})(x-x_i)(x-x_{i+1})|$$

where $x_{i-1} \le x \le x_{i+1}$ and $M_3 = \max\limits_{a \le x \le b} |f'''(x)|$

Substitute $t = \frac{x - x_i}{h}$ then,

$$x - x_{i-1} = x - (x_i - h) = x - x_i + h = th + h = (t+1)h$$
$$x - x_{i+1} = x - (x_i + h) = x - x_i - h = th - h = (t-1)h$$

and $\quad (x-x_{i-1})(x-x_i)(x-x_{i+1}) = (t+1)\,t(t-1)h^3 = t(t^2-1)h^3 = g(t)$

Setting $g'(t) = 0$, we get

$$3t^2 - 1 = 0 \quad \Rightarrow \quad t = \pm\frac{1}{\sqrt{3}}.$$

For both these values of t, we obtain

$$\max |(x-x_{i-1})(x-x_i)(x-x_{i+1})| = h^3 \max_{-1 \le t \le 1} |t(t^2-1)| = \frac{2h^3}{3\sqrt{3}}$$

Hence, the truncation error of the quadratic interpolation is bounded by

$$|E_2(f;x)| \le \frac{h^3}{9\sqrt{3}} M_3$$

or, $$|E_2(f;x)| \le \frac{h^3}{9\sqrt{3}} \max |f'''(\xi)|.$$

Example 2. *Determine the step size that can be used in the tabulation of f(x) = sin x in the interval* $\left[0, \frac{\pi}{4}\right]$ *at equally spaced nodal points so that the truncation error of the quadratic interpolation is less than* 5×10^{-8}.

Sol. From Ex. 1, we have

$$|E_2(f;x)| \le \frac{h^3}{9\sqrt{3}} M_3$$

For $\quad f(x) = \sin x$, we get $f'''(x) = -\cos x$ and $M_3 = \max\limits_{0 \le x \le \pi/4} |\cos x| = 1$

Hence the step size h is given by

$$\frac{h^3}{9\sqrt{3}} \le 5 \times 10^{-8} \quad \text{or} \quad h \approx 0.009$$

Example 3. *Using Lagrange's interpolation formula, find the value of sin* $\left(\frac{\pi}{6}\right)$ *from the following data:*

x:	*0*	$\pi/4$	$\pi/2$
y = sin x:	*0*	*0.70711*	*1.0*

Also estimate the error in the solution.

Sol.
$$\sin\left(\frac{\pi}{6}\right) = \frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}(0.70711) + \frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{4}\right)}{\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}(1) = 0.51743$$

Now, $y(x) = \sin x,\quad y'(x) = \cos x,\quad y''(x) = -\sin x,\quad y'''(x) = -\cos x$

Hence, $|y'''(\xi)| < 1$

when $x = \pi/6$,

$$|R_n(x)| \le \left|\frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{3!}\right| = 0.02392$$

which agrees with the actual error in problem.

4.37 EXPRESSION OF RATIONAL FUNCTION AS A SUM OF PARTIAL FRACTIONS

Let $$f(x) = \frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$$

Consider $\phi(x) = 3x^2 + x + 1$ and tabulate its values for $x = 1, 2, 3$, we get

x:	1	2	3
$\phi(x) = 3x^2 + x + 1$:	5	15	31

Using Lagrange's interpolation formula, we get

$$\phi(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{-1}(15) + \frac{(x-1)(x-2)}{2}(31)$$

$$= \frac{5}{2}(x-2)(x-3) - 15(x-1)(x-3) + \frac{31}{2}(x-1)(x-2)$$

$$f(x) = \frac{5}{2(x-1)} - \frac{15}{x-2} + \frac{31}{2(x-3)}.$$

4.38 INVERSE INTERPOLATION

The process of estimating the value of x for the value of y not in the table is called *inverse interpolation*. When values of x are unevenly spaced, Lagrange's method is used by interchanging x and y.

EXAMPLE

Example. *From the given table:*

x:	*20*	*25*	*30*	*35*
y(x):	*0.342*	*0.423*	*0.5*	*0.65*

Find the value of x for y(x) = 0.390.

Sol. By inverse interpolation formula,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$
$$= \frac{(0.39-0.423)(0.39-0.5)(0.39-0.65)}{(0.342-0.423)(0.342-0.5)(0.342-0.65)}(20)$$
$$+ \frac{(0.39-0.342)(0.39-0.5)(0.39-0.65)}{(0.423-0.342))(0.423-0.5)(0.423-0.65)}(25)$$
$$+ \frac{(0.39-0.342)(0.39-0.423)(0.39-0.65)}{(0.5-0.342)(0.5-0.423)(0.5-0.65)}(30)$$
$$+ \frac{(0.39-0.342)(0.39-0.423)(0.39-0.5)}{(0.65-0.342)(0.65-0.423)(0.65-0.5)}(35)$$
$$= 22.84057797.$$

4.39 DIVIDED DIFFERENCES

Lagrange's interpolation formula has the disadvantage that if another interpolation point were added, the interpolation co-efficients will have to be recomputed.

We therefore seek an interpolation polynomial which has the property that a polynomial of higher degree may be derived from it by simply adding new terms.

Newton's general interpolation formula is one such formula and it employs divided differences.

If (x_0, y_0), (x_1, y_1), (x_2, y_2) ...... are given points then the first divided difference for the arguments x_0, x_1 is defined by

$$\underset{x_1}{\Delta} y_0 = [x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly, $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$ and so on.

The second divided difference for x_0, x_1, x_2 is defined as

$$\underset{x_1, x_2}{\Delta^2} y_0 = [x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

Third divided difference for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \text{ and so on.}$$

4.40 PROPERTIES OF DIVIDED DIFFERENCES

1. The divided differences are symmetrical in their arguments *i.e.,* independent of the order of arguments.

$$[x_0, x_1] = \frac{y_1}{x_1 - x_0} + \frac{y_0}{x_0 - x_1} = [x_1, x_0]$$

Also, $$[x_0, x_1, x_2] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$
$$= [x_2, x_0, x_1] \text{ or } [x_1, x_2, x_0]$$

2. The n^{th} divided differences of a polynomial of n^{th} degree are constant.

Let the arguments be equally spaced so that

$$x_1 - x_0 = x_2 - x_1 = \ldots.. = x_n - x_{n-1} = h$$

then, $$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{(x_2 - x_0)} = \frac{1}{2h}\left(\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right) = \frac{1}{2!} \cdot \frac{1}{h^2} (\Delta^2 y_0)$$

In general, $$[x_0, x_1, x_2, \ldots\ldots, x_n] = \frac{1}{n!} \cdot \frac{1}{h^n} \Delta^n y_0$$

If tabulated function is a n^{th} degree polynomial. $\therefore$ $\Delta^n y_0$ = constant

$\therefore$ n^{th} divided differences will also be constant.

4.41 NEWTON'S DIVIDED DIFFERENCE INTERPOLATION FORMULA

(*U.P.T.U. 2009*)

Let $y_0, y_1, \ldots\ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots\ldots, x_n$, then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that, $$y = y_0 + (x - x_0)\,[x, x_0] \qquad \ldots(1)$$

Again, $$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives, $$[x, x_0] = [x_0, x_1] + (x - x_1)\,[x, x_0, x_1] \qquad \ldots(2)$$

From (1) and (2), $$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x, x_0, x_1] \qquad \ldots(3)$$

Also, $$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

which gives $$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)\,[x, x_0, x_1, x_2] \qquad \ldots(4)$$

From (3) and (4), $$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$y = f(x) = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3] + \ldots.. + (x - x_0)(x - x_1)(x - x_2)$$
$$\ldots..(x - x_{n-1})\,[x_0, x_1, x_2, x_3, \ldots\ldots, x_n] + (x - x_0)(x - x_1)(x - x_2)$$
$$\ldots..(x - x_n)\,[x, x_0, x_1, x_2, \ldots\ldots, x_n]$$

which is called Newton's general interpolation formula with divided differences, the last term being the remainder term after $(n + 1)$ terms.

Newton's divided difference formula can also be written as

$$\boxed{\begin{aligned} y = y_0 &+ (x - x_0)\,\Delta y_0 + (x - x_0)(x - x_1)\,\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 y_0 \\ &+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta^4 y_0 + \ldots\ldots + (x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1})\,\Delta^n y_0 \end{aligned}}$$

EXAMPLES

Example 1. (*i*) *Find the third divided difference with arguments 2, 4, 9, 10 of the function* $f(x) = x^3 - 2x$.

(*ii*) *If* $f(x) = \dfrac{1}{x^2}$, *find the first divided differences* $f(a, b)$, $f(a, b, c)$, $f(a, b, c, d)$.

(*iii*) *If* $f(x) = g(x)\,h(x)$, *prove that* $f(x_1, x_2) = g(x_1)\,h(x_1, x_2) + g(x_1, x_2)\,h(x_2)$.

Sol. (*i*)

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		$\dfrac{56 - 4}{4 - 2} = 26$		
4	56		$\dfrac{131 - 26}{9 - 2} = 15$	
		$\dfrac{711 - 56}{9 - 4} = 131$		$\dfrac{23 - 15}{10 - 2} = 1$
9	711		$\dfrac{269 - 131}{10 - 4} = 23$	
		$\dfrac{980 - 711}{10 - 9} = 269$		
10	980			

Hence third divided difference is 1.

(*ii*)

x	$f(x) = \dfrac{1}{x^2}$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	$\dfrac{1}{a^2}$			
		$\dfrac{\left(\dfrac{1}{b^2} - \dfrac{1}{a^2}\right)}{b - a} = \boxed{-\left(\dfrac{a + b}{a^2 b^2}\right)}$		
b	$\dfrac{1}{b^2}$		$\boxed{\dfrac{ab + bc + ca}{a^2 b^2 c^2}}$	
		$-\left(\dfrac{b + c}{b^2 c^2}\right)$		$\boxed{-\left(\dfrac{abc + acd + abd + bcd}{a^2 b^2 c^2 d^2}\right)}$
c	$\dfrac{1}{c^2}$		$\dfrac{bc + cd + db}{b^2 c^2 d^2}$	
		$-\left(\dfrac{c + d}{c^2 d^2}\right)$		
d	$\dfrac{1}{d^2}$			

From the above divided difference table, we observe that first divided differences,

$$f(a, b) = -\left(\frac{a+b}{a^2b^2}\right)$$

$$f(a, b, c) = \frac{ab+bc+ca}{a^2b^2c^2} \quad \text{and} \quad f(a, b, c, d) = -\left(\frac{abc+acd+abd+bcd}{a^2b^2c^2d^2}\right)$$

(*iii*) $$\text{RHS} = g(x_1)\,\frac{h(x_2)-h(x_1)}{x_2-x_1} + \frac{g(x_2)-g(x_1)}{x_2-x_1}\,h(x_2)$$

$$= \frac{1}{x_2-x_1}\,[\{g(x_1)\,h(x_2) - g(x_1)h(x_1)\} + \{g(x_2)\,h(x_2) - g(x_1)\,h(x_2)\}]$$

$$= \frac{g\,(x_2)\,h\,(x_2) - g\,(x_1)\,h\,(x_1)}{x_2-x_1} = \underset{x_2}{\Delta}\, g(x_1)\,h(x_1) = \underset{x_2}{\Delta}\, f(x_1) = f(x_1, x_2) = \text{LHS}.$$

Example 2. (*i*) *Prove that* $\underset{bcd}{\Delta}^3\left(\frac{1}{a}\right) = -\frac{1}{abcd}$

(*ii*) *Show that the* n^{th} *divided differences* $[x_0, x_1, \ldots, x_n]$ *for* $u_x = \frac{1}{x}$ *is* $\left[\frac{(-1)^n}{x_0\,x_1\,\ldots\,x_n}\right]$.

Sol. (*i*)

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	$\frac{1}{a}$			
		$\frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ba}$		
b	$\frac{1}{b}$		$(-1)^2\,\frac{1}{abc}$	
		$\frac{\frac{1}{c}-\frac{1}{b}}{c-b} = -\frac{1}{bc}$		$(-1)^3\,\frac{1}{abcd}$
c	$\frac{1}{c}$		$(-1)^2\,\frac{1}{bdc}$	
		$\frac{\frac{1}{d}-\frac{1}{c}}{d-c} = -\frac{1}{dc}$		
d	$\frac{1}{d}$			

From the table, we observe that $\underset{bcd}{\Delta}^3\left(\frac{1}{a}\right) = -\frac{1}{abcd}$. ...(1)

(*ii*) From (1), we see that

$$\underset{bcd}{\Delta}^3\left(\frac{1}{a}\right) = -\frac{1}{abcd} = (-1)^3\,f(a, b, c, d)$$

∴ In general,

$$\underset{x_0, x_1, \ldots, x_n}{\Delta}^n\left(\frac{1}{x}\right) = (-1)^n\,f\,(x_0, x_1, x_2, \ldots, x_n) = \left[\frac{(-1)^n}{x_0\;x_1\;x_2 \ldots x_n}\right].$$

Example 3. *Using Newton's divided difference formula, find a polynomial function satisfying the following data:*

x:	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

Hence find f(1). (U.P.T.U. 2014)

Sol. The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	$\boxed{1245}$				
		$\boxed{-404}$			
-1	33		$\boxed{94}$		
		-28		$\boxed{-14}$	
0	5		10		$\boxed{3}$
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = 1245 + (x + 4)(-404) + (x + 4)(x + 1)\,94$$
$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

Hence, $f(1) = 3 - 5 + 6 - 14 + 5 = -5$.

Example 4. *Find $f'(10)$ from the following data:*

x:	3	5	11	27	34
$f(x)$:	-13	23	899	17315	35606.

Sol. Divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
3	$\boxed{-13}$				
		$\boxed{18}$			
5	23		$\boxed{16}$		
		146		$\boxed{1}$	
11	899		40		0
		1026		1	
27	17315		69		
		2613			
34	35606				

By Newton's divided difference formula,

$$f(x) = -13 + (x - 3)\,18 + (x - 3)(x - 5)16 + (x - 3)(x - 5)(x - 11)1$$

$\therefore$ $$f'(x) = 3x^2 - 6x - 7$$

Put $x = 10$, $f'(10) = 3(10)^2 - 6(10) - 7 = 233$.

Example 5. *Given the following table, find f(x) as a polynomial in powers of (x – 5)*

x:	*0*	*2*	*3*	*4*	*7*	*9*
f(x):	*4*	*26*	*58*	*112*	*466*	*922.*

Sol. Divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	$\boxed{4}$			
		$\boxed{11}$		
2	26		$\boxed{7}$	
		32		$\boxed{1}$
3	58		11	
		54		1
4	112		16	
		118		1
7	466		22	
		228		
9	922			

By Newton's divided difference formula, we get

$$f(x) = 4 + (x - 0)(11) + (x - 0)(x - 2)7 + (x - 0)(x - 2)(x - 3)1$$
$$= x^3 + 2x^2 + 3x + 4$$

In order to express it in power of $(x - 5)$, we use synthetic division, as

5	1	2	3	4
		5	35	190
5	1	7	38	194
		5	60	
5	1	12	98	
		5		
	1	17		

$\therefore \qquad 2x^2 + x^3 + 3x + 4 = (x - 5)^3 + 17(x - 5)^2 + 98\ (x - 5) + 194.$

Example 6. *By means of Newton's divided difference formula, find the values of f(8) and f(15) from the following table*:

x:	*4*	*5*	*7*	*10*	*11*	*13*
f(x):	*48*	*100*	*294*	*900*	*1210*	*2028.*

[G.B.T.U. 2011, U.P.T.U. (MCA) 2009]

Sol. Newton's divided difference formula, using the arguments 4, 5, 7, 10, 11 and 13 is

$$f(x) = f(4) + (x - 4) \underset{5}{\Delta} f(4) + (x - 4)(x - 5) \underset{5,7}{\Delta}^2 f(4) + (x - 4)(x - 5)(x - 7) \underset{5,7,10}{\Delta}^3 f(4) + \ldots(1)$$

The divided difference table is as under:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	$\boxed{48}$				
		$\frac{100-48}{5-4} = \boxed{52}$			
5	100		$\frac{97-52}{7-4} = \boxed{15}$		
		$\frac{294-100}{7-5} = 97$		$\frac{21-15}{10-4} = \boxed{1}$	
7	294		$\frac{202-97}{10-5} = 21$		0
		$\frac{900-294}{10-7} = 202$		$\frac{27-21}{11-5} = 1$	
10	900		$\frac{310-202}{11-7} = 27$		0
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$	
11	1210		$\frac{409-310}{13-10} = 33$		
		$\frac{2028-1210}{13-11} = 409$			
13	2028				

Substituting the values of the divided differences in (1),

$$f(x) = 48 + (x-4) \times 52 + (x-4)(x-5) \times 15 + (x-5)(x-4)(x-7) \times 1$$

Putting $x = 8$ and 15, we get, $f(8) = 448$ and $f(15) = 3150$.

Example 7. *Given that $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 7 = 0.8451$, find the value of $\log_{10} 33$.*

Sol. $\log 30 = 1.4771$, $\log 32 = 5 \log 2 = 5 \times 0.3010 = 1.5050$

$\log 36 = 2(\log 2 + \log 3) = 2 \times (0.3010 + 0.4771) = 1.5562$

$$\log 35 = \log \frac{70}{2} = \log 70 - \log 2 = 1.8451 - 0.3010 = 1.5441.$$

Divided difference table is as under:

x	$10^4 \log_{10} x$	$10^4 \Delta \log_{10} x$	$10^4 \Delta^2 \log_{10} x$	$10^4 \Delta^3 \log_{10} x$
30	$\boxed{14771}$			
		$\frac{279}{2} = \boxed{139.5}$		
32	15050		$\frac{9.2}{5} = \boxed{-1.84}$	
		$\frac{391}{3} = 130.3$		$\frac{0.48}{6} = \boxed{-0.08}$
35	15441		$-\frac{9.3}{7} = -2.32$	
		$\frac{129}{1} = 121$		
36	15562			

Applying Newton's divided difference formula, we get

$$10^4 \log_{10} x = 14771 + (x - 30)(139.5) + (x - 30)(x - 32)(-1.84) + (x - 30)(x - 32)(x - 35)(-0.08)$$

Putting $x = 33$

$$10^4 \log_{10} 33 = 15184.46$$

$$\therefore \quad \log_{10} 33 = 1.5184.$$

Example 8. *The mode of a certain frequency curve $y = f(x)$ is very near to $x = 9$ and the values of frequency density $f(x)$ for $x = 8.9$, 9.0 and 9.3 are respectively equal to 0.30, 0.35 and 0.25. Calculate the approximate value of mode.*

Sol. Divided difference table is as under:

x	$100\, f(x)$	$100\, \Delta f(x)$	$100\, \Delta^2 f(x)$
8.9	$\boxed{30}$		
		$\frac{5}{0.9} = \boxed{\frac{50}{9}}$	
9.0	35		$-\frac{350}{9 \times 0.4} = \boxed{-\frac{3500}{36}}$
		$-\frac{10}{0.3} = -\frac{100}{3}$	
9.3	25		

Applying Newton's divided difference formula

$$100\, f(x) = 30 + (x - 8.9) \times \frac{50}{9} + (x - 8.9)(x - 9)\left(-\frac{3500}{36}\right)$$

$$\therefore \quad f(x) = -0.9722x^2 + 17.45833x - 17.597217$$

$$f'(x) = -1.9444\, x + 17.45833$$

Putting $f'(x) = 0$, we get $\quad x = \dfrac{17.45833}{1.9444} = 8.9788$

Also, $\quad f''(x) = -1.9444$ *i.e.,* (–)ve

$\therefore \quad f(x)$ is maximum at $x = 8.9788$. Hence mode is 8.9788.

ASSIGNMENT

1. Construct Newton's interpolation polynomial for the data shown in the following table:

x:	0	2	3	4
$f(x)$:	7	11	28	63

[*U.P.T.U. MCA (SUM) 2009*]

2. (*i*) Given the values:

x:	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Evaluate $f(9)$ using Newton's divided difference formula.

(*ii*) Apply Newton's divided difference formula to find the value of $f(8)$ if

$$f(1) = 3,\ f(3) = 31,\ f(6) = 223,\ f(10) = 1011,\ f(11) = 1343.$$

3. Obtain the Newton's divided difference interpolating polynomial and hence find $f(6)$, $f(5)$ and $f(8)$.

x:	3	7	9	10
$f(x)$:	168	120	72	63

[*G.B.T.U. 2012, U.P.T.U. 2007*]

4. Given that

x:	1	3	4	6	7
y_x:	1	27	81	729	2187

Find y_5. Why does it differ from 3^5 ?

5. Use Newton's divided difference formula to find the interpolating polynomial and hence evaluate $y(9.5)$ from the given data:

x:	7	8	9	10
y:	3	1	1	9

[*G.B.T.U. 2011 ; G.B.T.U. (C.O.) 2011*]

6. (*i*) The following table is given:

x:	0	1	2	5
y:	2	3	12	147

What is the form of the function?

(*ii*) Develop the divided-difference table from the data given below and obtain the interpolation polynomial $f(x)$:

x:	1	3	5	7	11
$f(x)$:	5	11	17	23	29

Also, find the value of $f(19.5)$. (*U.P.T.U. 2009*)

7. Find the function u_x in powers of $x - 1$ given that $u_0 = 8$, $u_1 = 11$, $u_4 = 68$, $u_5 = 123$.

8. For the following table, find $f(x)$ as a polynomial in x using Newton's divided difference formula.

x:	5	6	9	11
$f(x)$:	12	13	14	16

(*U.P.T.U. 2006*)

9. (*i*) Apply Newton's divided difference method to obtain an interpolatory polynomial for the following data:

x :	3	5	7	9	11	13
$f(x)$:	31	51	17	19	90	110

(*U.P.T.U. 2015*)

(*ii*) Find the Newton's divided difference interpolation polynomial for:

x:	0.5	1.5	3.0	5.0	6.5	8.0
$f(x)$:	1.625	5.875	31.0	131.0	282.125	521.0

10. Using Newton's divided difference formula, calculate $f(6)$ from the following data:

x:	1	2	7	8
$f(x)$:	1	5	5	4

11. (*i*) Using Newton divided difference method, find the interpolating polynomial and hence compute $f(3)$ from the following table: [*M.T.U. 2014, G.B.T.U. 2013*]

x:	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

(*ii*) Use Newton's divided difference method to compute $f(5.5)$ from the following data:

x:	0	1	4	5	6
$f(x)$:	1	14	15	6	3

(*G.B.T.U. 2010*)

12. Given the data $f(1) = 4$, $f(2) = 5$, $f(7) = 5$, $f(8) = 4$. Find the the value of $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum. (*M.T.U. 2013*)

13. For the following table, find $f(x)$ as a polynomial in x using Newton's divided difference formula:

x:	– 1	0	3	6	9
$f(x)$:	3	– 6	39	822	1611

Hence compute $f(1)$. (*M.T.U. 2012; U.P.T.U. 2009*)

14. Using the Newton's divided difference formula, find a polynomial which takes the values 3, 12, 15, – 21 when x has the values 3, 2, 1, – 1 respectively. (*U.P.T.U. 2008*)

15. Find Newton's divided differences polynomial for the following data:

x:	-3	-1	0	3	5
$f(x)$:	-30	-22	-12	330	3458

(*M.T.U. 2013*)

16. Compute $f'(3)$ from the following table:

x:	1	2	4	8	10
$f(x)$:	0	1	5	21	27

(*G.B.T.U. 2011*)

17. Find $f'''(5)$ from the data given below:

x:	2	4	9	13	16	21	29
$f(x)$:	57	1345	66340	402052	1118209	4287844	21242820

Answers

1. $x^3 - 2x + 7$ **2.** (*i*) 810 (*ii*) 521

3. $x^3 - 21x^2 + 119x - 27$; 147, 168, 93 **4.** 208.82222, 3^x is not a polynomial

5. $x^3 - 23x^2 + 174x - 431$, 3.625

6. (*i*) $x^3 + x^2 - x + 2$ (*ii*) $-$ 112.3955078; $\dfrac{-x^4}{320} + \dfrac{x^3}{20} - \dfrac{43x^2}{160} + \dfrac{71}{20}x + \dfrac{107}{64}$

7. $(x-1)^3 + 2(x-1)^2 + 4(x-1) + 11$ **8.** $\dfrac{x^3}{20} - \dfrac{7x^2}{6} + \dfrac{557x}{60} - 11.5$

9. (*i*) $f(x) = -0.025\,x^5 + 0.7265625\,x^4 - 6.3125\,x^3 + 9.796875\,x^2 + 79.3375\,x - 177.5234375$
(*ii*) $x^3 + x + 1$ **10.** 6.2381

11. (*i*) $x^3 - 9x^2 + 21x + 1$, 10 (*ii*) 3.096875 **12.** 5.66, $x = 4.5$

13. $f(x) = -\dfrac{257}{270}x^4 + \dfrac{1703}{135}x^3 - \dfrac{377}{30}x^2 - \dfrac{527}{15}x - 6$, $-$ 42.037

14. $f(x) = x^3 - 9x^2 + 17x + 6$ **15.** $5x^4 + 9x^3 - 27x^2 - 21x - 12$

16. 1.97916 **17.** 1626.

TEST YOUR KNOWLEDGE

1. What is meant by convergence of iterative method? (*M.T.U. 2012*)

2. Isolate the roots of the equation $x^3 - 4x + 1 = 0$. (*M.T.U. 2012*)

3. Derive Newton-Raphson formula to find approximate root of the equation $f(x) = 0$. (*M.T.U. 2013*)

4. What is the order of convergence of Newton-Raphson method?

5. Find the missing values in the following table:

x:	0	5	10	15	20	25
y:	6	10	—	17	—	31

6. Prove that:

(*i*) $(\Delta + 1)(1 - \nabla) \equiv 1$ (*ii*) $\delta E^{1/2} \equiv \Delta$

(*iii*) $\left(\frac{1}{2} + \frac{\delta^2}{4}\right)^{1/2} + \frac{\delta}{2} = E^{1/2}$ (*iv*) $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \nabla + \Delta$.

7. Construct the forward difference table for $f(x) = x^3 - 2x^2 + 4x + 5$ for $x = 1, 3, 5, 7$. (*M.T.U. 2012*)

8. Define central difference operator, shift operator and the average operator. (*G.B.T.U. 2012*)

9. Verify that $\nabla E \equiv \Delta$. (*M.T.U. 2012*)

10. Prove that: $e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$ (*M.T.U. 2013*)

11. Show that: $hD \equiv \sinh^{-1}(\mu\delta)$. (*M.T.U. 2013*)

12. Find the value of $\Delta^2(ab^{cx})$. (*M.T.U. 2013*)

13. Show that $E \equiv 1 + \Delta$ and $\Delta = \nabla(1 - \nabla)^{-1}$. [*M.T.U. (MCA) 2012*]

14. Show that the Regula-Falsi method has linear rate of convergence.

15. Define the order of convergence of an iterative method.

16. What do you mean by the rate of convergence of an iterative method? (*M.T.U. 2014*)

17. Show that $\delta = E^{1/2} - E^{-1/2}$. (*M.T.U. 2014*)

18. Define rate of convergence. (*M.T.U. 2014*)

Answers

2. 0.2541, 1.860, – 2.1147 **4.** 2 **5.** 13.25, 22.5.

12. $a(b^{ch} - 1)^2 b^{cx}$

UNIT 5

Numerical Techniques-II

5.1 SOLUTION OF SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS

The systems of simultaneous linear equations arise, both directly in modelling physical situations and indirectly in the numerical solution of other mathematical models. Problems such as determining the potential in certain electrical networks, stresses in a building frame, flow rates in a hydraulic system etc., are all reduced to solving a set of algebraic equations simultaneously.

Linear algebraic systems are also involved in the optimization theory, least squares fitting of data, numerical solution of boundary value problems for ordinary and partial differential equations, statistical inference etc.

Consider a non-homogeneous system of n simultaneous linear algebraic equations in n unknowns as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= b_3 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n &= b_n \end{aligned} \qquad \text{...(1)}$$

Using matrix notation, the above system (1) can be written as

$$AX = B \qquad \text{...(2)}$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$

By finding a solution of the system, we mean to obtain the values of n unknowns $x_1, x_2, x_3, \dots, x_n$ such that they satisfy the given equations. If B = O, then the system is called homogeneous.

The methods of solution of linear algebraic eqns. (1) may be classified into two types:

(*i*) Direct methods. These methods yield the exact solution after a finite number of steps in absence of round-off errors. In these methods, the amount of computation involved can be specified in advance. They are independent of the desired accuracy.

(*ii*) Iterative methods. These methods give a sequence of approximations which converges when the number of steps tend to infinity.

In some cases, both the direct and iterative methods are combined. First, we may use a direct method and then the solution may be improved by using iterative methods.

Here, we will study a direct and then an iterative method to solve the system (1).

5.2 DIRECT METHODS

Consider a system of linear equations

$$AX = B \quad \text{...(1)}$$

where A is coefficient matrix, X is column matrix of unknowns and B is column matrix of constants. The necessary and sufficient condition for the system (1) to be consistent is

$$\rho(A) = \rho(A : B)$$

where $\rho(A)$ denotes the rank of A and $\rho(A : B)$ denotes the rank of augmented matrix (A : B).

Further if $\rho(A) = \rho(A : B)$ = number of unknowns, the system (1) will have unique solution and if $\rho(A) = \rho(A : B) <$ number of unknowns, the system (1) will have infinite no. of solutions.

Again, the system (1) of equations can also be solved by using the inverse of matrix.

If A is non singular square matrix then premultiplying (1) by A^{-1}, we get

$$A^{-1} (AX) = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}B$$

The inverse of A can be obtained by either $A^{-1} = \dfrac{\text{adj}. A}{|A|}$ or by using elementary row operations or by using Cayley-Hamilton theorem.

Also, Cramer's rule can be used to solve (1). But this method is satisfactory only for a small number of simultaneous equations. Although this method involves simple calculation yet large amount of computation is required to evaluate large order determinants. For an $n \times n$ system, the method involves evaluation of $(n + 1)$ determinants each of n^{th} order. Each determinant involves large number of multiplications. The addition or subtraction time on computers is usually small compared to the time taken for multiplications or divisions time. The method which requires the least multiplication time is the one which is preferred. In addition to this, there are no simple methods to evaluate large order determinants. For example, to evaluate 10×10 determinants, we will have to carry out 10 times evaluation of determinants, each one of which require the evaluation of 9×9 determinants etc. and they naturally need large amount of computation. In view of above, the Cramer's rule is impractical for large systems of equations.

Similarly, the computation of inverse of a matrix is the main problem on computers and it becomes tougher for large system. Hence the method of finding unknowns by $X = A^{-1}B$ is also not practical for large systems.

To avoid these unnecessary computations, some simpler and less time consuming procedures were developed and suggested.

5.3 LU DECOMPOSITION METHOD

This method is also known as the *Triangularization method* or *method of factorization.*

In this method, the coefficient matrix A of the system of equations $AX = B$ is decomposed into the product of a lower triangular matrix L and an upper triangular matrix U so that

$$A = LU \quad \text{...(1)}$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ l_{31} & l_{32} & l_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \dots & l_{mn} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

Using the matrix multiplication and comparing corresponding elements in (1), we obtain

$$l_{i1}\, u_{1j} + l_{i2}\, u_{2j} + \ldots + l_{in}\, u_{nj} = a_{ij},\ 1 \le j \le n$$

where $\quad l_{ij} = 0, \quad j > i \quad$ and $\quad u_{ij} = 0, \quad i > j$

To produce a unique solution, it is convenient to choose either $u_{ii} = 1$ or $l_{ii} = 1$; $1 \le i \le n$.

(i) When we choose $l_{ii} = 1$, the method is called the **Doolittle's method.**

(ii) When we choose $u_{ii} = 1$, the method is called the **Crout's method.**

The given system of equations is

$$AX = B \qquad \ldots(2)$$

$$\Rightarrow \qquad LUX = B \qquad \ldots(3) \mid \text{Using (1)}$$

Let UX = Y then eqn. (3) becomes

$$LY = B \qquad \ldots(4)$$

The unknowns $y_1, y_2, y_3, \ldots, y_n$ in (4) are determined by forward substitution and the unknowns $x_1, x_2, x_3, \ldots, x_n$ in UX = Y are obtained by back substitution.

Note **1.** The method fails if any of the diagonal elements l_{ii} or u_{ii} is zero.

2. LU decomposition is guaranteed when the matrix A is positive definite.

EXAMPLES

Example 1. *Solve, by Crout's method, the following system of equations:*

$$x + y + z = 3$$
$$2x - y + 3z = 16$$
$$3x + y - z = -3. \qquad \text{[G.B.T.U. (M. Tech.) 2010]}$$

Sol. We choose $u_{ii} = 1$ and write

$$A = LU$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}\, u_{12} + l_{22} & l_{21}u_{13} + l_{22}\, u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}\, u_{13} + l_{32}\, u_{23} + l_{33} \end{bmatrix}$$

Equating, we get

$$l_{11} = 1,\ \ l_{21} = 2,\ l_{31} = 3$$
$$l_{11}\, u_{12} = 1 \quad \Rightarrow \quad u_{12} = 1$$
$$l_{11}\, u_{13} = 1 \quad \Rightarrow \quad u_{13} = 1$$
$$l_{21}\, u_{12} + l_{22} = -1 \quad \Rightarrow \quad l_{22} = -3$$
$$l_{31}\, u_{12} + l_{32} = 1 \quad \Rightarrow \quad l_{32} = -2$$
$$l_{31}\, u_{13} + l_{22}\, u_{23} = 3 \quad \Rightarrow \quad u_{23} = -\frac{1}{3}$$
$$l_{31}\, u_{13} + l_{32}\, u_{23} + l_{33} = -1 \quad \Rightarrow \quad l_{33} = -\frac{14}{3}$$

Thus, we get

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 3 & -2 & -\frac{14}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

The given system is

$$AX = B$$

$$\Rightarrow \qquad LUX = B \qquad \text{...(1)}$$

Let UX = Y so that (1) becomes

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 3 & -2 & -\frac{14}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ -3 \end{bmatrix}$$

which gives

$$y_1 = 3$$

$$2y_1 - 3y_2 = 16$$

$$3y_1 - 2y_2 - \frac{14}{3} y_3 = -3$$

$$\Rightarrow \qquad y_1 = 3, y_2 = -\frac{10}{3}, y_3 = 4 \qquad | \text{ By forward substitution}$$

Now, $\qquad UX = Y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -10/3 \\ 4 \end{bmatrix}$$

which gives, $\qquad x + y + z = 3$

$$y - \frac{1}{3} z = -\frac{10}{3}$$

$$z = 4$$

By back substitution

$$x = 1, y = -2, z = 4.$$

Example 2. *Solve the following system of equations by the LU factorization method:*

$$2x + 3y + z = 9$$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8.$$

Sol. We choose $l_{ii} = 1$ and write

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21} u_{11} & l_{21} u_{12} + u_{22} & l_{21} u_{13} + u_{23} \\ l_{31} u_{11} & l_{31} u_{12} + l_{32} u_{22} & l_{31} u_{13} + l_{32} u_{23} + u_{33} \end{bmatrix}$$

Equating, we get

$$u_{11} = 2, \quad u_{12} = 3, \quad u_{13} = 1$$
$$l_{21} u_{11} = 1 \quad \Rightarrow \quad l_{21} = 1/2$$
$$l_{31} u_{11} = 3 \quad \Rightarrow \quad l_{31} = 3/2$$
$$l_{21} u_{12} + u_{22} = 2 \quad \Rightarrow \quad u_{22} = 1/2$$
$$l_{21} u_{13} + u_{23} = 3 \quad \Rightarrow \quad u_{23} = 5/2$$
$$l_{31} u_{12} + l_{32} u_{22} = 1 \quad \Rightarrow \quad l_{32} = -7$$
$$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 2 \quad \Rightarrow \quad u_{33} = 18$$

Thus, we get

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

The given system is

$$AX = B$$

$\Rightarrow$ $$LUX = B \qquad ...(1)$$

Let $\quad UX = Y$ so that (1) becomes

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

which gives

$$y_1 = 9$$
$$\frac{1}{2} y_1 + y_2 = 6$$
$$\frac{3}{2} y_1 - 7 y_2 + y_3 = 8$$

$\Rightarrow$ $\quad y_1 = 9, \quad y_2 = 3/2, \quad y_3 = 5 \qquad$ | By forward substitution

Now, $$UX = Y$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

which gives,

$$2x + 3y + z = 9$$
$$\frac{1}{2} y + \frac{5}{2} z = \frac{3}{2}$$
$$18 z = 5$$

By back substitution

$\Rightarrow$ $$x = \frac{35}{18}, \quad y = \frac{29}{18}, \quad z = \frac{5}{18}$$

Note. We may also choose $u_{ii} = 1$ to get the solution.

Example 3. *Show that the LU decomposition method fails to solve the system of equations*

$$x_1 + x_2 - x_3 = 2$$
$$2x_1 + 2x_2 + 5x_3 = -3$$
$$3x_1 + 2x_2 - 3x_3 = 6$$

whose exact solution is (1, 0, – 1).

Sol. Case I. If we write (taking $l_{ii} = 1$)

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

We obtain,

$$u_{11} = 1, \quad u_{12} = 1, \quad u_{13} = -1$$
$$l_{21}\, u_{11} = 2 \quad \Rightarrow \quad l_{21} = 2$$
$$l_{21}\, u_{12} + u_{22} = 2 \quad \Rightarrow \quad u_{22} = 0$$

Hence LU decomposition method fails as the pivot $u_{22} = 0$.

Case II. If we write (taking $u_{ii} = 1$)

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

We obtain,

$$l_{11} = 1, \quad l_{21} = 2, \quad l_{31} = 3$$
$$l_{11}\, u_{12} = 1 \quad \Rightarrow \quad u_{12} = 1$$
$$l_{11}\, u_{13} = -1 \quad \Rightarrow \quad u_{13} = -1$$
$$l_{21}\, u_{12} + l_{22} = 2 \quad \Rightarrow \quad l_{22} = 0$$

Hence LU decomposition method fails as again the pivot $l_{22} = 0$.

ASSIGNMENT

1. Solve the following system(s) of linear equations by Crout's method.

(*i*) $x_1 + x_2 + x_3 = 1$, $4x_1 + 3x_2 - x_3 = 6$, $3x_1 + 5x_2 + 3x_3 = 4$

(*ii*) $10x + y + z = 12$, $2x + 10y + z = 13$, $2x + 2y + 10z = 14$

(*iii*) $3x - y + 2z = 12$, $x + 2y + 3z = 11$, $2x - 2y - z = 2$ [*M.T.U. 2012, G.B.T.U. 2011, 2013*]

(*iv*) $x_1 + x_2 + x_3 = 1$, $3x_1 + x_2 - 3x_3 = 5$, $x_1 - 2x_2 - 5x_3 = 10$ (*M.T.U. 2013*)

(*v*) $$\begin{bmatrix} 2 & 1 & -4 & 1 \\ -4 & 3 & 5 & -2 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 2 \\ -1 \end{bmatrix}.$$ (*U.P.T.U. 2007*)

2. Solve the following system of linear equations by LU decomposition method.

(*i*) $x_1 + 2x_2 + 3x_3 = 14$
$2x_1 + 5x_2 + 2x_3 = 18$
$3x_1 + x_2 + 5x_3 = 20$

(*ii*) $\begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$ (*U.P.T.U. 2009*)

(*iii*) $5x - 2y + z = 4$
$7x + y - 5z = 8$
$3x + 7y + 4z = 10.$ (*G.B.T.U. 2012*)

3. Solve the following system of equations by triangularization method.

(*i*) $2x - 3y + 10z = 3$
$-x + 4y + 2z = 20$
$5x + 2y + z = -12$

(*ii*) $x_1 + 2x_2 - x_3 = 3$
$x_1 - x_2 + x_3 = -1$
$2x_1 - 2x_2 + 3x_3 = 2$

(*iii*) $x_1 + x_2 - x_3 = 2$
$2x_1 + 3x_2 + 5x_3 = -3$
$3x_1 + 2x_2 - 3x_3 = 6.$

4. Obtain the LU decomposition of the matrix

$$\begin{bmatrix} 2 & -6 & 10 \\ 1 & 5 & 1 \\ -1 & 15 & -5 \end{bmatrix}$$

so that for $1 \le i \le 3$,

(*i*) $u_{ii} = 2$ (*ii*) $l_{ii} = 2.$

5. Decompose $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & -2 \\ 3 & 2 & -4 \end{bmatrix}$ in the form LU where L is lower triangular matrix and U is the upper triangular matrix and hence solve the system of equations:

$$4x_1 + x_2 + x_3 = 4$$
$$x_1 + 4x_2 - 2x_3 = 4$$
$$3x_1 + 2x_2 - 4x_3 = 6$$

(*U.P.T.U. 2014*)

Answers

1. (*i*) $x_1 = 1, x_2 = 0.5, x_3 = -0.5$ (*ii*) $x = 1, y = 1, z = 1$
(*iii*) $x = 3, y = 1, z = 2$ (*iv*) $x_1 = 6, x_2 = -7, x_3 = 2$
(*v*) $x_1 = 1, x_2 = -1, x_3 = -1, x_4 = -1$

2. (*i*) $x_1 = 1, x_2 = 2, x_3 = 3$ (*ii*) $x_1 = 5, x_2 = 6, x_3 = -10, x_4 = 8$
(*iii*) $x = 1.11926, y = 0.8685, z = 0.14067$

3. (*i*) $x = -4, y = 3, z = 2$ (*ii*) $x_1 = -1, x_2 = 4, x_3 = 4$
(*iii*) $x_1 = 1, x_2 = 0, x_3 = -1$

4. (*i*) $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 4 & 0 \\ -1/2 & 6 & 3 \end{bmatrix}, U = \begin{bmatrix} 2 & -6 & 10 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ (*ii*) $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{bmatrix}, U = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 4 & -2 \\ 0 & 0 & 3 \end{bmatrix}$

5. $x_1 = 1, x_2 = 0.5, x_3 = -0.5$

5.4 CHOLESKY METHOD

This method is also called square-root method. Consider a system of equations

$$AX = B \quad ...(1)$$

If the coefficient matrix A is symmetric and positive definite then A can be decomposed as

$$A = LL^T \quad ...(2)$$

where L is a lower triangular matrix.

A may also be decomposed as $A = UU^T$ where U is an upper triangular matrix.

From (1) and (2),

$$LL^TX = B$$

$$\Rightarrow \quad LY = B \quad ...(3)$$

where $\quad L^TX = Y \quad ...(4)$

The values y_i, $1 \le i \le n$ can be obtained by forward substitution and the solution x_i, $1 \le i \le n$ are obtained by back substitution.

Example. *Solve the following system of equations*

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

using Cholesky method.

Sol. Let $\quad A = LL^T \quad$ | $\because$ A is symmetric

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Equating the corresponding elements, we get

$$\begin{aligned} l_{11}^2 &= 1 & \Rightarrow \quad l_{11} &= 1 \\ l_{11}\, l_{21} &= 2 & \Rightarrow \quad l_{21} &= 2 \\ l_{11}\, l_{31} &= 3 & \Rightarrow \quad l_{31} &= 3 \\ l_{21}^2 + l_{22}^2 &= 8 & \Rightarrow \quad l_{22} &= 2 \\ l_{31}\, l_{21} + l_{32}\, l_{22} &= 22 & \Rightarrow \quad l_{32} &= 8 \\ l_{31}^2 + l_{32}^2 + l_{33}^2 &= 82 & \Rightarrow \quad l_{33} &= 3 \end{aligned}$$

Hence, $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}$

The given system can be written as $LL^TX = B$

or $LY = B$ where $L^TX = Y$

From $LY = B$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$\Rightarrow$ $y_1 = 5, y_2 = -2, y_3 = -3$ | By forward substitution

From $L^TX = Y$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

$\Rightarrow$ $x_1 = 2, \quad x_2 = 3, \quad x_3 = -1.$ | By backward substitution

ASSIGNMENT

1. Solve the system of equations

$$\begin{aligned} 4x_1 - x_2 &= 1 \\ -x_1 + 4x_2 - x_3 &= 0 \\ -x_2 + 4x_3 &= 0 \end{aligned}$$

by the Cholesky method.

Answer

1. $x_1 = \dfrac{15}{56}, \quad x_2 = \dfrac{1}{14}, \quad x_3 = \dfrac{1}{56}.$

5.5 ITERATIVE METHODS

The iterative or indirect methods start from an approximation to the true solution and if convergent, derive a sequence of closer approximations. The cycle of computations is repeated till the desired accuracy is attained. In these methods, the amount of computation depends on the accuracy required.

Generally, direct methods are preferred for the solution of a linear system of equations but in case of matrices having large number of zero elements, iterative methods are preferred since they preserve those elements.

The direct method discussed so far involve many subtractions. When the terms involved in subtractions are nearly equal, their difference is nearly zero and hence causes inaccuracies. The inaccuracies due to this inherent weakness of the direct methods cannot be completely avoided whereas the iterative methods are free from such inaccuracies.

The disadvantage of the iterative methods is that they can not be applied to all systems of equations. The sufficient condition for their use is that the system of equations should be diagonally dominant. In other words, after rearranging the equations, if necessary, the larger coefficients must be along the leading diagonal of the coefficient matrix. Successful use of iteration process requires that the moduli of diagonal coefficients of given systems should be large in comparison with the moduli of nondiagonal coefficients.

Among the iterative methods, we will first discuss Jacobi's method and then Gauss-Seidel method. It is known that for a given system of equations, the Gauss-Seidel method converges more rapidly than the Jacobi's method.

5.6 JACOBI'S ITERATIVE METHOD OR METHOD OF SIMULTANEOUS DISPLACEMENTS

Consider a system of n simultaneous linear equations in n unknowns as

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}\,x_n &= b_n \end{aligned}\right\} \quad \ldots(1)$$

where a_{ii} is the largest coefficient in the i^{th} equation $(1 \le i \le n)$.

System (1) of equations can be rewritten as

$$\left.\begin{aligned} x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}\,x_2 - \frac{a_{13}}{a_{11}}\,x_3 - \ldots - \frac{a_{1n}}{a_{11}}\,x_n \\ x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}\,x_1 - \frac{a_{23}}{a_{22}}\,x_3 - \ldots - \frac{a_{2n}}{a_{22}}\,x_n \\ &\vdots \quad\quad\quad \vdots \quad\quad\quad \vdots \quad\quad\quad \vdots \\ x_n &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}\,x_1 - \frac{a_{n2}}{a_{nn}}\,x_2 - \ldots - \frac{a_{n(n-1)}}{a_{nn}}\,x_{n-1} \end{aligned}\right\} \quad \ldots(2)$$

Let the first approximations to the unknowns $x_1, x_2, \ldots, x_n$ be $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$. Put the first approximation in RHS of (2) to get the results as $x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots, x_n^{(2)}$, a system of second approximations.

This entire process is repeated until the values of $x_1, x_2, \ldots, x_n$ are obtained to desired accuracy *i.e.,* until successive values of each of the unknowns are sufficiently close.

Jacobi's method, if suitable to a system of equations, converges for any value of initial approximations. Generally they are taken as $x_1^{(0)} = 0, x_2^{(0)} = 0, x_3^{(0)} = 0, \ldots$ and so on in absence of any better estimates.

Example. *Solve the following system of equations by Jacobi's method:*

$$\begin{aligned} 8x - 3y + 2z &= 20 \\ 6x + 3y + 12z &= 35 \\ 4x + 11y - z &= 33. \end{aligned}$$

Sol. The coefficient matrix of the given system is not diagonally dominant. Hence we rearrange the equations such that the elements in the coefficient matrix are diagonally dominant.

$$\left.\begin{aligned} 8x - 3y + 2z &= 20 \\ 4x + 11y - z &= 33 \\ 6x + 3y + 12z &= 35 \end{aligned}\right\} \quad \ldots(1)$$

Now, we write the equations in the form

$$\left.\begin{aligned} x &= \frac{1}{8}\,(20 + 3y - 2z) \\ y &= \frac{1}{11}\,(33 - 4x + z) \\ z &= \frac{1}{12}\,(35 - 6x - 3y) \end{aligned}\right\} \quad \ldots(2)$$

We start from an initial approximation $x^{(0)} = 0$, $y^{(0)} = 0$ and $z^{(0)} = 0$. Substituting them in (2), we get

First approximation

$$x^{(1)} = \frac{1}{8}\,[20 + 3(0) - 2(0)] = 2.5$$

$$y^{(1)} = \frac{1}{11}\,[33 - 4(0) + (0)] = 3$$

$$z^{(1)} = \frac{1}{12}\,[35 - 6(0) - 3(0)] = 2.9166667$$

Second approximation

$$x^{(2)} = \frac{1}{8}\,[20 + 3y^{(1)} - 2z^{(1)}] = 2.895833$$

$$y^{(2)} = \frac{1}{11}\,[33 - 4x^{(1)} + z^{(1)}] = 2.3560606$$

$$z^{(2)} = \frac{1}{12}\,[35 - 6x^{(1)} - 3y^{(1)}] = 0.9166666$$

Third approximation

$$x^{(3)} = \frac{1}{8}\,[20 + 3y^{(2)} - 2z^{(2)}] = 3.1543561$$

$$y^{(3)} = \frac{1}{11}\,[33 - 4x^{(2)} + z^{(2)}] = 2.030303$$

$$z^{(3)} = \frac{1}{12}\,[35 - 6x^{(2)} - 3y^{(2)}] = 0.8797348$$

Proceeding in this way, after 12^{th} approximation, we find that the two successive approximations are same upto four decimal places. Hence at this stage, we get

$$x = 3.0167,\ y = 1.9858 \text{ and } z = 0.9118.$$

ASSIGNMENT

1. Solve the following system of equations by Jacobi iteration method:

(*i*)
$$\begin{aligned} 3x + 4y + 15z &= 54.8 \\ x + 12y + 3z &= 39.66 \\ 10x + y - 2z &= 7.74 \end{aligned}$$

(*ii*)
$$\begin{aligned} 54x + y + z &= 110 \\ 2x + 15y + 6z &= 72 \\ -x + 6y + 27z &= 85 \end{aligned}$$

(*iii*)
$$\begin{aligned} 10x_1 + 2x_2 + x_3 &= 9 \\ x_1 + 10x_2 - x_3 &= -22 \\ -2x_1 + 3x_2 + 10x_3 &= 22 \end{aligned}$$

(*iv*)
$$\begin{aligned} 30x - 2y + 3z &= 75 \\ x + 17y - 2z &= 48 \\ x + y + 9z &= 15 \end{aligned}$$

(*v*)
$$\begin{aligned} 10x_1 - x_2 &= 17 \\ -x_1 + 10x_2 - x_3 &= 24 \\ -x_1 + 10x_3 - x_4 &= 32 \\ -x_3 + 10x_4 &= 46 \end{aligned}$$

2. Solve the following system of equations using Jacobi's method:

$$5x - y + z = 10$$
$$2x + 4y = 12$$
$$x + y + 5z = -1$$

Start with the solution (2, 3, 0).

Answers

1. (*i*) $x = 1.074, y = 2.524, z = 2.765$ (*ii*) $x = 1.926, y = 3.573, z = 2.425$

(*iii*) $x_1 = 1, x_2 = -2, x_3 = 3$ (*iv*) $x = 2.580, y = 2.798, z = 1.069$

(*v*) $x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 5$ **2.** $x = 2.555, y = 1.722, z = -1.05$

5.7 GAUSS-SEIDEL ITERATIVE METHOD *OR* METHOD OF SUCCESSIVE DISPLACEMENTS

Consider a system of n simultaneous linear eqns. in n unknowns as

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots(1)$$

where a_{ii} is the largest coefficient in the i^{th} equation $(1 \le i \le n)$.

We solve each equation of the system (1) for the unknown with the largest coefficient interms of the remaining unknown. System (1) can be written as

$$\left.\begin{array}{l} x_1 = \dfrac{b_1}{a_{11}} - \dfrac{a_{12}}{a_{11}} x_2 - \dfrac{a_{13}}{a_{11}} x_3 - \ldots - \dfrac{a_{1n}}{a_{11}} x_n \\[2ex] x_2 = \dfrac{b_2}{a_{22}} - \dfrac{a_{21}}{a_{22}} x_1 - \dfrac{a_{23}}{a_{22}} x_3 - \ldots - \dfrac{a_{2n}}{a_{22}} x_n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ x_n = \dfrac{b_n}{a_{nn}} - \dfrac{a_{n1}}{a_{nn}} x_1 - \dfrac{a_{n2}}{a_{nn}} x_2 - \ldots - \dfrac{a_{n(n-1)}}{a_{nn}} x_{n-1} \end{array}\right\} \quad \ldots(2)$$

In the first equation of (2), we substitute the first approximation $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)}$ into RHS. and denote the result as $x_1^{(2)}$.

In the second equation, we substitute $x_1^{(2)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)}$ and denote the result as $x_2^{(2)}$. In the third equation, we substitute $x_1^{(2)}, x_2^{(2)}, x_3^{(1)}, \ldots, x_n^{(1)}$ and denote the result as $x_3^{(2)}$.

In this manner, we complete the first stage of iteration and the entire process is recpeated till the values of $x_1, x_2, x_3, \ldots, x_n$ are obtained to the accuracy required.

This method uses an improved component as soon as it is available therefore it is also called the **method of successive displacements.**

For any choice of first approximation $x_j^{(1)}$ $(1 \le j \le n)$, Gauss-Seidel method converges if every equation of (2) satisfies the condition $\sum_{j=1, j\ne i}^{n} \left| \dfrac{a_{ij}}{a_{ii}} \right| \le 1, (1 \le i \le n)$, where '<' sign should be valid in case of at least one equation.

EXAMPLES

Example 1. *Use Gauss-Seidel iterative method to solve the following system of simultaneous equations:*

$$9x + 4y + z = -17$$
$$x - 2y - 6z = 14$$
$$x + 6y = 4$$

Perform four iterations. (U.P.T.U. 2014)

Sol. The given system is not diagonally dominant. Hence rearranging the equations as

$$9x + 4y + z = -17$$
$$x + 6y = 4$$
$$x - 2y - 6z = 14$$

Now, Gauss-Seidel's iterative method can be applied.

From the above equations, we get

$$x = \frac{1}{9}(-17 - 4y - z) \qquad ...(1)$$

$$y = \frac{1}{6}(4 - x) \qquad ...(2)$$

$$z = \frac{1}{6}(x - 2y - 14) \qquad ...(3)$$

First approximation

Starting with $y = 0 = z$, we obtain

$$x^{(1)} = -\frac{17}{9} = -1.8888$$

Now, putting $x = -1.888$ and $z = 0$ in eqn. (2), we get

$$y^{(1)} = 0.9815$$

Again putting $x = -1.888$ and $y = 0.9815$ in eqn. (3), we get

$$z^{(1)} = -2.9753$$

Second approximation

$$x^{(2)} = \frac{1}{9}\{-17 - 4y^{(1)} - z^{(1)}\} = -1.9945$$

$$y^{(2)} = \frac{1}{6}\{4 - x^{(2)}\} = 0.9991$$

$$z^{(2)} = \frac{1}{6}\{x^{(2)} - 2y^{(2)} - 14\} = -2.9988$$

Third approximation

$$x^{(3)} = -1.9997$$
$$y^{(3)} = 0.9999$$
$$z^{(3)} = -2.9999$$

Fourth approximation

$$x^{(4)} = -1.9999 \approx -2$$
$$y^{(4)} = 0.9999 \approx 1$$
$$z^{(4)} = -2.9999 \approx -3$$

Hence after four iterations, we obtain

$$x = -2, y = 1, z = -3.$$

Example 2. *Solve the following system of equations using Gauss-Seidel iterative method:*

$$2x + 10y + z = 51$$
$$10x + y + 2z = 44$$
$$x + 2y + 10z = 61$$

[U.P.T.U. (MCA) 2009]

Sol. The given system is not diagonally dominant. Hence rearranging the equations as

$$10x + y + 2z = 44$$
$$2x + 10y + z = 51$$
$$x + 2y + 10z = 61$$

Now, Gauss-Seidel's iterative method can be applied.

From the above equations, we get

$$x = \frac{1}{10}(44 - y - 2z) \quad ...(1)$$

$$y = \frac{1}{10}(51 - 2x - z) \quad ...(2)$$

$$z = \frac{1}{10}(61 - x - 2y) \quad ...(3)$$

First approximation.

Starting with $y = 0 = z$, we obtain from (1),

$$x^{(1)} = 4.4$$

Now, putting $x = 4.4$ and $z = 0$ in equation (2), we get

$$y^{(1)} = 4.22$$

Again, putting $x = 4.4$ and $y = 4.22$ in equation (3), we get

$$z^{(1)} = 4.816$$

Second approximation.

$$x^{(2)} = 3.0148$$
$$y^{(2)} = 4.01544$$
$$z^{(2)} = 4.995432$$

Third approximation

$$x^{(3)} = 2.9993696$$
$$y^{(3)} = 4.00058288$$
$$z^{(3)} = 4.999946464$$

Fourth approximation.

$$x^{(4)} = 2.999952419 \approx 3$$
$$y^{(4)} = 4.00001487 \approx 4$$
$$z^{(4)} = 5.000001784 \approx 5$$

Hence after four iterations, we obtain

$$x = 3, y = 4 \quad \text{and} \quad z = 5$$

Example 3. *Solve the following system of equations by Gauss-Seidel iterative method:*

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

[M.T.U. 2013; U.P.T.U. 2009]

Sol. The given system is diagonally dominant hence, from the given equations, we obtain

$$x = \frac{1}{20}(17 - y + 2z) \quad \text{...(1)}$$

$$y = \frac{1}{20}(-18 - 3x + z) \quad \text{...(2)}$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad \text{...(3)}$$

First Approximation

Starting with $y = 0 = z$, we obtain from (1),

$$x^{(1)} = 0.85$$

Now, putting $x = 0.85$, $z = 0$ in equation (2), we get

$$y^{(1)} = -1.0275$$

Again, putting $x = 0.85$ and $y = -1.0275$ in eqn. (3), we get

$$z^{(1)} = 1.010875$$

Second Approximation

$$x^{(2)} = 1.0024625$$
$$y^{(2)} = -0.999825625$$
$$z^{(2)} = 0.999779906$$

Third Approximation

$$x^{(3)} = 0.999969271 \approx 1$$
$$y^{(3)} = -1.000006395 \approx -1$$
$$z^{(3)} = 1.000002114 \approx 1$$

Hence after three approximations, we obtain,

$$x = 1, y = -1, z = 1$$

ASSIGNMENT

1. Test if the following system of equations is diagonally dominant and hence solve this system using Gauss-Seidel method:

$$2x_1 + x_2 + 4x_3 = 7$$
$$3x_1 + x_2 + 2x_3 = 6$$
$$-x_1 + 4x_2 + 2x_3 = 5$$

(*G.B.T.U. 2011*)

2. Define diagonally dominant system of equations. Solve the following system of equations using Gauss-Seidel method:

$$10x + 15y + 3z = 14$$
$$-30x + y + 5z = 17$$
$$x + y + 4z = 3$$

(*G.B.T.U. 2012*)

3. (*i*) Obtain the solution of the following system of equations using Gauss-Seidel iterative method:

$$2x - 7y - 10z = -17$$
$$5x + y + 3z = 14$$
$$x + 10y + 9z = 7$$

Perform three iterations. [*G.B.T.U. (MCA) 2011*]

(*ii*) Solve the following system of linear equations using Gauss-Seidel method:

$$10x + 3y + 7z = 41$$
$$3x + 20y + 17z = 101$$
$$x + 19y + 23z = 201.$$

Perform three iterations. (*U.P.T.U. 2015*)

4. Solve by Gauss-Seidel iterative method:

(*i*) $10x + 2y + z = 9$, $2x + 20y - 2z = -44$, $-2x + 3y + 10z = 22$

(*ii*) $3x_1 + 2x_2 - x_3 = 7$, $x_1 - 3x_2 + 2x_3 = 4$, $-x_1 + x_2 - 3x_3 = -1$

(*iii*) $x + 0.01y - 0.02z = 3.3354$, $0.02x + y - 0.05z = 4.8241$, $0.03x - 0.01y + z = 7.341$

(*iv*) $10x_1 - x_2 + 2x_3 = 4$, $x_1 + 10x_2 + x_3 = 7$, $2x_1 + 3x_2 + 20x_3 = 13$

(*v*) $6x_1 - 3x_2 + x_3 = 11$, $x_1 - 7x_2 + x_3 = 10$, $2x_1 + x_2 - 8x_3 = -15$ (*G.B.T.U. 2013, 2011*)

(*vi*) $8x - 3y + 2z = 20$, $6x + 3y + 12z = 35$, $4x + 11y - z = 33$ [*G.B.T.U. (M. Tech.) 2010*]

(*vii*) $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$ (*U.P.T.U. 2015*)

(*viii*) $83x + 11y - 4z = 95$, $7x + 52y + 13z = 104$, $3x + 8y + 29z = 71$ (*M.T.U. 2014*)

5. Describe a method for solving a system of linear equations. Solve the following system of linear equations using Gauss-Seidel method:

$$23x_1 + 13x_2 + 3x_3 = 29$$
$$5x_1 + 23x_2 + 7x_3 = 37$$
$$11x_1 + x_2 + 23x_3 = 43.$$

(*U.P.T.U. 2009*)

6. Solve the following equations by Gauss-Seidel procedure:

$$9x_1 + 2x_2 + 4x_3 = 20$$
$$x_1 + 10x_2 + 4x_3 = 6$$
$$2x_1 - 4x_2 + 10x_3 = -15$$

The answer should be correct to 3 significant digits. [*U.P.T.U. MCA (SUM) 2009*]

7. Using Gauss-Seidel method, solve the following system of linear algebraic equations

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

correct up to four decimal places. Perform seven iterations.

Answers

1. $x_1 = 1, x_2 = 1, x_3 = 1$ **2.** $x = -0.4328, y = 1.1055, z = 0.5818$

3. (*i*) $x = 1, y = -3, z = 4$ (*ii*) $x = -3.9429, y = -6.4887, z = 14.2708$

4. (*i*) $x = 1, y = -2, z = 3$ (*ii*) $x_1 = 2.72, x_2 = -1.04, x_3 = -0.92$

(*iii*) $x = 3.429, y = 5.119, z = 7.289$ (*iv*) $x_1 = 0.3567, x_2 = 0.6120, x_3 = 0.5225$
(*v*) $x_1 = 1, x_2 = -1, x_3 = 2$ (*vi*) $x = 3.016, y = 1.985, z = 0.9118$
(*vii*) $x = 2.425, y = 3.573, z = 1.925$ (*viii*) $x = 1.057, y = 1.367, z = 1.961$.

5. $x_1 = 0.4765, x_2 = 1.0189, x_3 = 1.5973$ **6.** $x_1 = 2.74, x_2 = 0.987, x_3 = -1.65$

7. $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0$

5.8 INTRODUCTION TO NUMERICAL DIFFERENTIATION AND INTEGRATION

Consider a function of a single variable $y = f(x)$. If $f(x)$ is defined as an expression, its derivative or integral may often be determined using the techniques of calculus.

However, when $f(x)$ is a complicated function or when it is given in a tabular form, we use numerical methods.

Here, we will discuss numerical methods for approximating the derivative(s) $f^{(r)}(x)$, $r \geq 1$ of a given function $f(x)$ and for the evaluation of the integral $\int_a^b f(x)\,dx$ where a, b may be finite or infinite.

The accuracy attainable by these methods would depend on the given function and the order of the polynomial used. If the polynomial fitted is exact then the error would be theoretically zero. In practice, however, rounding errors will introduce errors in the calculated values.

The error introduced in obtaining derivatives is in general much worse than that introduced in determining integrals. It may be observed that any errors in approximating a function are amplified while taking the derivative whereas they are smoothed out in integration.

Thus numerical differentiation should be avoided if an alternative exists.

5.9 NUMERICAL DIFFERENTIATION [*M.T.U. 2012, U.P.T.U. 2008*]

In the case of numerical data, the functional form of $f(x)$ is not known in general. First we have to find an appropriate form of $f(x)$ and then obtain its derivatives. So "**Numerical differentiation**" is concerned with the method of finding the successive derivatives of a function at a given argument, using the given table of entries corresponding to a set of arguments, equally or unequally spaced. Using the theory of interpolation, a suitable interpolating polynomial can be chosen to represent the function to a good degree of approximation in the given interval of the argument.

For the proper choice of interpolation formula, the criterion is same as in case of interpolation problems. In case of equidistant values of x, if the derivative is to be found at a point near the beginning or the end of the given set of values, we should use Newton's forward or backward difference formula accordingly. Also if the derivative is to be found at a point near the middle of the given set of values, then we should use any one of the central difference formulae. However, if the values of the function are not known at equidistant values of x, we shall use Newton's divided difference or Lagrange's formula.

5.10 FORMULAE FOR DERIVATIVES

(1) **Newton's forward difference interpolation formula is**

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \quad \ldots(1)$$

where

$$u = \frac{x-a}{h} \quad \ldots(2)$$

Differentiating eqn. (1) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \ldots \quad \ldots(3)$$

Differentiating eqn. (2) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h} \quad \ldots(4)$$

We know that

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \ldots\right] \quad \ldots(5)$$

Expression (5) provides the value of $\frac{dy}{dx}$ at any x which is not tabulated.

Formula (5) becomes simple for tabulated values of x, in particular when $x = a$ and $u = 0$

Putting $u = 0$ in (5), we get

$$\boxed{\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \ldots\right]} \quad \ldots(6)$$

(*M.T.U. 2013*)

Differentiating eqn. (5) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \ldots\right]\frac{1}{h}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \ldots\right] \quad \ldots(7)$$

Putting $u = 0$ in eqn. (7), we get

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \ldots\right)} \quad \ldots(8)$$

Similarly, we get

$$\boxed{\left(\frac{d^3y}{dx^3}\right)_{x=a} = \frac{1}{h^3}\left(\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right)} \quad \ldots(9)$$

and so on.

Formulae for computing higher derivatives may be obtained by successive differentiation.

Aliter: We know that

$$\mathrm{E} = e^{h\mathrm{D}} \quad \Rightarrow \quad 1 + \Delta = e^{h\mathrm{D}}$$

$$\therefore \qquad h\mathrm{D} = \log(1+\Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots$$

$$\Rightarrow \qquad \mathrm{D} = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]$$

[*U.P.T.U. (MCA) 2007*]

Similarly, $$D^2 = \frac{1}{h^2}\left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right)^2 = \frac{1}{h^2}\left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \ldots\right)$$

and $$D^3 = \frac{1}{h^3}\left(\Delta^3 - \frac{3}{2}\Delta^4 + \ldots\right)$$

(2) **Newton's backward difference interpolation formula is**

$$y = y_n + u\,\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots \qquad \ldots(10)$$

where $u = \dfrac{x - x_n}{h}$ ...(11)

Differentiating eqn. (10) with respect to u, we get

$$\frac{dy}{du} = \nabla y_n + \left(\frac{2u+1}{2}\right)\nabla^2 y_n + \left(\frac{3u^2 + 6u + 2}{6}\right)\nabla^3 y_n + \ldots \qquad \ldots(12)$$

Differentiating eqn. (11) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h} \qquad \ldots(13)$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\nabla y_n + \left(\frac{2u+1}{2}\right)\nabla^2 y_n + \left(\frac{3u^2 + 6u + 2}{6}\right)\nabla^3 y_n + \ldots\right] \qquad \ldots(14)$$

Expression (14) provides us the value of $\dfrac{dy}{dx}$ at any x which is not tabulated.

At $x = x_n$, we have $u = 0$

∴ Putting $u = 0$ in eqn. (14), we get

$$\boxed{\left(\frac{dy}{dx}\right)_{x = x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right)} \qquad \ldots(15)$$

Differentiating eqn. (14) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\,\nabla^3 y_n + \left(\frac{6u^2 + 18u + 11}{12}\right)\nabla^4 y_n + \ldots\right] \qquad \ldots(16)$$

Putting $u = 0$ in eqn. (16), we get

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x = x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right)} \qquad \ldots(17)$$

Similarly, we get

$$\boxed{\left(\frac{d^3y}{dx^3}\right)_{x = x_n} = \frac{1}{h^3}\left(\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right)} \qquad \ldots(18)$$

and so on.

Formulae for computing higher derivatives may be obtained by successive differentiation.

Aliter: We know that

$$E^{-1} = 1 - \nabla$$

$$e^{-hD} = 1 - \nabla$$

$$\therefore \quad -hD = \log(1-\nabla) = -\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right)$$

$$\Rightarrow \quad D = \frac{1}{h}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \ldots\right)$$

Also, $$D^2 = \frac{1}{h^2}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\right)^2 = \frac{1}{h^2}\left(\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots\right)$$

Similarly, $D^3 = \dfrac{1}{h^3}\left(\nabla^3 + \dfrac{3}{2}\nabla^4 + \ldots\right)$ and so on.

(3) **Stirling's central difference interpolation formula is**

$$y = y_0 + \frac{u}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots \quad \ldots(19)$$

where $$u = \frac{x-a}{h} \quad \ldots(20)$$

Differentiating eqn. (19) with respect to u, we get

$$\frac{dy}{du} = \frac{\Delta y_0 + \Delta y_{-1}}{2} + u\Delta^2 y_{-1} + \left(\frac{3u^2-1}{6}\right)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \left(\frac{4u^3-2u}{4!}\right)\Delta^4 y_{-2} + \left(\frac{5u^4 - 15u^2 + 4}{5!}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots \quad \ldots(21)$$

Differentiating eqn. (20) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h} \quad \ldots(22)$$

Now, $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + u\Delta^2 y_{-1} + \left(\frac{3u^2-1}{6}\right)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)\right.$$

$$\left. + \left(\frac{4u^3-2u}{4!}\right)\Delta^4 y_{-2} + \left(\frac{5u^4 - 15u^2 + 4}{5!}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots\right] \quad \ldots(23)$$

Expression (23) provides us the value of $\dfrac{dy}{dx}$ at any x which is not tabulated.

Put $x = a$, we have $u = 0$

$\therefore$ Putting $u = 0$ in eqn. (23), we get

$$\boxed{\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) - \ldots\right]}$$

$$\ldots(24)$$

Differentiating eqn. (23) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_{-1} + u\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \left(\frac{6u^2 - 1}{12}\right)\Delta^4 y_{-2} + \left(\frac{2u^3 - 3u}{12}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots\right] \qquad \ldots(25)$$

Putting $u = 0$ in eqn. (25), we get

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \ldots\right)} \qquad \ldots(26)$$

and so on.

Formulae for computing higher derivatives may be obtained by successive differentiation.

(4) **Bessel's central difference interpolation formula is**

$$y = \left(\frac{y_0 + y_1}{2}\right) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{u(u-1)\left(u - \frac{1}{2}\right)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(u+1)\,u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \frac{(u+1)\,u(u-1)(u-2)\left(u - \frac{1}{2}\right)}{5!}\Delta^5 y_{-2}$$

$$+ \frac{(u+2)(u+1)\,u(u-1)(u-2)(u-3)}{6!}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots \qquad \ldots(27)$$

where $\quad u = \dfrac{x-a}{h} \qquad \ldots(28)$

Differentiating eqn. (27) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \left(\frac{2u-1}{2!}\right)\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \left(\frac{3u^2 - 3u + \frac{1}{2}}{3!}\right)\Delta^3 y_{-1}$$

$$+ \left(\frac{4u^3 - 6u^2 - 2u + 2}{4!}\right)\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \left(\frac{5u^4 - 10u^3 + 5u - 1}{5!}\right)\Delta^5 y_{-2}$$

$$+ \left(\frac{6u^5 - 15u^4 - 20u^3 + 45u^2 + 8u - 12}{6!}\right)\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots \qquad \ldots(29)$$

Differentiating eqn. (28) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h}$$

Now, $\qquad \dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$$= \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2!}\right)\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \left(\frac{3u^2 - 3u + \frac{1}{2}}{3!}\right)\Delta^3 y_{-1}\right.$$

$$+ \left(\frac{4u^3 - 6u^2 - 2u + 2}{4!}\right)\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \left(\frac{5u^4 - 10u^3 + 5u - 1}{5!}\right)\Delta^5 y_{-2}$$

$$\left. + \left(\frac{6u^5 - 15u^4 - 20u^3 + 45u^2 + 8u - 12}{6!}\right)\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots\right] \quad \ldots(30)$$

Expression (30) provides us the value of $\frac{dy}{dx}$ at any x which is not tabulated.

Put $x = a$, we have $u = 0$

$\therefore$ Putting $u = 0$ in eqn. (30), we get

$$\boxed{\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{60}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots\right]} \quad \ldots(31)$$

Differentiating eqn. (30) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \left(\frac{2u-1}{2}\right)\Delta^3 y_{-1} + \left(\frac{6u^2 - 6u - 1}{12}\right)\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)\right.$$

$$+ \left(\frac{4u^3 - 6u^2 + 1}{24}\right)\Delta^5 y_{-2}$$

$$\left. + \left(\frac{15u^4 - 30u^3 - 30u^2 + 45u + 4}{360}\right)\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots\right] \quad \ldots(32)$$

Putting $u = 0$ in eqn. (32), we get

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left[\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) - \frac{1}{2}\Delta^3 y_{-1} - \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \frac{1}{24}\Delta^5 y_{-2} + \frac{1}{90}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots\right]} \quad \ldots(33)$$

and so on.

(5) **For unequally spaced values of the argument**

(*i*) **Newton's divided difference formula is**

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)$$
$$(x - x_2)\,\Delta^3 f(x_0) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta^4 f(x_0) + \ldots \quad \ldots(34)$$

$f'(x)$ is given by

$$f'(x) = \Delta f(x_0) + \{2x - (x_0 + x_1)\}\ \Delta^2 f(x_0) + \{3x^2 - 2x(x_0 + x_1 + x_2) + (x_0x_1 + x_1x_2 + x_2x_0)\}\ \Delta^3 f(x_0) + \ldots \quad \ldots(35)$$

(*ii*) **Lagrange's interpolation formula is**

$$f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} f(x_1) + \ldots \quad \ldots(36)$$

$f'(x)$ can be obtained by differentiating $f(x)$ in eqn. (36).

Note 1. Formula (8) can be extended as

$$\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \frac{137}{180}\Delta^6 - \frac{7}{10}\Delta^7 + \frac{363}{560}\Delta^8 + \ldots\right) y_0$$

2. Formula (17) can be extended as

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \frac{137}{180}\nabla^6 + \frac{7}{10}\nabla^7 + \frac{363}{560}\nabla^8 + \ldots\right) y_n.$$

5.11 MAXIMA AND MINIMA OF A TABULATED FUNCTION

Since maxima and minima of $y = f(x)$ can be found by equating $\frac{dy}{dx}$ to zero and solving the equation for the argument x, the same method can be used to determine maxima and minima of tabulated function by differentiating the interpolating polynomial.

e.g., if Newton's forward difference formula is used, we have

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \quad \ldots(1)$$

Differentiating eqn. (1) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \ldots$$

For maxima or minima,

$$\frac{dy}{du} = 0$$

$$\Rightarrow \qquad \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \ldots = 0 \quad \ldots(2)$$

If we terminate LHS series after third differences for convenience, eqn. (2) being a quadratic in u gives two values of u.

Corresponding to these values, $x = a + uh$ will give the corresponding x at which function may be maximum or minimum.

For maximum, $\frac{d^2y}{du^2}$ = (–)ve; For minimum, $\frac{d^2y}{du^2}$ = (+)ve.

EXAMPLES

Example 1. *Find* $\frac{dy}{dx}$ *at* $x = 0.1$ *from the following table:*

x:	0.1	0.2	0.3	0.4
y:	0.9975	0.9900	0.9776	0.9604.

Sol. Take $a = 0.1$. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	0.9975			
		$\boxed{-0.0075}$		
0.2	0.9900		$\boxed{-0.0049}$	
		– 0.0124		$\boxed{0.0001}$
0.3	0.9776		– 0.0048	
		– 0.0172		
0.4	0.9604			

Here $\qquad h = 0.1$ and $y_0 = 0.9975$

$$\left[\frac{dy}{dx}\right]_{x=0.1} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right]$$

$$= \frac{1}{0.1}\left[-0.0075 - \frac{1}{2}(-0.0049) + \frac{1}{3}(0.0001)\right] = -0.050167.$$

Example 2. *The table given below reveals the velocity 'v' of a body during the time 't' specified. Find its acceleration at t = 1.1.* (U.P.T.U. 2014)

t:	1.0	1.1	1.2	1.3	1.4
v:	43.1	47.7	52.1	56.4	60.8.

Sol. The difference table is

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
1.0	43.1				
		4.6			
1.1	47.7		– 0.2		
		$\boxed{4.4}$		0.1	
1.2	52.1		$\boxed{-0.1}$		0.1
		4.3		$\boxed{0.2}$	
1.3	56.4		0.1		
		4.4			
1.4	60.8				

Let $a = 1.1$,

$\therefore$ $v_0 = 47.7$ and $h = 0.1$

Acceleration at $t = 1.1$ is given by

$$\left[\frac{dv}{dt}\right]_{t=1.1} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0\right] = \frac{1}{0.1}\left[4.4 - \frac{1}{2}(-0.1) + \frac{1}{3}(0.2)\right] = 45.1667$$

Hence the required acceleration is **45.1667.**

Example 3. *Find f′(1.1) and f″(1.1) from the following table:*

x:	*1.0*	*1.2*	*1.4*	*1.6*	*1.8*	*2.0*
f(x):	*0.0*	*0.1280*	*0.5540*	*1.2960*	*2.4320*	*4.000.*

(G.B.T.U. 2010)

Sol. Since we are to find $f'(x)$ and $f''(x)$ for non-tabular value of x, we proceed as follows :

Newton's forward difference formula is

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots \quad \ldots(1)$$

where $$u = \frac{x-a}{h} \quad \ldots(2)$$

Differentiating eqn. (1) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{2u^3-9u^2+11u-3}{12}\right)\Delta^4 y_0 + \ldots \quad \ldots(3)$$

Differentiating eqn. (2) with respect to x

$$\frac{du}{dx} = \frac{1}{h} \quad \ldots(4)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{2u^3-9u^2+11u-3}{12}\right)\Delta^4 y_0 + \ldots\right] \quad \ldots(5)$$

Also, at $x = 1.1$, $u = \dfrac{1.1-1.0}{0.2} = \dfrac{1}{2}$ | Here $a = 1.0$ and $h = 0.2$

Forward difference table is as follows:

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	0.0					
		$\boxed{0.1280}$				
1.2	0.1280		$\boxed{0.298}$			
		0.4260		$\boxed{0.018}$		
1.4	0.5540		0.316		$\boxed{0.06}$	
		0.7420		0.078		$\boxed{-0.1}$
1.6	1.2960		0.394		– 0.04	
		1.1360		0.038		
1.8	2.4320		0.432			
		1.5680				
2.0	4.000					

From eqn. (5),

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{2u^3-9u^2+11u-3}{12}\right)\Delta^4 y_0 \right.$$
$$\left. + \left(\frac{5u^4-40u^3+105u^2-100u+24}{120}\right)\Delta^5 y_0 + \ldots\right] \qquad \ldots(6)$$

At $x = 1.1$, we get

$$f'(1.1) = \left(\frac{dy}{dx}\right)_{x=1.1} = \frac{1}{0.2}\left[0.1280 + \left\{\frac{2\left(\frac{1}{2}\right)-1}{2}\right\}(0.298)\right.$$
$$+ \left\{\frac{3\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right)+2}{6}\right\}(0.018) + \left\{\frac{2\left(\frac{1}{2}\right)^3 - 9\left(\frac{1}{2}\right)^2 + 11\left(\frac{1}{2}\right)-3}{12}\right\}(0.06)$$
$$\left. + \left\{\frac{5\left(\frac{1}{2}\right)^4 - 40\left(\frac{1}{2}\right)^3 + 105\left(\frac{1}{2}\right)^2 - 100\left(\frac{1}{2}\right)+24}{120}\right\}(-0.1)\right]$$
$$= 0.66724.$$

Differentiating eqn. (6), with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\,\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0\right.$$
$$\left. + \left(\frac{2u^3-12u^2+21u-10}{12}\right)\Delta^5 y_0 + \ldots\right]$$

At $x = 1.1$, we get

$$f''(1.1) = \left(\frac{d^2y}{dx^2}\right)_{x=1.1} = 8.13125.$$

Example 4. *The distance covered by an athlete for the 50 metre race is given in the following table:*

Time (sec):	*0*	*1*	*2*	*3*	*4*	*5*	*6*
Distance (metre):	*0*	*2.5*	*8.5*	*15.5*	*24.5*	*36.5*	*50*

Determine the speed of the athlete at t = 5 sec correct to two decimals. (U.P.T.U. 2009)

Sol. Here we are to find derivative at $t = 5$ which is near the end of the table hence we shall use formula obtained from Newton's backward difference formula. Backward difference table is as follows:

t	s	∇s	$\nabla^2 s$	$\nabla^3 s$	$\nabla^4 s$	$\nabla^5 s$	$\nabla^6 s$
0	0						
		2.5					
1	2.5		3.5				
		6		– 2.5			
2	8.5		1		3.5		
		7		1		$\boxed{-3.5}$	
3	15.5		2		$\boxed{0}$		1
		9		$\boxed{1}$		– 2.5	
4	24.5		$\boxed{3}$		– 2.5		
		$\boxed{12}$		– 1.5			
5	36.5		1.5				
		13.5					
6	50						

Speed of athlete at $t = 5$ sec is given by

$$\left(\frac{ds}{dt}\right)_{t=5} = \frac{1}{h}\left[\nabla s_5 + \frac{1}{2}\nabla^2 s_5 + \frac{1}{3}\nabla^3 s_5 + \frac{1}{4}\nabla^4 s_5 + \frac{1}{5}\nabla^5 s_5\right]$$

$$= \frac{1}{1}\left[12 + \frac{1}{2}(3) + \frac{1}{3}(1) + \frac{1}{4}(0) + \frac{1}{5}(-3.5)\right]$$

$$= 13.1333 \approx 13.13 \text{ metre/sec.}$$

Example 5. *A rod is rotating in a plane. The following table gives the angle* θ *(in radians) through which the rod has turned for various values of time t (in seconds)*

t:	*0*	*0.2*	*0.4*	*0.6*	*0.8*	*1.0*	*1.2*
θ*:*	*0*	*0.12*	*0.49*	*1.12*	*2.02*	*3.20*	*4.67.*

Calculate the angular velocity and angular acceleration of the rod at t = 0.6 sec.

[G.B.T.U. (C.O.) 2011]

Sol. Forward difference table is

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	0.49		0.26		0
		0.63		0.01	
0.6	1.12		$\boxed{0.27}$		0
		$\boxed{0.9}$		$\boxed{0.01}$	
0.8	2.02		$\boxed{0.28}$		0
		1.18		0.01	
1.0	3.20		0.29		
		1.47			
1.2	4.67				

Here $a = 0.6$

$\therefore$ $\theta_0 = 1.12$ and $h = 0.2$

Since we are to find derivatives at $t = 0.6$ sec which is in the middle of the table, hence we shall use formula obtained from Stirling's or Bessel's central difference formula.

Let us choose formula obtained from Bessel's central difference formula.

Angular velocity at $t = 0.6$ sec is given by

$$\left(\frac{d\theta}{dt}\right)_{t=0.6} = \frac{1}{h}\left[\Delta\theta_0 - \frac{1}{2}\left(\frac{\Delta^2\theta_{-1} + \Delta^2\theta_0}{2}\right) + \frac{1}{12}\Delta^3\theta_{-1}\right]$$

$$= \frac{1}{0.2}\left[0.9 - \frac{1}{2}\left(\frac{0.27+0.28}{2}\right) + \frac{1}{12}(0.01)\right] = 3.81667 \text{ rad./sec.}$$

Angular acceleration at $t = 0.6$ sec is given by

$$\left(\frac{d^2\theta}{dt^2}\right)_{t=0.6} = \frac{1}{h^2}\left[\left(\frac{\Delta^2\theta_{-1} + \Delta^2\theta_0}{2}\right) - \frac{1}{2}\Delta^3\theta_{-1}\right]$$

$$= \frac{1}{(0.2)^2}\left[\left(\frac{0.27+0.28}{2}\right) - \frac{1}{2}(0.01)\right] = 6.75 \text{ rad./sec}^2.$$

Note. In case we choose formula obtained from Stirling's formula, at $t = 0.6$ sec.,

angular velocity $$\left(\frac{d\theta}{dt}\right) = \frac{1}{h}\left[\left(\frac{\Delta\theta_0 + \Delta\theta_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3\theta_{-1} + \Delta^3\theta_{-2}}{2}\right)\right]$$

$$= \frac{1}{0.2}\left[\left(\frac{0.9+0.63}{2}\right) - \frac{1}{6}\left(\frac{0.01+0.01}{2}\right)\right] = 3.81667 \text{ rad./sec.}$$

and angular acceleration $$\left(\frac{d^2\theta}{dt^2}\right) = \frac{1}{h^2}(\Delta^2\theta_{-1}) = \frac{1}{(0.2)^2}(0.27) = 6.75 \text{ rad./sec}^2.$$

Example 6. *From the following table of values of x and y, obtain* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *for x = 1.2, 2.2 and 1.6.*

x:	*1.0*	*1.2*	*1.4*	*1.6*	*1.8*	*2.0*	*2.2*
y:	*2.7183*	*3.3201*	*4.0552*	*4.9530*	*6.0496*	*7.3891*	*9.0250.*

Sol. The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		$\boxed{0.0001}$
		1.0966		0.0441		$\boxed{0.0014}$	
1.8	6.0496		0.2429		$\boxed{0.0094}$		
		1.3395		$\boxed{0.0535}$			
2.0	7.3891		$\boxed{0.2964}$				
		$\boxed{1.6359}$					
2.2	9.0250						

(*i*) Here $a = 1.2$

$\therefore$ $y_0 = 3.3201; \quad h = 0.2$

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.008) + \frac{1}{5}(0.0014)\right]$$

$$= 3.3205$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{(0.2)^2}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right] = 3.318$$

(*ii*) Here $a = 2.2$,

$\therefore$ $y_n = 9.02$ and $h = 0.2$

$$\left[\frac{dy}{dx}\right]_{x=2.2} = \frac{1}{0.2}\left[1.6359 + \frac{1}{2}(0.2964) + \frac{1}{3}(0.0535) + \frac{1}{4}(0.0094) + \frac{1}{5}(0.0014) + \frac{1}{6}(0.0001)\right] = 9.0229$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=2.2} = \frac{1}{0.04}\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094) + \frac{5}{6}(0.0014) + \frac{137}{180}(0.0001)\right] = 8.9939.$$

(*iii*) Here $a = 1.6$

$\therefore$ $y_0 = 4.9530, y_{-1} = 4.0552, y_{-2} = 3.3201, y_{-3} = 2.7183$ and $h = 0.2$

By using Stirling formula for derivatives, we get

$$\left[\frac{dy}{dx}\right]_{x=1.6} = \frac{1}{0.2}\left[\left(\frac{1.0966+0.8978}{2}\right) - \frac{1}{6}\left(\frac{0.0441+0.0361}{2}\right) + \frac{1}{30}\left(\frac{0.0014+0.0013}{2}\right)\right] = 4.9530$$

and $$\left[\frac{d^2y}{dx^2}\right]_{x=1.6} = \frac{1}{0.04}\left[0.1988 - \frac{1}{12}(0.0080) + \frac{1}{90}(0.0001)\right] = 4.9525.$$

Example 7. *The table below gives the result of an observation.* θ *is the observed temperature in degrees centigrade of a vessel of cooling water, t is the time in minutes from the beginning of observations:*

t:	*1*	*3*	*5*	*7*	*9*
θ:	*85.3*	*74.5*	*67.0*	*60.5*	*54.3*

Find the approximate rate of cooling at t = 3 and 3.5.

Sol. Forward difference table is

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
1	85.3				
		– 10.8			
3	74.5		3.3		
		$\boxed{-7.5}$		– 2.3	
5	67.0		$\boxed{1.0}$		1.6
		– 6.5		$\boxed{-0.7}$	
7	60.5		0.3		
		– 6.2			
9	54.3				

(*i*) When $t = 3$, $\theta_0 = 74.5$. Here $h = 2$

Rate of cooling $= \dfrac{d\theta}{dt}$

$$\therefore \quad \left(\frac{d\theta}{dt}\right)_{t=3} = \frac{1}{h}\left[\Delta\theta_0 - \frac{1}{2}\Delta^2\theta_0 + \frac{1}{3}\Delta^3\theta_0 - \frac{1}{4}\Delta^4\theta_0\right]$$

$$= \frac{1}{2}\left[-7.5 - \frac{1}{2}(1) + \frac{1}{3}(-0.7)\right] = -4.11667°\text{C/min.}$$

(*ii*) $t = 3.5$ is the non-tabular value of t so, we have from Newton's forward difference formula,

$$\frac{d\theta}{dt} = \frac{1}{h}\left[\Delta\theta_0 + \left(\frac{2u-1}{2}\right)\Delta^2\theta_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3\theta_0 + \left(\frac{2u^3-9u^2+11u-3}{12}\right)\Delta^4\theta_0 + \ldots\right] \quad \ldots(1)$$

At $t = 3.5$, $\quad u = \dfrac{3.5 - 3.0}{2} = \dfrac{0.5}{2} = 0.25$ | Here $a = 3.0$ and $h = 2$

From (1),

$$\left(\frac{d\theta}{dt}\right)_{t=3.5} = \frac{1}{2}\left[-7.5 + \left\{\frac{2(0.25)-1}{2}\right\}(1) + \left\{\frac{3(0.25)^2 - 6(0.25) + 2}{6}\right\}(-0.7)\right]$$

$$= -3.9151°\text{C/min.}$$

Example 8. *Find x for which y is maximum and find this value of y*

x:	*1.2*	*1.3*	*1.4*	*1.5*	*1.6*
y:	*0.9320*	*0.9636*	*0.9855*	*0.9975*	*0.9996.*

Sol. The difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.2	$\boxed{0.9320}$				
		$\boxed{0.0316}$			
1.3	0.9636		$\boxed{-0.0097}$		
		0.0219		– 0.0002	
1.4	0.9855		– 0.0099		0.0002
		0.0120		0	
1.5	0.9975		– 0.0099		
		0.0021			
1.6	0.9996				

Let $\quad y_0 = 0.9320$ and $a = 1.2$

By Newton's forward difference formula,

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \ldots$$

$$= 0.9320 + 0.0316\,u + \frac{u(u-1)}{2}(-0.0097)$$ | Neglecting higher differences

$$\frac{dy}{du} = 0.0316 + \left(\frac{2u-1}{2}\right)(-0.0097)$$

At a maximum, $\dfrac{dy}{du} = 0$

$\Rightarrow \quad 0.0316 = \left(u - \dfrac{1}{2}\right)(0.0097) \quad \Rightarrow \quad u = 3.76$

$\therefore \quad x = a + hu = 1.2 + (0.1)(3.76) = 1.576$

To find $y_{\text{max.}}$, we use backward difference formula,

$$x = x_n + hu$$

$\Rightarrow \quad 1.576 = 1.6 + (0.1)u \quad \Rightarrow \quad u = -0.24$

$$y(1.576) = y_n + u\,\nabla y_n + \frac{u(u+1)}{2}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$= 0.9996 - (0.24 \times 0.0021) + \frac{(-0.24)(1-0.24)}{2}(-0.0099)$$

$$= 0.9999988 = 0.9999 \text{ nearly}$$

$\therefore$ Maximum y = 0.9999 **approximately.**

Example 9. *Derive the formula* $y_0' = (y_{-2} - 8y_{-1} + 8y_1 - y_2)/12h$ *for numerical differentiation at mid-point of a table.* (U.P.T.U. 2009)

Sol. Take $a = 0$ Difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	y_{-2}				
		$y_{-1} - y_{-2}$			
-1	y_{-1}		$y_0 - 2y_{-1} + y_{-2}$		
		$\boxed{y_0 - y_{-1}}$		$\boxed{y_1 - 3y_0 + 3y_{-1} - y_{-2}}$	
0	y_0		$y_1 - 2y_0 + y_{-1}$		$y_2 - 4y_1 + 6y_0 - 4y_{-1} + y_{-2}$
		$\boxed{y_1 - y_0}$		$\boxed{y_2 - 3y_1 + 3y_0 - y_{-1}}$	
1	y_1		$y_2 - 2y_1 + y_0$		
		$y_2 - y_1$			
2	y_2				

By using Stirling's formula for first derivative, we get

$$y_0' = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots\right]$$

$$= \frac{1}{h}\left[\left(\frac{y_1 - y_0 + y_0 - y_{-1}}{2}\right) - \frac{1}{12}(y_2 - 3y_1 + 3y_0 - y_{-1} + y_1 - 3y_0 + 3y_{-1} - y_{-2})\right]$$

$$= \frac{1}{h}\left[\left(\frac{y_1 - y_{-1}}{2}\right) - \frac{1}{12}(y_2 - 2y_1 + 2y_{-1} - y_{-2})\right]$$

$$\Rightarrow \quad y_0' = \frac{1}{12h}[8y_1 - 8y_{-1} - y_2 + y_{-2}]$$

Example 10. *State the three different finite difference approximations to the first derivative $f'(x_0)$ together with the order of their truncation errors. Derive the forward difference approximation and its leading error term.*

Sol. (*i*) Newton's forward difference approximation is given by

$$f(x) = f_0 + u\,\Delta f_0 + \frac{u(u-1)}{2}\Delta^2 f_0$$

where $\quad u = \dfrac{x - x_0}{h} \quad$ and $\quad E = \dfrac{1}{6}u(u-1)(u-2)h^3 f'''(\xi)$

We have, $\quad f'(x) = \dfrac{df}{du}\cdot\dfrac{du}{dx} = \dfrac{1}{h}\left[\Delta f_0 + \dfrac{1}{2}(2u-1)\Delta^2 f_0\right]$

and $\quad |E'(x_0)| = |E'(u=0)| \le \dfrac{h^2}{3} M_3 \quad$ where $\quad M_3 = \max\limits_{x_0 \le x \le x_2} |f'''(x)|$

(*ii*) Newton's backward difference approximation is given by

$$f(x) = f_2 + u\,\nabla f_2 + \frac{1}{2}\,u(u+1)\,\nabla^2 f_2$$

where $u = \dfrac{x - x_2}{h}$ and $E = \dfrac{1}{6}\,u\,(u+1)\,(u+2)\,h^3 f'''(\xi)$

We have, $$f'(x) = \frac{1}{h}\left[\nabla f_2 + \frac{1}{2}(2u+1)\,\nabla^2 f_2\right]$$

and $$|E'(x_2)| = |E'(u=0)| \le \frac{h^2}{3}\,M_3$$

(*iii*) Central difference approximation is given by

$$f(x) = f_0 + \frac{u}{2}\,(\delta f_{1/2} + \delta f_{-1/2}) \quad \text{where} \quad u = \frac{x - x_0}{h}.$$

We have $$f'(x) = \frac{1}{2h}(\delta f_{1/2} + \delta f_{-1/2}) = \frac{1}{2h}[(f_1 - f_0) + (f_0 - f_{-1})] = \frac{1}{2h}(f_1 - f_{-1})$$

and $$|E'(x)| \le \frac{h^2}{6}\,M_3.$$

ASSIGNMENT

1. (*i*) Find $y'(0)$ and $y''(0)$ from the given table:

x:	0	1	2	3	4	5
y:	4	8	15	7	6	2

(*ii*) Find the first and second derivatives for the function tabulated below at the point $x = 3.0$:

x:	3	3.2	3.4	3.6	3.8	4.0
y:	–14	–10.032	–5.296	0.256	6.672	14

2. (*i*) From the following table, estimate $y'(1.05)$:

x:	1	1.05	1.10	1.15	1.20	1.25
y:	1.1	1.1347	1.1688	1.1564	1.2344	1.2345

(*G.B.T.U. 2013*)

(*ii*) Find the derivative of $f(x)$ at $x = 0.4$ from the following table:

x:	0.1	0.2	0.3	0.4
$f(x)$:	1.10517	1.22140	1.34986	1.49182

(*M.T.U. 2013*)

3. Find $\dfrac{d}{dx}(J_0)$ and $\dfrac{d^2}{dx^2}(J_0)$ at $x = 0.1$ from the following table:

x:	0.0	0.1	0.2	0.3	0.4
$J_0(x)$:	1	0.9975	0.99	0.9776	0.9604.

4. (*i*) Find the numerical value of $y'(10°)$ for $y = \sin x$ given that:

$\sin 0° = 0.000$, $\sin 10° = 0.1736$, $\sin 20° = 0.3420$, $\sin 30° = 0.5000$, $\sin 40° = 0.6428$.

(*ii*) Compute $y'(1.1)$ from the following table:

x:	1.0	1.1	1.2	1.3	1.4	1.5
$y(x)$:	7.989	8.403	8.781	9.129	9.451	9.750

(*G.B.T.U. 2012*)

5. (*i*) Find $f'(1.5)$ and $f''(1.5)$ from the following table:

x:	0.0	0.5	1.0	1.5	2.0
$f(x)$:	0.3989	0.3521	0.2420	0.1245	0.0540

[*U.P.T.U. (MCA) 2006*]

(*ii*) The population of a certain town is given below. Find the rate of growth of the population in 1961:

Year:	1931	1941	1951	1961	1971	
Population: (*in lacs*)	40.62	60.80	71.95	103.56	132.65	(*M.T.U. 2012*)

6. Find first and second derivative of the function tabulated below at $x = 0.6$ [*G.B.T.U. (C.O.) 2010*]

x:	0.4	0.5	0.6	0.7	0.8
y:	1.5836	1.7974	2.0442	2.3275	2.6511.

7. Find the values of $f'(5)$, $f''(5)$ and $f''(0.5)$ from the following table: (*G.B.T.U. 2012, U.P.T.U. 2007*)

x:	0	1	2	3	4	5
$f(x)$:	4930	5026	5122	5217	5312	5407

8. (*i*) A slider in a machine moves along a fixed straight rod. Its distance x cm along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when $t = 0.3$ second.

t:	0	0.1	0.2	0.3	0.4	0.5	0.6
x:	30.13	31.62	32.87	33.64	33.95	33.81	33.24.

(*ii*) A slider in a machine moves along fixed straight rod. Its distances x (m.) along the rod are given at various times (sec.)

t:	1	1.1	1.2	1.3	1.4	1.5
x:	16.40	19.01	21.96	25.29	29.03	33.21

Find the velocity of the slider at $t = 1.1$ sec.

9. A slider in a machine moves along a fixed straight rod. Its distance x(in cm) along the rod is given at various times t (in secs).

t:	0	0.1	0.2	0.3	0.4	0.5	0.6
x:	30.28	31.43	32.98	33.54	33.97	33.48	32.13

Evaluate $\frac{dx}{dt}$ at $t = 0.1$ and at $t = 0.5$.

10. Given the following table:

x:	1	1.05	1.1	1.15	1.2	1.25	1.3
$f(x) = \sqrt{x}$:	1	1.0247	1.04881	1.07238	1.09544	1.11803	1.14014

Apply the above results to find $f'(1)$, $f''(1)$ and $f'''(1)$.

11. (*i*) Using Newton's divided difference formula, find $f'(10)$ from the following data:

x:	3	5	11	27	34
$f(x)$:	– 13	23	899	17315	35606

(*ii*) Given the following data, find $y'(6)$:

x:	0	2	3	4	7	8
$f(x)$:	4	26	58	112	466	922

(*U.P.T.U. 2008*)

(*iii*) Find $f'(6)$ from the following table:

x:	0	1	3	4	5	7	9
$f(y)$:	150	108	0	–54	–100	–144	–84

12. (*i*) From the table below, for what value of x, y is minimum? Also find this value of y.

x:	3	4	5	6	7	8
y:	0.205	0.240	0.259	0.262	0.250	0.224.

(*ii*) Find the minimum value of y from the following table:

x:	0.2	0.3	0.4	0.5	0.6	0.7
y:	0.9182	0.8975	0.8873	0.8862	0.8935	0.9086

13. If $y = f(x)$ and y_n denotes $f(x_0 + nh)$, prove that, if powers of h above h^6 be neglected,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{3}{4h}\left[y_1 - y_{-1} - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3})\right]$$ (*U.P.T.U. 2006*)

14. The following table gives values of pressure P and specific volume V of saturated steam:

P:	105	42.7	25.3	16.7	13
V:	2	4	6	8	10

Find

(*a*) the rate of change of pressure w.r.t. volume at V = 2

(*b*) the rate of change of volume w.r.t. pressure at P = 105.

15. y is a function of x satisfying the equation $xy'' + ay' + (x - b)y = 0$, where a and b are integers. Find the values of constants a and b if y is given by the following table:

x:	0.8	1	1.2	1.4	1.6	1.8	2	2.2
y:	1.73036	1.95532	2.19756	2.45693	2.73309	3.02549	2.3333	3.65563.

16. (*a*) When does the need of numerical differentiation arise? (*U.P.T.U. 2008*)

(*b*) Prove that $D = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \ldots\right)$, the symbols used have usual meanings.

[*U.P.T.U.* (*MCA*) *2007*]

Answers

1. (*i*) – 27.9, 117.67 (*ii*) 18, 18

2. (*i*) 3.5856 (*ii*) 1.4913 **3.** – 0.05, – 0.5

4. (*i*) 0.9848 (*ii*) 3.945833

5. (*i*) – 0.2051, 0.194 (*ii*) 5.167 lacs/year **6.** 2.64442, 3.6475

7. 94.15, – 3.4166, 1.6458 **8.** (*i*) 5.33 cm/sec, – 45.6 cm/sec^2 (*ii*) 27.7 m/sec.

9. 32.44166 cm/sec., – 24.05833 cm/sec. **10.** 0.50024, – 0.256, 0.4

11. (*i*) 232.869 (*ii*) 125.4744 (*iii*) – 23

12. (*i*) 5.6875, 0.2628 (*ii*) 0.4623

14. (*a*) – 52.4 (*b*) – 0.01908 **15.** $a = 8$, $b = 6$.

5.12 ERRORS IN NUMERICAL DIFFERENTIATION

In numerical differentiation, the error in the higher order derivatives occurs due to the reason that, although the tabulated function and its approximating polynomial would agree at the set of data points, their slopes at these points may vary considerably. Numerical differentiation is, therefore, an unsatisfactory process and should be used only in rare cases.

The numerical computation of derivatives involves two types of errors: **truncation errors** and **rounding errors.**

The truncation error is caused by replacing the tabulated function by means of an interpolating polynomial.

Truncation error in first derivative is

$$= \frac{1}{6h}\left|\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right|.$$

Truncation error in second derivative is

$$= \frac{1}{12h^2} | \Delta^4 y_{-2} |.$$

The rounding error is $\propto \frac{1}{h}$ in case of first derivatives while it is $\propto \frac{1}{h^2}$ in case of second derivatives, and so on.

Max. Rounding error in first derivative is $= \frac{3}{2}\frac{\varepsilon}{h}$

Max. Rounding error in second derivative is $= \frac{4\varepsilon}{h^2}$

where ε is the maximum error in the value of y_i.

Example. *Assume the table of values given in previous. Example 6 and the function values are correct to the accuracy given, estimate the errors in* $\frac{dy}{dx}$ *at x = 1.6.*

Sol. Since the values are correct to four decimals, it follows that $\varepsilon = 0.5 \times 10^{-4}$

Truncation error $$= \frac{1}{6h} \left| \frac{\Delta^3 y_{-1} + \Delta^3 y_0}{2} \right|$$

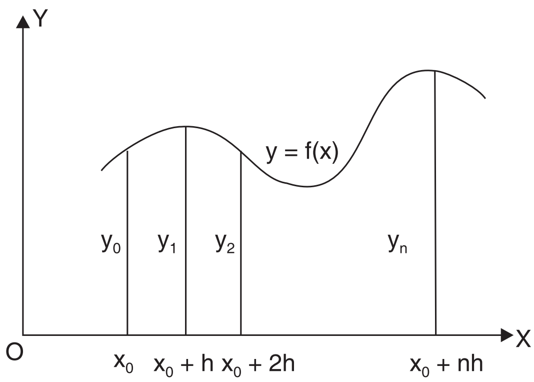
$$= \frac{1}{1.2} \left(\frac{0.0361 + 0.0441}{2} \right)$$ | See difference table in Example 6

$$= 0.03342$$

Rounding error $$= \frac{3\varepsilon}{2h} = \frac{3 \times 0.5 \times 10^{-4}}{2 \times 0.2} = 0.00038.$$

5.13 NUMERICAL INTEGRATION

Given a set of tabulated values of the integrand $f(x)$, to determine the value of $\int_{x_0}^{x_n} f(x)\, dx$ is called numerical integration. We subdivide the given interval of integration into a large number of subintervals of equal width h and replace the function tabulated at the points of subdivision by any one of the interpolating polynomials like Newton-Gregory's, Stirling's, Bessel's over each of the subintervals and evaluate the integral. We have several formulae for numerical integration which we shall derive in the sequel.



5.14 NEWTON-COTE'S QUADRATURE FORMULA

[*M.T.U. 2013, U.P.T.U. (MCA) 2009*]

Let $\quad \text{I} = \int_a^b y\, dx$, where y takes the values $y_0, y_1, y_2, \ldots\ldots, y_n$ for $x = x_0, x_1, x_2, \ldots\ldots, x_n$.

Let the interval of integration (a, b) be divided into n equal sub-intervals, each of width $h = \dfrac{b-a}{n}$ so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ..., $x_n = x_0 + nh = b$.

$$\therefore \qquad \text{I} = \int_{x_0}^{x_0 + nh} f(x)\, dx$$

Since any x is given by $x = x_0 + rh$ and $dx = hdr$

$$\therefore \qquad \text{I} = h \int_0^n f(x_0 + rh)\, dr$$

$$= h \int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \ldots \right] dr$$

[By Newton's forward interpolation formula]

$$= h \left[ry_0 + \frac{r^2}{2} \Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2} \right) \Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2 \right) \Delta^3 y_0 + \ldots \right]_0^n$$

$$\text{I} = nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \ldots \right] \qquad \ldots(1)$$

This is a **general quadrature formula** and is known as **Newton-Cote's quadrature formula.** A number of important deductions *viz.* Trapezoidal rule, Simpson's one-third and three-eighth rules, can be immediately deduced by putting $n = 1$, 2 and 3 respectively, in formula (1).

5.15 TRAPEZOIDAL RULE (n = 1)

[*U.P.T.U. MCA (SUM) 2008*]

Putting $n = 1$ in formula (1) and taking the curve through (x_0, y_0) and (x_1, y_1) as a polynomial of degree one so that differences of order higher than one vanish, we get

$$\int_{x_0}^{x_0 + h} f(x)\, dx = h \left(y_0 + \frac{1}{2} \Delta y_0 \right) = \frac{h}{2} [2y_0 + (y_1 - y_0)] = \frac{h}{2} (y_0 + y_1)$$

Similarly, for the next sub-interval $(x_0 + h, x_0 + 2h)$, we get

$$\int_{x_0 + h}^{x_0 + 2h} f(x)\, dx = \frac{h}{2} (y_1 + y_2), \ldots\ldots, \int_{x_0 + (n-1)h}^{x_0 + nh} f(x)\, dx = \frac{h}{2} (y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\boxed{\int_{\mathbf{x_0}}^{\mathbf{x_0 + nh}} \mathbf{f(x)\, dx} = \frac{\mathbf{h}}{\mathbf{2}} [(\mathbf{y_0} + \mathbf{y_n}) + \mathbf{2}(\mathbf{y_1} + \mathbf{y_2} + \ldots\ldots + \mathbf{y_{n-1}})]}$$

which is known as **Trapezoidal rule.** By increasing the number of subintervals, thereby making h very small, we can improve the accuracy of the value of the given integral.

5.16 SIMPSON'S ONE-THIRD RULE (n = 2) (*G.B.T.U. 2013*)

Putting $n = 2$ in formula (1) and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a polynomial of degree two so that differences of order higher than two vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{2h}{6}\,[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)] = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, $\displaystyle\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\,(y_2 + 4y_3 + y_4)$, ... ,

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\boxed{\int_{\mathbf{x_0}}^{\mathbf{x_0+nh}} \mathbf{f(x)\,dx} = \frac{\mathbf{h}}{\mathbf{3}}\,[(\mathbf{y_0} + \mathbf{y_n}) + 4(\mathbf{y_1} + \mathbf{y_3} + \ldots + \mathbf{y_{n-1}}) + 2(\mathbf{y_2} + \mathbf{y_4} + \ldots + \mathbf{y_{n-2}})]}$$

which is known as **Simpson's one-third rule.**

While using this formula, the given interval of integration must be divided into an even number of sub-intervals, since we find the area over two sub-intervals at a time.

5.17 SIMPSON'S THREE-EIGHTH RULE (n = 3) (*U.P.T.U. 2015*)

Putting $n = 3$ in formula (1) and taking the curve through (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) as a polynomial of degree three so that differences of order higher than three vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}\,[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)]$$

$$= \frac{3h}{8}\,[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly, $\displaystyle\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\,[y_3 + 3y_4 + 3y_5 + y_6]$, ...

$$\int_{x_0+(n-3)h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\,[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding the above integrals, we get

$$\boxed{\begin{aligned}\int_{\mathbf{x_0}}^{\mathbf{x_0+nh}} \mathbf{f(x)\,dx} = \frac{\mathbf{3h}}{\mathbf{8}}\,[(\mathbf{y_0} + \mathbf{y_n}) + 3(\mathbf{y_1} + \mathbf{y_2} + \mathbf{y_4} + \mathbf{y_5} + \ldots\ldots + \mathbf{y_{n-2}} + \mathbf{y_{n-1}})\\ + 2(\mathbf{y_3} + \mathbf{y_6} + \ldots\ldots + \mathbf{y_{n-3}})]\end{aligned}}$$

which is known as **Simpson's three-eighth rule.**

While using this formula, the given interval of integration must be divided into sub-intervals whose number n is a multiple of 3.

5.18 ERRORS IN QUADRATURE FORMULAE

If y_p is a polynomial representing the function $y = f(x)$ in the interval $[a, b]$ then error in the quadrature formulae is given by

$$E = \int_a^b f(x)\,dx - \int_a^b y_p\,dx \quad ...(1)$$

5.18.1. Error in Trapezoidal Rule [*U.P.T.U. (MCA) 2006, U.P.T.U. MCA (SUM) 2008*]

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series, we get

$$y = y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!}\, y_0'' + ... \quad ...(2)$$

$$\therefore \quad \int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_0+h} \left[y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!}\, y_0'' + ... \right] dx$$

$$= hy_0 + \frac{h^2}{2!}\, y_0'' + \frac{h^3}{3!}\, y_0''' + ... \quad ...(3)$$

Now, area of the first trapezium in the interval $[x_0, x_1] = A_1 = \frac{h}{2}\,(y_0 + y_1)$ ...(4)

Putting $x = x_0 + h$, $y = y_1$ in (2),

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}\, y_0'' + ... \quad ...(5)$$

From (4) and (5), we get

$$A_1 = \frac{h}{2}\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!}\, y_0'' + ... \right] = hy_0 + \frac{h^2}{2!}\, y_0' + \frac{h^3}{2\,2!}\, y_0'' + ... \quad ...(6)$$

Subtracting eqn. (6) from eqn. (3) gives the error in $[x_0, x_1]$,

$$\int_{x_0}^{x_1} y\,dx - A_1 = \left(\frac{1}{3!} - \frac{1}{2\,2!} \right) h^3\, y_0'' + ... \approx -\frac{h^3}{12}\, y_0'' \quad \text{| Neglecting other terms}$$

Similarly, the error in $[x_1, x_2]$ is $-\frac{h^3}{12}\, y_1''$ and in $[x_{n-1}, x_n]$ is $\frac{-h^3}{12}\, y_{n-1}''$.

Hence the total error is

$$E = \frac{-h^3}{12}\,(y_0'' + y_1'' + \cdots + y_{n-1}'')$$

Let $y''(\xi)$, $a < \xi < b$ be the maximum of $|\,y_0''\,|, |\,y_1''\,|, ..., |\,y_{n-1}''\,|$ then, we have

$$E < \frac{-nh^3}{12}\, y''(\xi) = -\frac{(b-a)}{12}\, h^2\, y''(\xi) \qquad |\because \quad b - a = nh$$

Hence the error in the Trapezoidal rule is of order h^2.

5.18.2. Error in Simpson's 1/3rd Rule. (*U.P.T.U. 2008*)

Integrating eqn. (2) w.r.t. x between the limits x_0 and x_2.

$$\int_{x_0}^{x_2} y\,dx = \int_{x_0}^{x_0+2h}\left[y_0 + (x - x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \ldots\right] dx$$

$$= 2hy_0 + 2h^2 y_0' + \frac{8h^3}{3!}\,y_0'' + \frac{16h^4}{4!}\,y_0''' + \frac{32h^5}{5!}\,y_0^{(iv)} + \ldots \qquad \ldots(7)$$

Now, $$A_1 = \frac{h}{3}\,(y_0 + 4y_1 + y_2) \qquad \ldots(8)$$

where A_1 is the area of the curve in the interval $[x_0, x_2]$.

Putting $x = x_0 + h$, $y = y_1$ in (2), we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}\,y_0'' + \frac{h^3}{3!}\,y_0''' + \ldots \qquad \ldots(9)$$

Putting $x = x_0 + 2h$, $y = y_2$ in (2), we get

$$y_2 = y_0 + 2h\,y_0' + \frac{4h^2}{2!}\,y_0'' + \frac{8h^3}{3!}\,y_0''' + \ldots \qquad \ldots(10)$$

Substituting eqns. (9) and (10) in eqn. (8), we get

$$A_1 = 2h\,y_0 + 2h^2\,y_0' + \frac{4h^3}{3}\,y_0'' + \frac{2h^4}{3}\,y_0''' + \frac{5h^5}{18}\,y_0^{(iv)} + \ldots \qquad \ldots(11)$$

Now, the error in interval $[x_0, x_2]$ is given by

$$\int_{x_0}^{x_2} y\,dx - A_1 = \left(\frac{4}{15} - \frac{5}{18}\right) h^5\,y_0^{(iv)} + \ldots$$

$$\approx -\frac{h^5}{90}\,y_0^{(iv)} \qquad | \text{ Neglecting terms of order } h^6, h^7, \ldots$$

Similarly, the principal part of error in interval $[x_2, x_4]$ is $= -\dfrac{h^5}{90}\,y_2^{(iv)}$ and so on.

Hence total principal error is

$$E = -\frac{h^5}{90}\,[y_0^{(iv)} + y_2^{(iv)} + \ldots + y_{2(n-1)}^{(iv)}]$$

Let $y^{(iv)}(\xi)$ be the maximum of $|\,y_0^{(iv)}\,|, |\,y_2^{(iv)}\,|, \ldots, |\,y_{2n-1}^{(iv)}\,|$.

Then, we have, $E < -\dfrac{h^5}{90}\,y^{(iv)}(\xi) = \dfrac{-(b-a)\,h^4}{180}\,y^{(iv)}(\xi)$

Hence, the error in the Simpson's (1/3)rd rule is of order h^4.

Note. Similarly, the principal part of the error for Simpson's (3/8)th rule is $-\dfrac{3h^5}{80}\,y^{(iv)}$ in the interval $[x_0, x_3]$.

EXAMPLES

Example 1. *Evaluate* $\int_{0.6}^{2} y\,dx$*, where y is given by the following table:*

x:	*0.6*	*0.8*	*1.0*	*1.2*	*1.4*	*1.6*	*1.8*	*2.0*
y:	*1.23*	*1.58*	*2.03*	*4.32*	*6.25*	*8.36*	*10.23*	*12.45.*

Sol. Here the no. of sub-intervals is 7 which is neither even nor a multiple of 3. Also this number is neither a multiple of 4 nor a multiple of 6 hence using Trapezoidal rule, we get

$$\int_{0.6}^{2} y\,dx = \frac{h}{2}\,[(y_0 + y_7) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)]$$

$$= \frac{0.2}{2}\,[(1.23 + 12.45) + 2(1.58 + 2.03 + 4.32 + 6.25 + 8.36 + 10.23)]$$

| Here $h = 0.2$

$$= 7.922.$$

Example 2. *Find, from the following table, the area bounded by the curve and the x-axis from x = 7.47 to x = 7.52.*

x:	*7.47*	*7.48*	*7.49*	*7.50*	*7.51*	*7.52*
f(x):	*1.93*	*1.95*	*1.98*	*2.01*	*2.03*	*2.06.*

Sol. We know that

Area = $\int_{7.47}^{7.52} f(x)\,dx$ with $h = 0.01$, the trapezoidal rule gives,

$$\text{Area} = \frac{0.01}{2}\,[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)] = 0.09965.$$

Example 3. *Use Simpson's rule for evaluating* $\int_{-0.6}^{0.3} f(x)\,dx$ *from the table given below:*

x:	*– 0.6*	*– 0.5*	*– 0.4*	*– 0.3*	*– 0.2*	*– 0.1*	*0*	*0.1*	*0.2*	*0.3*
f(x):	*4*	*2*	*5*	*3*	*– 2*	*1*	*6*	*4*	*2*	*8*

Sol. Since the no. of sub-intervals is 9(a multiple of 3) hence we will use Simpson's 3/8th rule here.

$$\therefore \quad \int_{-0.6}^{0.3} f(x)\,dx = \frac{3(0.1)}{8}\,[(4 + 8) + 3\{2 + 5 + (-2) + 1 + 4 + 2\} + 2(3 + 6)] = 2.475.$$

Example 4. *A river is 80 m wide. The depth 'y' of the river at a distance 'x' from one bank is given by the following table:*

x:	*0*	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*
y:	*0*	*4*	*7*	*9*	*12*	*15*	*14*	*8*	*3*

Find the approximate area of cross section of the river using Simpson's 1/3rd rule.

Sol. The required area of the cross section of the river $= \int_{0}^{80} y\,dx$ ...(1)

Here no. of sub-intervals is 8. *By Simpson's 1/3rd rule,*

$$\int_0^{80} y\,dx = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{10}{3}[(0 + 3) + 4(4 + 9 + 15 + 8) + 2(7 + 12 + 14)] = 710$$

Hence the required area of the cross section of the river = 710 sq. m.

Example 5. *The speed, v metres per second, of a car, t seconds after it starts, is shown in the following table:*

t	0	12	24	36	48	60	72	84	96	108	120
v	0	3.60	10.08	18.90	21.60	18.54	10.26	5.40	4.50	5.40	9.00

Using Simpson's $\left(\frac{1}{3}\right)^{rd}$ *rule, find the distance travelled by the car in 2 minutes.*

Sol. If s metres is the distance covered in t seconds, then

$$\frac{ds}{dt} = v$$

$$\therefore \qquad \Big[s\Big]_{t=0}^{t=120} = \int_0^{120} v\,dt$$

Since the number of sub-intervals is **10 (even)**, hence, by using Simpson's 1/3rd rule,

$$\int_0^{120} v\,dt = \frac{h}{3}[(v_0 + v_{10}) + 4(v_1 + v_3 + v_5 + v_7 + v_9) + 2(v_2 + v_4 + v_6 + v_8)]$$

$$= \frac{12}{3}[(0 + 9) + 4(3.6 + 18.9 + 18.54 + 5.4 + 5.4)$$

$$+ 2(10.08 + 21.6 + 10.26 + 4.5)]$$

$$= 1236.96 \text{ metres.}$$

Hence, the distance travelled by car in 2 minutes is 1236.96 metres.

Example 6. *A train is moving at the speed of 30 m/sec. Suddenly brakes are applied. The speed of the train per second after t seconds is given by*

Time (t):	0	5	10	15	20	25	30	35	40	45
Speed (v):	30	24	19	16	13	11	10	8	7	5

Apply Simpson's three-eighth rule to determine the distance moved by the train in 45 seconds.

Sol. If s metres is the distance covered in t seconds, then

$$\frac{ds}{dt} = v$$

$$\Rightarrow \qquad \Big[s\Big]_{t=0}^{t=45} = \int_0^{45} v\,dt \qquad \text{...(1)}$$

Since the no. of sub-intervals is **9 (a multiple of 3)** hence by using Simpson's $\left(\frac{3}{8}\right)^{\text{th}}$ rule,

$$\int_0^{45} v\,dt = \frac{3h}{8}\,[(v_0 + v_9) + 3(v_1 + v_2 + v_4 + v_5 + v_7 + v_8) + 2(v_3 + v_6)]$$

$$= \frac{15}{8}\,[(30 + 5) + 3(24 + 19 + 13 + 11 + 8 + 7) + 2(16 + 10)]$$

$$= 624.375 \text{ metres.}$$

Hence the distance moved by the train in 45 seconds is **624.375** metres.

Example 7. *Evaluate* $\int_0^1 \frac{dx}{1+x^2}$ *using*

(*i*) *Simpson's* $\frac{1}{3}$ *rule taking* $h = \frac{1}{4}$ [U.P.T.U. (MCA) 2007]

(*ii*) *Simpson's* $\frac{3}{8}$ *rule taking* $h = \frac{1}{6}$ [G.B.T.U. 2010, U.P.T.U. (MCA) 2006]

Hence compute an approximate value of π *in each case.*

Sol. (*i*) The values of $f(x) = \frac{1}{1+x^2}$ at $x = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$ are given below:

x:	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x)$:	1	$\frac{16}{17}$	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\frac{1}{3}$ rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\,[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1}{12}\left[(1 + 0.5) + 4\left\{\frac{16}{17} + 0.64\right\} + 2(0.8)\right] = 0.785392156$$

Also $$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}$$

$\therefore$ $$\frac{\pi}{4} \simeq 0.785392156 \quad \Rightarrow \quad \pi \simeq 3.1415686$$

(*ii*) The values of $f(x) = \frac{1}{1+x^2}$ at $x = 0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1$ are given below:

x:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$f(x)$:	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8} \, [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\left(\frac{1}{6}\right)}{8}\left[\left(1+\frac{1}{2}\right)+3\left\{\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right\}+2\left(\frac{4}{5}\right)\right] = 0.785395862$$

Also, $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$

$\therefore$ $\frac{\pi}{4} = 0.785395862 \quad \Rightarrow \quad \pi = 3.141583$

Example 8. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using*

(*i*) *Simpson's one-third rule* [M.T.U. (MCA) 2012]

(*ii*) *Simpson's three-eighth rule*

(*iii*) *Trapezoidal rule.*

Sol. Divide the interval (0, 6) into six parts each of width $h = 1$.

The values of $f(x) = \frac{1}{1+x^2}$ are given below:

x:	0	1	2	3	4	5	6
$f(x)$:	1	0.5	0.2	0.1	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(*i*) By Simpson's one-third rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3} \, [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}\left[\left(1+\frac{1}{37}\right)+4\left(0.5+0.1+\frac{1}{26}\right)+2\left(0.2+\frac{1}{17}\right)\right] = 1.366173413.$$

(*ii*) By Simpson's three-eighth rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8} \, [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}\left[\left(1+\frac{1}{37}\right)+3\left(0.5+0.2+\frac{1}{17}+\frac{1}{26}\right)+2(0.1)\right] = 1.357080836.$$

(*iii*) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2} \, [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{37}\right)+2\left(0.5+0.2+0.1+\frac{1}{17}+\frac{1}{26}\right)\right] = 1.410798581.$$

Example 9. *Evaluate* $\int_0^1 \frac{dx}{1+x}$ *by dividing the interval of integration into 8 equal parts. Hence find* $\log_e 2$ *approximately.* [U.P.T.U. (MCA) 2008]

Sol. Since the interval of integration is divided into an even number of sub-intervals, we shall use Simpson's one-third rule.

Here, $y = \frac{1}{1+x} = f(x)$

$$y_0 = f(0) = \frac{1}{1+0} = 1, \quad y_1 = f\left(\frac{1}{8}\right) = \frac{1}{1+\frac{1}{8}} = \frac{8}{9}, \quad y_2 = f\left(\frac{2}{8}\right) = \frac{4}{5}$$

$$y_3 = f\left(\frac{3}{8}\right) = \frac{8}{11}, \quad y_4 = f\left(\frac{4}{8}\right) = \frac{2}{3}, \quad y_5 = f\left(\frac{5}{8}\right) = \frac{8}{13}$$

$$y_6 = f\left(\frac{6}{8}\right) = \frac{4}{7}, \quad y_7 = f\left(\frac{7}{8}\right) = \frac{8}{15} \quad \text{and} \quad y_8 = f(1) = \frac{1}{2}$$

Hence the table of values is

x:	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	1
y:	1	$\frac{8}{9}$	$\frac{4}{5}$	$\frac{8}{11}$	$\frac{2}{3}$	$\frac{8}{13}$	$\frac{4}{7}$	$\frac{8}{15}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's 1/3rd rule,

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\,[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]$$

$$= \frac{1}{24}\left[\left(1+\frac{1}{2}\right)+4\left(\frac{8}{9}+\frac{8}{11}+\frac{8}{13}+\frac{8}{15}\right)+2\left(\frac{4}{5}+\frac{2}{3}+\frac{4}{7}\right)\right] \quad | \text{ Here } h = 1/8$$

$$= 0.69315453$$

Since, $\int_0^1 \frac{dx}{1+x} = \left[\log_e(1+x)\right]_0^1 = \log_e 2$

$\therefore \quad \log_e 2 = 0.69315453.$

Example 10. *Using Simpson's* $3/8^{th}$ *rule on integration, evaluate*

$$\int_0^6 \frac{1}{1+x}\,dx$$ [G.B.T.U. 2009; G.B.T.U. (M.Tech.) 2011]

Sol. The table of values is as follows: (Here $h = 1$)

x:	0	1	2	3	4	5	6
$f(x) = \frac{1}{1+x}$:	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8th rule,

$$\int_0^6 \frac{1}{1+x}\, dx = \frac{3h}{8}\, [(y_0 + y_6) + 3\,(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}\left[\left(1+\frac{1}{7}\right)+3\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}\right)+2\left(\frac{1}{4}\right)\right] = 1.96607.$$

Example 11. *Find* $\int_0^6 \frac{e^x}{1+x}\, dx$ *approximately using Simpson's 3/8th rule on integration.*

[U.P.T.U. (MCA) 2009, U.P.T.U. 2006, 2014]

Sol. Divide the given integral of integration into 6 equal sub-intervals, the arguments are 0, 1, 2, 3, 4, 5, 6; $h = 1$.

$$f(x) = \frac{e^x}{1+x}\ ; \qquad y_0 = f(0) = 1$$

$$y_1 = f(1) = \frac{e}{2}, \qquad y_2 = f(2) = \frac{e^2}{3}, \qquad y_3 = f(3) = \frac{e^3}{4},$$

$$y_4 = f(4) = \frac{e^4}{5}, \qquad y_5 = f(5) = \frac{e^5}{6}, \qquad y_6 = f(6) = \frac{e^6}{7}$$

The table is as below:

x:	0	1	2	3	4	5	6
y:	1	$\frac{e}{2}$	$\frac{e^2}{3}$	$\frac{e^3}{4}$	$\frac{e^4}{5}$	$\frac{e^5}{6}$	$\frac{e^6}{7}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Applying Simpson's three-eighth rule, we have

$$\int_0^6 \frac{e^x}{1+x}\, dx = \frac{3h}{8}\, [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}\left[\left(1+\frac{e^6}{7}\right)+3\left(\frac{e}{2}+\frac{e^2}{3}+\frac{e^4}{5}+\frac{e^5}{6}\right)+2\,\frac{e^3}{4}\right]$$

$$= \frac{3}{8}\, [(1 + 57.6327) + 3(1.3591 + 2.463 + 10.9196 + 24.7355 + 2(5.0214)]$$

$$= 70.1652.$$

Note. It is not possible to evaluate $\int_0^6 \frac{e^x}{1+x}\, dx$ by using usual calculus method. Numerical integration comes to our rescue in such situations.

Example 12. *Evaluate* $\int_1^2 e^{-\frac{1}{2}x}\, dx$ *using four intervals.*

Sol. The table of values is:

x:	1	1.25	1.5	1.75	2
$y = e^{-x/2}$:	0.60653	0.53526	0.47237	0.41686	0.36788
	y_0	y_1	y_2	y_3	y_4

Since we have four (even) sub-intervals here, we will use Simpson's 1/3rd rule.

$$\therefore \quad \int_1^2 e^{-\frac{1}{2}x}\, dx = \frac{h}{3}\,[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{0.25}{3}\,[(0.60653 + 0.36788) + 4(0.53526) + 0.41686) + 2(0.47237)]$$

$$= 0.4773025.$$

Example 13. *Compute* $I_p = \int_0^1 \frac{x^p}{x^3 + 10}\, dx$ *for* $p = 0, 1.$

Use Trapezoidal rule with number of points 3, 5 and 9.

Sol. For $p = 0$

$$I_0 = \int_0^1 \frac{1}{x^3 + 10}\, dx$$

(i) Dividing the interval (0, 1) into 2 equal parts, each of width $h = \frac{1-0}{2} = \frac{1}{2}$, the values of $f(x) = \frac{1}{x^3 + 10}$ are given below:

x:	0	$\frac{1}{2}$	1
$f(x)$:	$\frac{1}{10}$	$\frac{8}{81}$	$\frac{1}{11}$

By Trapezoidal rule,

$$I_0 = \frac{h}{2}\,[(y_0 + y_2) + 2y_1] = \frac{1}{4}\left[\left(\frac{1}{10} + \frac{1}{11}\right) + \frac{16}{81}\right] = 0.0971099$$

(ii) Dividing the interval (0, 1) into 4 equal parts, each of width $h = \frac{1-0}{4} = \frac{1}{4}$, the values of $f(x) = \frac{1}{x^3 + 10}$ are given below:

x:	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x)$:	$\frac{1}{10}$	$\frac{64}{641}$	$\frac{8}{81}$	$\frac{64}{667}$	$\frac{1}{11}$

By Trapezoidal rule,

$$I_0 = \frac{h}{2}\,[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{8}\left[\left(\frac{1}{10} + \frac{1}{11}\right) + 2\left(\frac{64}{641} + \frac{8}{81} + \frac{64}{667}\right)\right] = 0.0975039$$

(*iii*) Dividing the interval (0, 1) into 8 equal parts, each of width $h = \frac{1}{8}$, the values of

$f(x) = \dfrac{1}{x^3 + 10}$ are given below:

x:	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
$f(x)$:	$\frac{1}{10}$	$\frac{512}{5121}$	$\frac{64}{641}$	$\frac{512}{5147}$	$\frac{8}{81}$	$\frac{512}{5245}$	$\frac{64}{667}$	$\frac{512}{5463}$	$\frac{1}{11}$

By Trapezoidal rule,

$$I_0 = \frac{h}{2}\,[(y_0 + y_8) + 2\,(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{1}{16}\left[\left(\frac{1}{10} + \frac{1}{11}\right) + 2\left(\frac{512}{5121} + \frac{64}{641} + \frac{512}{5147} + \frac{8}{81} + \frac{512}{5245} + \frac{64}{667} + \frac{512}{5463}\right)\right]$$

$$= 0.0976012$$

Repeating the above process for $p = 1$ so that

$$I_1 = \int_0^1 \frac{x}{x^3 + 10}\,dx$$

and dividing the interval (0, 1) into 2, 4 and 8 equal parts respectively, we get I_1 as 0.0480733, 0.0481145 and 0.0481164 respectively.

Example 14. *A solid of revolution is formed by rotating about x-axis, the lines x = 0 and x = 1 and a curve through the points with the following co-ordinates.*

x:	*0*	*0.25*	*0.5*	*0.75*	*1*
y:	*1*	*0.9896*	*0.9589*	*0.9089*	*0.8415*

Estimate the volume of the solid formed using Simpson's rule.

Sol. If V is the volume of the solid formed then we know that

$$V = \pi \int_0^1 y^2\,dx$$

Hence we need the values of y^2 and these are tabulated below correct to four decimal places

x	0	0.25	0.5	0.75	1
y^2	1	0.9793	0.9195	0.8261	0.7081

With $h = 0.25$, Simpson's rule gives

$$V = \pi\,\frac{(0.25)}{3}\,[(1 + 0.7081) + 4(0.9793 + 0.8261) + 2(0.9195)] = 2.8192.$$

Example 15. *A tank is discharging water through an orifice at a depth of x metre below the surface of the water whose area is A m^2. Following are the values of x for the corresponding values of A.*

A:	*1.257*	*1.39*	*1.52*	*1.65*	*1.809*	*1.962*	*2.123*	*2.295*	*2.462*	*2.650*	*2.827*
x:	*1.5*	*1.65*	*1.8*	*1.95*	*2.1*	*2.25*	*2.4*	*2.55*	*2.7*	*2.85*	*3*

Using the formula (0.018) $T = \int_{1.5}^{3.0} \frac{A}{\sqrt{x}}\, dx$, *calculate T, the time (in seconds) for the level of the water to drop from 3.0 m to 1.5 m above the orifice.*

Sol. Here $h = 0.15$

The table of values of x and the corresponding values of $\frac{A}{\sqrt{x}}$ is

x	1.5	1.65	1.8	1.95	2.1	2.25	2.4	2.55	2.7	2.85	3
$y = \frac{A}{\sqrt{x}}$	1.025	1.081	1.132	1.182	1.249	1.308	1.375	1.438	1.498	1.571	1.632

Using Simpson's 1/3rd rule, we get

$$\int_{1.5}^{3} \frac{A}{\sqrt{x}}\, dx = \frac{0.15}{3}[(1.025 + 1.632) + 4(1.081 + 1.182 + 1.308 + 1.438 + 1.571) + 2(1.132 + 1.249 + 1.375 + 1.498)]$$

$$= 1.9743$$

Using the formula

$$(0.018)\mathrm{T} = \int_{1.5}^{3} \frac{A}{\sqrt{x}}\, dx$$

We get $0.018\mathrm{T} = 1.9743 \Rightarrow \mathrm{T} = 110$ sec. (approximately).

ASSIGNMENT

1. (*i*) Evaluate $\int_1^2 \frac{1}{x}\, dx$ by Simpson's 1/3rd rule with four strips and determine the error by direct integration. [*G.B.T.U. (MCA) 2010*]

(*ii*) Evaluate the integral $\int_0^1 e^{x+1}\, dx$ using Simpson's 1/3 rule by dividing the interval of integration into 8 equal parts. (*G.B.T.U. 2013, 2011*)

2. Compute $\int_0^{\pi/2} \sin x\, dx$ using Simpson's three-eighth rule of integration, taking $h = \frac{\pi}{18}$. (*U.P.T.U. 2007*)

3. Evaluate $\int_4^{5.2} \log_e x\, dx$ by Simpson's 3/8th rule. [*G.B.T.U. (C.O.) 2011*]

4. (*i*) Find the value of $\int_0^{\pi/2} \sqrt{1 - 0.162 \sin^2 x}\; dx$ using Simpson's one-third rule taking 6 sub-intervals. (*U.P.T.U. 2007*) (*ii*) Find the approximate value of the following integral using Simpson's rule by dividing the interval into six equal parts: $\int_0^{\pi} \sqrt{1+3\cos^2 x}\, dx$. (*M.T.U. 2012*)

5. (*i*) Use Simpson's 1/3 rule to evaluate the integral of $f(x) = \frac{\sin(x)}{x}$ with respect to x between $x = 0$ and $x = 1$ with a value of $h = 0.25$. [*G.B.T.U. MCA (SUM) 2010*]

(*ii*) Evaluate $\int_0^8 x \sec x\, dx$ using eight intervals by Trapezoidal rule. (*U.P.T.U. 2009*)

(*iii*) State the need and scope of numerical integration. Use the Trapezoidal rule to estimate the integral $\int_0^2 e^{x^2}\, dx$ taking the number of intervals 10. [*G.B.T.U. (C.O.) 2010, U.P.T.U. 2008*]

6. (*a*) Evaluate using Trapezoidal rule

(*i*) $\int_0^{\pi} t \sin t\, dt$ (*ii*) $\int_{-2}^{2} \frac{t\, dt}{5+2t}$

(*b*) Evaluate the integral $\int_0^{2\pi} e^{-t} \sin 10\, t\, dt$ using

(*i*) Simpson's 3/8 rule. (*M.T.U. 2012*)

(*ii*) Simpson's rule with eight intervals. (*U.P.T.U. 2014*)

7. (*i*) Evaluate $\int_0^4 e^x\, dx$ by Simpson's rule, given that $e = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.6$ and compare it with the actual value.

(*ii*) Using Simpson's 1/3 rule, evaluate $\int_0^3 (2x - x^2)\, dx$ by taking $n = 6$. (*G.B.T.U. 2013*)

8. (*i*) The velocities of a car running on a straight road at intervals of 2 minutes are given below:

Time (*in minutes*:	0	2	4	6	8	10	12
Velocity (*in km/hr*:	0	22	30	27	18	7	0

Apply Simpson's rule to find the distance covered by the car. (*G.B.T.U. 2011*)

(*ii*) The velocity v of a particle at distance s from a point on its path is given by the table below:

s (*in meter*):	0	10	20	30	40	50	60
v (*m/sec*):	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by using Simpson's $\frac{1}{3}$ rule. Compare the result with Simpson's 3/8 rule. (*U.P.T.U. 2007, 2015 ; G.B.T.U. (C.O.) 2011*]

(*iii*) The velocity 'v' of a particle at distance 's' from a point on its linear path is given in the following table:

s (*m*):	0	2.5	5	7.5	10	12.5	15	17.5	20
v (*m/sec*):	16	19	21	22	20	17	13	11	9

Apply Simpson's rule to estimate the time taken by the particle to traverse the distance of 20 meters. (*G.B.T.U. 2011*)

9. The velocity of a train which starts from rest is given by the following table, time being reckoned in minutes from the start and speed in kilometres per hour:

Minutes:	0	2	4	6	8	10	12	14	16	18	20
Speed (km/hr):	0	10	18	25	29	32	20	11	5	2	0

Estimate the total distance in 20 minutes. $\left[\textbf{Hint.}\ \text{Here step-size } h = \frac{2}{60}\right]$

10. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's 1/3rd rule, find the velocity of the rocket at $t = 80$ seconds. (*G.B.T.U. 2012*)

$t(sec)$:	0	10	20	30	40	50	60	70	80
$f(cm/sec^2)$:	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67.

11. (*i*) Find an approximate value of $\log_e 5$ by calculating to 4 decimal places, by Simpson's 1/3rd rule, $\int_0^5 \frac{dx}{4x+5}$ dividing the range into 10 equal parts.

(*ii*) Find the value of $\log_e 2$ from $\int_0^{1} \frac{x^2}{1+x^3}\, dx$ using Simpson's 1/3rd rule by dividing the range of integration into four equal parts. Also find the error.

12. In an experiment, a quantity G was measured as follows:

G(20) = 95.9, G(21) = 96.85, G(22) = 97.77, G(23) = 98.68, G(24) = 99.56, G(25) = 100.41, G(26) = 101.24.

Compute $\int_{20}^{26} G(x)\, dx$ by Simpson's rule.

13. Using the data of the following table, compute the integral $\int_{0.5}^{1.1} xy\, dx$ by Simpson's rule:

x:	0.5	0.6	0.7	0.8	0.9	1.0	1.1
y:	0.4804	0.5669	0.6490	0.7262	0.7985	0.8658	0.9281

14. Apply Simpson's 1/3rd rule to evaluate the integral $I = \int_0^1 e^x\, dx$ by choosing step size $h = 0.1$

Show that this step size is sufficient to obtain the result correct to five decimal places.

15. (*i*) Use Simpson's rule dividing the range into ten equal parts to show that:

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2}\, dx = 0.173$$

(*ii*) Evaluate $\int_0^2 \frac{dx}{x^2+x+1}$ to three decimals, dividing the range of integration into 8 equal parts.

(*U.P.T.U. 2007*)

(*iii*) Compute $I = \int_{0.2}^{1.5} e^{-x^2}\, dx$ using Simpson's $\left(\frac{1}{3}\right)^{\text{rd}}$ rule with four subdivisions.

(*iv*) Using Simpson's 1/3rd formula, evaluate $\int_0^1 \frac{dx}{x^3+x+1}$ choose with steplength 0.25.

(*U.P.T.U. 2009*)

16. (*i*) Using 3/8th Simpson's rule, Evaluate: $\int_0^6 \frac{dx}{1+x^4}$.

(*ii*) Evaluate $\int_0^9 \frac{1}{1+x^3}\, dx$ by using Simpson's three-eighth rule. [*G.B.T.U. (M.Tech.) 2010*] (*iii*) State Simpson's three-eighth rule. Using this rule, evaluate the following integral

$$\int_0^6 \frac{x}{1+x^5}\, dx$$ (*U.P.T.U. 2015*)

17. A solid of revolution is formed by revolving the area $y = f(x)$ from $x = 1$ to $x = 3$ about the x-axis. The following table gives relation between x and y for this curve:

x:	1	1.5	2	2.5	3
y:	1	0.9896	0.9589	0.9089	0.8415

Using Simpson's one-third rule, estimate the volume of the solid formed. (*M.T.U. 2013*)

18. (*i*) Find the distance between two stations from the following data consisting of the speeds $v(t)$ of an electric train at various times t after leaving one station until it stops at the next station. Apply Simpson's rule:

v (miles/hr):	0	13	33	39.5	40	40	36	15	0
t (min):	0	0.5	1	1.5	2	2.5	3	3.5	4

$\left[\textbf{Hint.} \text{ Here } h = \frac{1}{120}\right]$ (*G.B.T.U. 2012*)

(*ii*) The speed of a train at various times after leaving one station until it stops at another station are given in the following table:

Speed (in mph):	0	13	33	39.5	40	40	36	15	0
Time (in minutes):	0	0.5	1	1.5	2	2.5	3	3.25	3.5

Find the distance between the two stations using trapezoidal rule, Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule. (*M.T.U. 2013*)

$$\left[\begin{array}{l}\textbf{Hint. } \text{Since time interval is not same, hence,} \\ s = \int_0^{3.5} v\, dt = \int_0^{3} v\, dt + \int_3^{3.5} v\, dt \quad (\text{taking} \quad h = 0.5 \text{ and } 0.25)\end{array}\right]$$

19. A curve is drawn to pass through the points given by the following table:

x:	1	1.5	2	2.5	3	3.5	4
y:	2	2.4	2.7	2.8	3	2.6	2.1

Find

(*i*) Centre of gravity of the area. (*ii*) Volume of the solid of revolution.

(*iii*) The area bounded by the curve, the x-axis and lines $x = 1$, $x = 4$.

20. (*i*) Obtain the global truncation error term of trapezoidal method of integration.

(*ii*) Derive an expression for error estimation in Simpson's one-third rule. (*U.P.T.U. 2008*)

(*iii*) Derive Simpson's $3/8^{\text{th}}$ rule. [*U.P.T.U. (MCA) 2009; U.P.T.U. 2009*]

Answers

1. (*i*) 0.69325 ; 0.0001 (*ii*) 4.67078 **2.** 0.974353 **3.** 1.8278

4. (*i*) 1.505103 (*ii*) 5.4059

5. (*i*) 0.946058 (*ii*) – 6.435936 (*iii*) 17.1702

6. (*a*) (*i*) 3.14 (*ii*) – 0.747 (*b*) (*i*) – 0.24245 (*ii*) 0.39466

7. (*i*) 53.87, 53.60 (*ii*) 0

8. (*i*) $3\frac{5}{9}$ km (*ii*) 1.063521 sec, 1.0643752 sec (*iii*) 1.26164 sec

9. 5.156 km **10.** 30.87 m/sec **11.** (*i*) 1.61 (*ii*) 0.693255; 0.0001078

12. 591.85333 **13.** 0.358487 **14.** 1.718282782

15. (*ii*) 0.824 (*iii*) 0.658596 (*iv*) 0.6305

16. (*i*) 1.019286497 (*ii*) 1.124955 (*iii*) 0.646382112

17. 5.638478 **18.** (*i*) 1.8 miles (*ii*) 1.666 miles, 1.666 miles, 1.6671875 miles

19. (*i*) (2.53, 1.31) (*ii*) 64.07 (*iii*) 7.7833

5.19 INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

A physical situation that concerns with the rate of change of one quantity with respect to another gives rise to a differential equation.

Consider the first order ordinary differential equation

$$\frac{dy}{dx} = f(x, y) \qquad ...(1)$$

with the initial condition $y(x_0) = y_0$ ...(2)

Many analytical techniques exist for solving such equations. But these methods can be applied to solve only a selected class of differential equations.

However, a majority of differential equations appearing in physical problems cannot be solved analytically. Thus it becomes imperative to discuss their solution by numerical methods.

In numerical methods, we donot proceed in the hope of finding a relation between variables but we find the numerical values of the dependent variable for certain values of independent variable.

5.20 INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS (*M.T.U. 2012*)

Problems in which all the conditions are specified at the initial point only are called **initial-value problems**. For example, the problem given by eqns. (1) and (2) is an initial value problem.

Problems involving second and higher order differential equations, in which the conditions at two or more points are specified, are called **boundary-value problems**.

To obtain a unique solution of n^{th} order ordinary differential equation, it is necessary to specify n values of the dependent variable and/or its derivative at specific values of independent variable.

5.21 SINGLE-STEP AND MULTI-STEP METHODS

The numerical solutions are obtained step-by-step through a series of equal intervals in the independent variable so that as soon as the solution y has been obtained at $x = x_i$, the next step consists of evaluating y_{i+1} at $x = x_{i+1}$. The methods which require only the numerical value y_i in order to compute the next value y_{i+1} for solving equation (1) given above are termed as **single-step methods.**

The methods which require not only the numerical value y_i but also at least one of the past values $y_{i-1}, y_{i-2}, \ldots\ldots$ are termed as **multi-step methods.**

5.22 COMPARISON OF SINGLE-STEP AND MULTI-STEP METHODS

The single-step method has obvious advantages over the multi-step methods that use several past values ($y_n, y_{n-1}, \ldots\ldots, y_{n-p}$) and that require initial values ($y_1, y_2, \ldots\ldots, y_n$) that have to be calculated by another method.

The major disadvantage of single-step method is that they use many more evaluations of the derivative to attain the same degree of accuracy compared with the multi-step methods.

5.23 NUMERICAL METHODS OF SOLUTION OF O.D.E.

Here, we will discuss various numerical methods of solving ordinary differential equations.

We must know that these methods will yield the solution in one of the two forms:

(a) A series for y in terms of powers of x from which the value of y can be obtained by direct substitution.

(b) A set of tabulated values of x and y.

Picard's method belong to class (a) while those of Euler's, Runge-Kutta, etc., belong to class (b). Methods which belong to class (b) are called **step-by-step methods** or **marching methods** because the values of y are computed by short steps ahead for equal intervals of the independent variable.

In Euler's and Runge-Kutta methods, the interval range h should be kept small hence they can be applied for tabulating y only over a limited range.

5.24 PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

Emile Picard (1856–1941) was a distinguished Professor of Mathematics at the University of Paris, France. He was famous for his researches on the Theory of Functions.

Consider the differential equation

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0 \qquad \ldots(1)$$

Integrating eqn. (1) between the limits x_0 and x and the corresponding limits y_0 and y, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

$$\Rightarrow \qquad y - y_0 = \int_{x_0}^{x} f(x, y)\, dx$$

or,

$$\boxed{y = y_0 + \int_{x_0}^{x} f(x, y)\, dx} \qquad \ldots(2)$$

In equation (2), the unknown function y appears under the integral sign. Such type of equation is called integral equation.

This equation can be solved by the method of successive approximations or iterations.

To obtain the first approximation, we replace y by y_0 in the RHS of eqn. (2).

Now, the first approximation is

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx \qquad ...(3)$$

The integrand is a function of x alone and can be integrated.

For a second approximation, replace y_0 by $y^{(1)}$ in $f(x, y_0)$ which gives

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\{x, y^{(1)}\}\, dx \qquad ...(4)$$

Proceeding in this way, we obtain $y^{(3)}, y^{(4)}, ..., y^{(n-1)}$ and $y^{(n)}$ where

$$y^{(n)} = y_0 + \int_{x_0}^{x} f\{x, y^{(n-1)}\}\, dx \text{ with } y(x_0) = y_0$$

As a matter of fact, the process is stopped when the two values of y *viz.* $y^{(n-1)}$ and $y^{(n)}$ are same to the desired degree of accuracy.

Picard's method is of theoretical value considerably. Practically, it is unsatisfactory because of the difficulties which arise in performing the necessary integrations. However, each step gives a better approximation of the required solution than the preceding one.

EXAMPLES

Example 1. *Use Picard's method to obtain y for x = 0.2. Given:*

$$\frac{dy}{dx} = x - y \text{ with initial condition } y = 1 \text{ when } x = 0.$$

Sol. Here $f(x, y) = x - y, \quad x_0 = 0, y_0 = 1$

We have first approximation,

$$y^{(1)} = y_0 + \int_0^x f(x, y_0)\, dx = 1 + \int_0^x (x - 1)\, dx = 1 - x + \frac{x^2}{2}$$

Second approximation,

$$y^{(2)} = y_0 + \int_0^x f\{x, y^{(1)}\}\, dx = 1 + \int_0^x \{x - y^{(1)}\}\, dx$$

$$= 1 + \int_0^x \left(x - 1 + x - \frac{x^2}{2}\right) dx = 1 - x + x^2 - \frac{x^3}{6}$$

Third approximation,

$$y^{(3)} = y_0 + \int_0^x f\{x, y^{(2)}\}\, dx = 1 + \int_0^x \{x - y^{(2)}\}\, dx$$

$$= 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{6}\right) dx = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}$$

Fourth approximation,

$$y^{(4)} = y_0 + \int_0^x f\{x, y^{(3)}\}\, dx = 1 + \int_0^x \{x - y^{(3)}\}\, dx$$

$$= 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{24} \right) dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

When $x = 0.2$, we get

$$y^{(1)} = 0.82,\ y^{(2)} = 0.83867,\ y^{(3)} = 0.83740,\ y^{(4)} = 0.83746$$

Since $y^{(3)}$ and $y^{(4)}$ are same upto four decimal places, therefore,

$$y(0.2) = 0.8374.$$

Example 2. *Using Picard's method of successive approximations, obtain a solution up to fifth approximation of the equation* $\frac{dy}{dx} = y + x$ *such that* $y = 1$ *when* $x = 0$.

Sol. We have,

$$f(x, y) = y + x,\ x_0 = 0,\ y_0 = 1$$

First approximation,

$$y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0)\, dx = y_0 + \int_0^x (x + y_0)\, dx$$

$$= 1 + \int_0^x (x + 1)\, dx = 1 + x + \frac{x^2}{2}$$

Second approximation,

$$y^{(2)} = y_0 + \int_0^x f\{x, y^{(1)}\}\, dx = 1 + \int_0^x \{x + y^{(1)}\}\, dx$$

$$= 1 + \int_0^x \left(x + 1 + x + \frac{x^2}{2} \right) dx = 1 + x + x^2 + \frac{x^3}{6}$$

Third approximation,

$$y^{(3)} = y_0 + \int_0^x f\{x, y^{(2)}\}\, dx = 1 + \int_0^x \{x + y^{(2)}\}\, dx$$

$$= 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6} \right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Fourth approximation,

$$y^{(4)} = y_0 + \int_0^x f\{x, y^{(3)}\}\, dx$$

$$= 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} \right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

Fifth approximation,

$$y^{(5)} = y_0 + \int_0^x f\{x, y^{(4)}\}\, dx$$

$$= 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120} \right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

Example 3. *If* $\frac{dy}{dx} = \frac{y - x}{y + x}$*, find the value of y at x = 0.1 using Picard's method. Given that y(0) = 1.*

Sol. First approximation,

$$y^{(1)} = y_0 + \int_0^x \frac{y_0 - x}{y_0 + x}\, dx = 1 + \int_0^x \left(\frac{1 - x}{1 + x} \right) dx$$

$$= 1 + \int_0^x \left(\frac{2}{1 + x} - 1 \right) dx = 1 - x + 2 \log (1 + x)$$

Second approximation,

$$y^{(2)} = 1 + x - 2 \int_0^x \frac{x\, dx}{1 + 2 \log (1 + x)}$$

which is difficult to integrate.

Thus, when $x = 0.1$, $y^{(1)} = 1 - 0.1 + 2 \log (1.1) = 1.09062$

Here in this example, only I approximation can be obtained and so it gives the approximate value of y for $x = 0.1$.

Example 4. *Given the differential equation* $\frac{dy}{dx} = \frac{x^2}{y^2 + 1}$ *with the initial condition y = 0 when x = 0. Use Picard's method to obtain y for x = 0.25, 0.5 and 1.0 correct to three decimal places.*

Sol. The given initial value problem is

$$\frac{dy}{dx} = f(x, y) = \frac{x^2}{y^2 + 1} \qquad \text{where} \quad y_0 = 0 \quad \text{at } x_0 = 0$$

We have first approximation,

$$y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0)\, dx = 0 + \int_0^x \frac{x^2}{0 + 1}\, dx = \frac{1}{3} x^3 \qquad \text{...(1)}$$

Second approximation,

$$y^{(2)} = y_0 + \int_{x_0}^x f\{x, y^{(1)}\}\, dx$$

$$= 0 + \int_0^x \frac{x^2}{\left(\frac{x^3}{3}\right)^2 + 1}\, dx = \tan^{-1} \frac{x^3}{3}$$

$$= \frac{1}{3}x^3 - \frac{1}{3}\left(\frac{1}{3}x^3\right)^3 + \ldots = \frac{1}{3}x^3 - \frac{1}{81}\, x^9 + \ldots \qquad \ldots(2)$$

From (1) and (2), we see that $y^{(1)}$ and $y^{(2)}$ agree to the first term $\frac{x^3}{3}$. To find the range of values of x so that the series with the term $\frac{1}{3}x^3$ alone will give the result correct to three decimal places, we put

$$\frac{1}{81}x^9 \le 0.0005$$

which gives, $\quad x^9 \le 0.0405 \quad$ or $\quad x \le 0.7$

Hence, $\quad y(0.25) = \frac{1}{3}\,(0.25)^3 = 0.005 \quad$ and $\quad y(0.5) = \frac{1}{3}(0.5)^3 = 0.042$

To find $y(1.0)$, we make use of eqn. (2) which gives,

$$y(1.0) = \frac{1}{3} - \frac{1}{81} = 0.321.$$

5.25 PICARD'S METHOD FOR SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

Let $\quad \frac{dy}{dx} = \phi(x, y, z) \quad$ and $\quad \frac{dz}{dx} = f(x, y, z)$

be the simultaneous differential equations with initial conditions $y(x_0) = y_0$; $z(x_0) = z_0$.

Picard's method gives

$$y^{(1)} = y_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx;\ z^{(1)} = z_0 + \int_{x_0}^{x} f(x, y_0, z_0)\, dx$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} \phi\{x, y^{(1)}, z^{(1)}\}\, dx;\ z^{(2)} = z_0 + \int_{x_0}^{x} f\{x, y^{(1)}, z^{(1)}\}\, dx$$

and so on as successive approximations.

5.26 PICARD'S METHOD FOR SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

By putting $\frac{dy}{dx} = z$, it can be reduced to two first order simultaneous differential equations:

$$\frac{dy}{dx} = z \quad \text{and} \quad \frac{dz}{dx} = f(x, y, z)$$

which can be solved easily as explained in art. 5.25.

EXAMPLES

Example 1. *Approximate y and z by using Picard's method for the particular solution of* $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$ *given that* $y = 2$, $z = 1$ *when* $x = 0$. (G.B.T.U. 2011)

Sol. Let $\phi(x, y, z) = x + z,\ f(x, y, z) = x - y^2$

Here, $x_0 = 0,\ y_0 = 2,\ z_0 = 1$

We have, $$\frac{dy}{dx} = \phi(x, y, z) \quad \Rightarrow \quad y = y_0 + \int_{x_0}^{x} \phi(x, y, z)\, dx \qquad ...(1)$$

Also, $$\frac{dz}{dx} = f(x, y, z) \quad \Rightarrow \quad z = z_0 + \int_{x_0}^{x} f(x, y, z)\, dx \qquad ...(2)$$

First approximation,

$$y^{(1)} = y_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = 2 + \int_0^x (x + z_0)\, dx$$

$$= 2 + \int_0^x (x + 1)\, dx = 2 + x + \frac{x^2}{2}$$

and

$$z^{(1)} = z_0 + \int_{x_0}^{x} f(x, y_0, z_0)\, dx = 1 + \int_0^x (x - y_0^2)\, dx$$

$$= 1 + \int_0^x (x - 4)\, dx = 1 - 4x + \frac{x^2}{2}$$

Second approximation,

$$y^{(2)} = y_0 + \int_{x_0}^{x} \phi\{x, y^{(1)}, z^{(1)}\}\, dx = 2 + \int_0^x \{x + z^{(1)}\}\, dx$$

$$= 2 + \int_0^x \left(x + 1 - 4x + \frac{x^2}{2}\right) dx = 2 + x - \frac{3}{2}x^2 + \frac{x^3}{6}$$

$$z^{(2)} = z_0 + \int_{x_0}^{x} f\{x, y^{(1)}, z^{(1)}\}\, dx$$

$$= 1 + \int_0^x \left[x - \left(2 + x + \frac{x^2}{2}\right)^2\right] dx$$

$$= 1 - 4x - \frac{3}{2}x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{20}.$$

Example 2. *Solve by Picard's method, the differential equations* $\frac{dy}{dx} = z$, $\frac{dz}{dx} = x^3 (y + z)$ *where* $y = 1$, $z = \frac{1}{2}$ *at* $x = 0$. *Obtain the values of y and z from III approximation when* $x = 0.2$ *and* $x = 0.5$.

Sol. Let $\phi(x, y, z) = z, \quad f(x, y, z) = x^3(y + z)$

Here $x_0 = 0, \quad y_0 = 1, \quad z_0 = \frac{1}{2}$

First approximation,

$$y^{(1)} = y_0 + \int_0^x \phi(x, y_0, z_0)\, dx = 1 + \int_0^x z_0\, dx = 1 + \frac{1}{2} x$$

$$z^{(1)} = z_0 + \int_0^x f(x, y_0, z_0)\, dx = \frac{1}{2} + \int_0^x x^3 (y_0 + z_0)\, dx = \frac{1}{2} + \frac{3}{8} x^4.$$

Second approximation,

$$y^{(2)} = 1 + \int_0^x z^{(1)}\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8} x^4\right) dx = 1 + \frac{x}{2} + \frac{3}{40} x^5$$

$$z^{(2)} = \frac{1}{2} + \int_0^x x^3 \{y^{(1)} + z^{(1)}\}\, dx$$

$$= \frac{1}{2} + \int_0^x x^3 \left(\frac{3}{2} + \frac{x}{2} + \frac{3}{8} x^4\right) dx = \frac{1}{2} + \frac{3}{8} x^4 + \frac{x^5}{10} + \frac{3}{64} x^8$$

Third approximation,

$$y^{(3)} = 1 + \int_0^x z^{(2)}\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right) dx$$

$$= 1 + \frac{x}{2} + \frac{3}{40} x^5 + \frac{x^6}{60} + \frac{x^9}{192}$$

$$z^{(3)} = \frac{1}{2} + \int_0^x x^3 \{y^{(2)} + z^{(2)}\}\, dx$$

$$= \frac{1}{2} + \int_0^x x^3 \left\{\frac{3}{2} + \frac{x}{2} + \frac{3}{8} x^4 + \frac{7}{40} x^5 + \frac{3}{64} x^8\right\} dx$$

$$= \frac{1}{2} + \frac{3}{8} x^4 + \frac{x^5}{10} + \frac{3}{64} x^8 + \frac{7}{360} x^9 + \frac{1}{256} x^{12}$$

When $x = 0.2$

$$y^{(3)} = 1 + 0.1 + \frac{3}{40}(0.2)^5 + \frac{(0.2)^6}{60} + \frac{1}{192}(0.2)^9$$

$$= 1.100024 \qquad \text{(leaving higher terms)}$$

$$z^{(3)} = \frac{1}{2} + \frac{3}{8}(0.2)^4 + \frac{(0.2)^5}{10} + \frac{3}{64}(0.2)^8 + \frac{7}{360}(0.2)^9 + \frac{1}{256}(0.2)^{12}$$

$$= 0.500632 \qquad \text{(leaving higher terms)}$$

When $x = 0.5$

$$y^{(3)} = 1 + \frac{0.5}{2} + \frac{3}{40}(0.5)^5 + \frac{(0.5)^6}{60} + \frac{1}{192}\ (0.5)^9 = 1.25234375$$

$$z^{(3)} = \frac{1}{2} + \frac{3}{8}(0.5)^4 + \frac{(0.5)^5}{10} + \frac{3}{64}(0.5)^8 + \frac{7}{360}(0.5)^9 + \frac{1}{256}(0.5)^{12}$$
$$= 0.5234375.$$

Example 3. *Use Picard's method to approximate y when x = 0.1 given that*

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0 \text{ and } y = 0.5,\ \frac{dy}{dx} = 0.1 \text{ when } x = 0.$$

Sol. We have, $$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0 \quad ...(1)$$

Let $\frac{dy}{dx} = z$ so that eqn. (1) becomes $\frac{dz}{dx} + 2xz + y = 0$

Now the equations to be solved are

$$\frac{dy}{dx} = f(x, y, z) = z \quad ...(2)$$

$$\frac{dz}{dx} = \phi(x, y, z) = -(2xz + y) \quad ...(3)$$

with the conditions $y_0 = 0.5, z_0 = 0.1$ at $x_0 = 0$

First approximation

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\, dx = 0.5 + \int_0^x z_0\, dx = 0.5 + 0.1\, x$$

$$z^{(1)} = z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = 0.1 + \int_0^x [-(2x\, z_0 + y_0)]\, dx$$

$$= 0.1 - \int_0^x (0.2x + 0.5)\, dx = 0.1 - 0.5x - 0.1x^2$$

Second approximation

$$y^{(2)} = y_0 + \int_0^x f\{x, y^{(1)}, z^{(1)}\}\, dx = 0.5 + \int_0^x z^{(1)}\, dx$$

$$= 0.5 + \int_0^x (0.1 - 0.5x - 0.1x^2)\, dx$$

$$= 0.5 + 0.1x - 0.25x^2 - \frac{0.1}{3}x^3$$

$$z^{(2)} = z_0 + \int_0^x \phi\, \{x, y^{(1)}, z^{(1)}\}\, dx = 0.1 - \int_0^x [2x\, z^{(1)} + y^{(1)}]\, dx$$

$$= 0.1 - \int_0^x [2x(0.1 - 0.5x - 0.1x^2) + (0.5 + 0.1x)]\, dx$$

$$= 0.1 - \int_0^x (0.3x - x^2 - 0.2x^3 + 0.5)\, dx$$

$$= 0.1 - 0.5x - 0.3\ \frac{x^2}{2} + \frac{x^3}{3} + \frac{0.1\,x^4}{2}$$

Third approximation

$$y^{(3)} = y_0 + \int_0^x f\{x, y^{(2)}, z^{(2)}\}\, dx = 0.5 + \int_0^x z^{(2)}\, dx$$

$$= 0.5 + \int_0^x \left(0.1 - 0.5x - 0.15x^2 + \frac{x^3}{3} + 0.05x^4\right) dx$$

$$= 0.5 + 0.1x - 0.25x^2 - 0.05x^3 + \frac{x^4}{12} + 0.01x^5$$

$$z^{(3)} = z_0 + \int_0^x \phi\ \{x, y^{(2)}, z^{(2)}\}\, dx = 0.1 - \int_0^x [2x\, z^{(2)} + y^{(2)}]\, dx$$

$$= 0.1 - \int_0^x \left[2x\left(0.1 - 0.5x - 0.15x^2 + \frac{x^3}{3} + 0.05x^4\right) + \left(0.5 + 0.1x - 0.25x^2 - \frac{0.1}{3}x^3\right)\right] dx$$

$$= 0.1 - 0.5x + 0.15x^2 - \frac{2.5}{6}x^3 + 0.2x^4 + \frac{2}{15}x^5 + \frac{0.1}{6}x^6$$

Now when $x = 0.1$, $y^{(1)} = 0.51$, $y^{(2)} = 0.50746667$, $y^{(3)} = 0.50745933$

Since $y^{(2)}$ and $y^{(3)}$ are same up to four decimal places hence $y(0.1) = 0.5074$.

ASSIGNMENT

1. Find the solution of $\frac{dy}{dx} = 1 + xy$, $y(0) = 1$ in the interval $(0, 0.5)$ correct to 3 decimal places taking $h = 0.1$.
2. Solve numerically $\frac{dy}{dx} = 2x - y$, $y(0) = 0.9$ at $x = 0.4$ by Picard's method with three iterations and compare the result with the exact value.
3. Apply Picard's method to find the solution of the initial value problem $\frac{dy}{dx} = y - x$, $y(0) = 2$. Show that the iterative solution approaches the exact solution. (*M.T.U. 2012*)
4. Use Picard's method to approximate the value of y when $x = 0.1$ given that $y = 1$ when $x = 0$ and $\frac{dy}{dx} = 3x + y^2$. (*M.T.U. 2013*)
5. (*i*) For the differential equation $\frac{dy}{dx} = x - y^2$, $y(0) = 0$, calculate $y(0.2)$ by Picard's method to third approximations and round off the value at 4th place of decimals.

   (*ii*) Find an approximate value of y when $x = 0.1$ if $\frac{dy}{dx} = x - y^2$ and $y = 1$ at $x = 0$ using Picard's method.

6. Employ Picard's method to find $y\ (0.2)$ and $y\ (0.4)$ given that $\dfrac{dy}{dx} = 1 + y^2$ and $y\ (0) = 0$.

7. Employ Picard's method to obtain the solution of $\dfrac{dy}{dx} = x^2 + y^2$ for $x = 0.4$ correct to four places of decimal given that $y = 0$ when $x = 0$.

8. Find $y(0.2)$ if $\dfrac{dy}{dx} = \log_{10}(x + y)$; $y(0) = 1$. Use Picard's method.

9. Solve by Picard's method: $\dfrac{d^2y}{dx^2} = x^3\left(\dfrac{dy}{dx} + y\right)$ where $y = 1$, $\dfrac{dy}{dx} = \dfrac{1}{2}$ when $x = 0$, obtain the results up to third approximation.

Answers

1.

x :	0	0.1	0.2	0.3	0.4	0.5
y :	1	1.105	1.222	1.355	1.505	1.677

2. 0.7432, 0.7439

3. $y = 2 + 2x + \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dfrac{x^4}{24}$ (up to third approx.); exact solution: $y = 1 + x + e^x$

4. $y = 1 + x + \dfrac{5}{2}x^2 + \dfrac{4}{3}x^3 + \dfrac{3}{4}x^4 + \dfrac{9}{20}x^5$ (up to second approx.)

5. (i) 0.019984, 0.0200 (ii) 0.9138 **6.** 0.2027, 0.4227

7. 0.0213 **8.** 1.0082

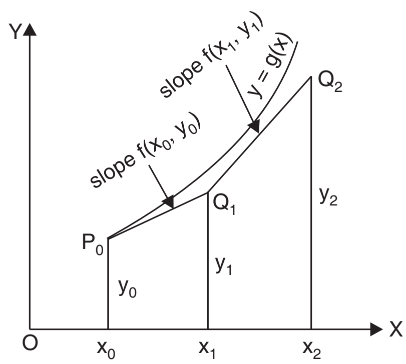
9. $y^{(1)} = 1 + \dfrac{x}{2}$, $y^{(2)} = 1 + \dfrac{x}{2} + \dfrac{3x^5}{40}$, $y^{(3)} = 1 + \dfrac{x}{2} + \dfrac{3x^5}{40} + \dfrac{x^6}{60} + \dfrac{x^9}{192}$.

5.27 EULER'S METHOD [*G.B.T.U. (MCA) 2010*]

It is the simplest one-step method and has a limited application because of its low accuracy. This method yields solution of an ordinary diff. eqn. in the form of a set of tabulated values.

In this method, we determine the change Δy is y corresponding to small increase in the argument x. Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \qquad ...(1)$$



Let $y = g(x)$ be the solution of (1). Let x_0, x_1, x_2 ...... be equidistant values of x.

In this method, we use the property that in a small interval, a curve is nearly a straight line. Thus at the point (x_0, y_0), we approximate the curve by the tangent at the point (x_0, y_0).

The equation of the tangent at $P_0(x_0, y_0)$ is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{P_0} (x - x_0) = f(x_0, y_0)\,(x - x_0)$$

$$\Rightarrow \qquad y = y_0 + (x - x_0)\, f(x_0, y_0) \qquad \text{...(2)}$$

This gives the y-coordinate of any point on the tangent. Since the curve is approximated by the tangent in the interval (x_0, x_1), the value of y on the curve corresponding to $x = x_1$ is given by the above value of y in eqn. (2) approximately.

Putting $x = x_1(= x_0 + h)$ in eqn. (2), we get

$$y_1 = y_0 + hf(x_0, y_0)$$

Thus Q_1 is (x_1, y_1)

Similarly, approximating the curve in the next interval (x_1, x_2) by a line through $Q_1(x_1, y_1)$ with slope $f(x_1, y_1)$, we get

$$y_2 = y_1 + hf(x_1, y_1)$$

In general, it can be shown that,

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

This is called Euler's formula.

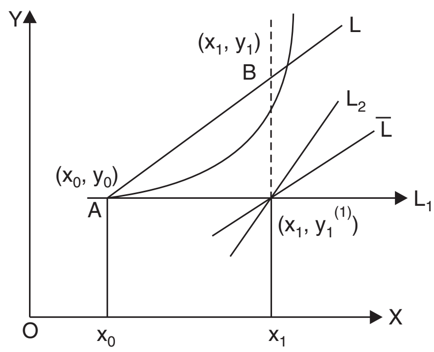
A great disadvantage of this method lies in the fact that if $\frac{dy}{dx}$ changes rapidly over an interval, its value at the beginning of the interval may give a poor approximation as compared to its average value over the interval and thus the value of y calculated from Euler's method may be in much error from its true value. These errors accumulate in the succeeding intervals and the value of y becomes much erroroneous ultimately.

Note. In Euler's method, the curve of actual solution $y = g(x)$ is approximated by a sequence of short lines. The process is very slow. If h is not properly chosen, the curve $P_0Q_1Q_2$ ...... of short lines representing numerical solution deviates significantly from the curve of actual solution.

5.28 IMPROVED EULER'S METHOD

The improved Euler's method gives greater improvement in accuracy over the original Euler's method. Here the core idea is that we use a line through (x_0, y_0) whose slope is the average of the slopes at (x_0, y_0) and $(x_1, y_1^{(1)})$ where $y_1^{(1)} = y_0 + hf(x_0, y_0)$. This line approximates the curve in the interval (x_0, x_1).

Geometrically, if L_1 is the tangent at (x_0, y_0), L_2 is a line through $(x_1, y_1^{(1)})$ of slope $f(x_1, y_1^{(1)})$ and $\overline{L}$ is the line through $(x_1, y_1^{(1)})$ but with a slope equal to the average of $f(x_0, y_0)$ and $f(x_1, y_1^{(1)})$ then the line L through (x_0, y_0) and parallel to $\overline{L}$ is used to approximate the curve in the interval (x_0, x_1). Thus the ordinate of the point B will give the value of y_1. Now, the eqn. of the line AL is given by



$$y_1 = y_0 + (x_1 - x_0)\left[\frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}\right]$$

$$= y_0 + h \left[\frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2} \right] \quad ...(1)$$

A generalized form of improved Euler's formula is

$$\boxed{y_1^{(n+1)} = y_0 + \frac{h}{2}\, [f(x_0, y_0) + f\{x_1, y_1^{(n)}\}]} \; ; n = 0, 1, 2, ... \quad ...(2)$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1.

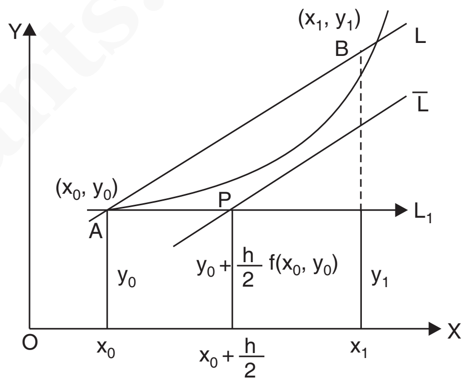
The above iteration formula can be started by choosing $y_1^{(1)}$ from Euler's formula

$$y_1^{(1)} = y_0 + hf(x_0, y_0) \quad ...(3)$$

Since this formula attempts to correct the values of y_{n+1} using the predicted value of y_{n+1} (by Euler's method), it is classified as a *one-step predictor-corrector method.*

5.29 MODIFIED EULER'S METHOD

In this method, the curve in the interval (x_0, x_1), where $x_1 = x_0 + h$ is approximated by the line through (x_0, y_0) with slope $f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$, which is the slope at the middle point whose abscissa is the average of x_0 and x_1.



In the adjoining figure, line AL through $A(x_0, y_0)$ which is parallel to the line $P\overline{L}$ with slope $f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$ approximates the curve in the interval (x_0, x_1). The ordinate at $x = x_1$, meeting the line L at B, will give the value of y_1.

The equation for line AL is

$$y - y_0 = (x - x_0) \left[f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\} \right] \quad ...(1)$$

Putting $x = x_1$ in (1), we get

$$y_1 = y_0 + (x_1 - x_0) \left[f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\} \right]$$

$$\Rightarrow \quad y_1 = y_0 + h\, f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$$

Proceeding in the same way, we obtain

$$\boxed{y_{n+1} = y_n + h\, f\left\{x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right\}} \quad ...(2)$$

which is the generalized form of **modified Euler's formula.**

EXAMPLES

Example 1. *Given* $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$ *with y = 1 for x = 0. Find y approximately for x = 0.1 by Euler's method.*

Sol. We have

$$\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x} \; ; x_0 = 0, y_0 = 1, h = 0.1$$

Hence the approximate value of y at $x = 0.1$ is given by

$$y_1 = y_0 + hf(x_0, y_0) \qquad | \text{ Using } y_{n+1} = y_n + hf(x_n, y_n)$$

$$= 1 + (0.1)\left(\frac{1-0}{1+0}\right) = 1.1$$

Much better accuracy is obtained by breaking up the interval 0 to 0.1 into five steps. The approximate value of y at $x_A = 0.02$ is given by,

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.02)\left(\frac{1-0}{1+0}\right) = 1.02$$

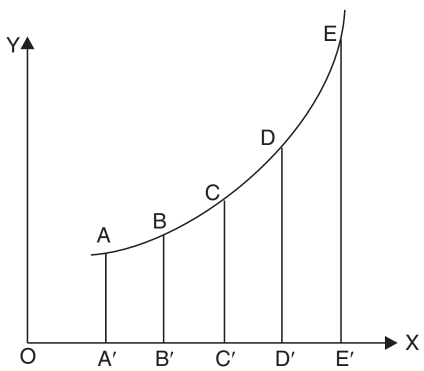
At $x_B = 0.04$, $\quad y_2 = y_1 + hf(x_1, y_1) = 1.02 + (0.02)\left(\dfrac{1.02-0.02}{1.02+0.02}\right) = 1.0392$

At $x_C = 0.06$, $\quad y_3 = 1.0392 + (0.02)\left(\dfrac{1.0392-0.04}{1.0392+0.04}\right) = 1.0577$

At $x_D = 0.08$, $\quad y_4 = 1.0577 + (0.02)\left(\dfrac{1.0577-0.06}{1.0577+0.06}\right) = 1.0756$

At $x_E = 0.1$, $\quad y_5 = 1.0756 + (0.02)\left(\dfrac{1.0756-0.08}{1.0756+0.08}\right) = 1.0928$

Hence $y = 1.0928$ when $x = 0.1$



Example 2. *Solve the equation* $\frac{dy}{dx} = 1 - y$ *with the initial condition x = 0, y = 0 using Euler's algorithm and tabulate the solutions at x = 0.1, 0.2, 0.3.*

Sol. Here, $f(x, y) = 1 - y$

Taking $h = 0.1$, $x_0 = 0$, $y_0 = 0$, we obtain

$$y_1 = y_0 + hf(x_0, y_0) = 0 + (0.1)(1 - 0) = 0.1$$

$\therefore$ $y(0.1) = 0.1$

Again, $y_2 = y_1 + hf(x_1, y_1) = 0.1 + (0.1)(1 - 0.1) = 0.1 + 0.09 = 0.19$

$\therefore$ $y(0.2) = 0.19$

Again, $y_3 = y_2 + hf(x_2, y_2) = 0.19 + (0.1)(1 - 0.19) = 0.271$

$\therefore$ $y(0.3) = 0.271$

Tabulated values are

x	$y(x)$
0	0
0.1	0.1
0.2	0.19
0.3	0.271

Example 3. *Use Euler's method to obtain an approximate value of y(0.4) for the equation* $y' = x + y$, $y(0) = 1$ *with* $h = 0.1$.

Sol. Here, $f(x, y) = x + y$

Taking $h = 0.1$, $x_0 = 0$, $y_0 = 1$, we obtain

$$y_1 = y_0 + h\, f(x_0, y_0) = 1 + (0.1)(0 + 1) = 1.1$$

$\therefore$ $y(0.1) = 1.1$

Again, $y_2 = y_1 + h\, f(x_1, y_1) = 1.1 + (0.1)(0.1 + 1.1) = 1.22$

$\therefore$ $y(0.2) = 1.22$

Again, $y_3 = y_2 + h\, f(x_2, y_2) = 1.22 + (0.1)(0.2 + 1.22) = 1.362$

$\therefore$ $y(0.3) = 1.362$

Again, $y_4 = y_3 + h\, f(x_3, y_3) = 1.362 + (0.1)(0.3 + 1.362) = 1.5282$

$\therefore$ $y(0.4) = 1.5282$

Example 4. *Solve the following differential equation using Euler's method from x = 0 to x = 0.2 when h = 0.05.*

$$\frac{dy}{dx} + xy = 0,\ y(0) = 1$$

Sol. Here, $f(x, y) = -xy$

Taking $h = 0.05$, $x_0 = 0$, $y_0 = 1$, we obtain

$$y_1 = y_0 + h\, f(x_0, y_0) = 1 + (0.05)\{-(0 \times 1)\} = 1$$

$\therefore$ $y(0.05) = 1$

Again, $y_2 = y_1 + h\, f(x_1, y_1) = 1 + (0.05)\{-(0.05 \times 1)\} = 0.9975$

$\therefore$ $y(0.1) = 0.9975$

Again, $y_3 = y_2 + h\, f(x_2, y_2) = 0.9975 + (0.05)\{-(0.1 \times 0.9975)\} = 0.9925125$

$\therefore$ $y(0.15) = 0.9925125$

Again, $y_4 = y_3 + h\, f(x_3, y_3) = 0.9925125 + (0.05)\, \{-(0.15)\,(0.9925125)\}$
$= 0.985068656$

$\therefore \quad y(0.2) = 0.985068656$

Hence, in tabular form, we can write

x	0	0.05	0.10	0.15	0.2
y	1	1	0.9975	0.9925125	0.985068656

Example 5. *Find y(0.1) using improved Euler's method and then y(0.2) by using modified Euler's method, given that* $\dfrac{dy}{dx} = \log(x + y),\ y(0) = 1.0$ (U.P.T.U. 2007)

Sol. Here $f(x, y) = \log(x + y)$

Also, $x_0 = 0,\ y_0 = 1,\ h = 0.1$

Using improved Euler's method, we get

$$y_1^{(1)} = y_0 + h\, f(x_0, y_0) = 1 + (0.1)\log(0 + 1) = 1$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f\{x_1, y_1^{(1)}\}]$$

$$= 1 + \frac{(0.1)}{2}\,[0 + \log(0.1 + 1)] = 1.0047655$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f\{x_1, y_1^{(2)}\}]$$

$$= 1 + \frac{(0.1)}{2}\,[0 + \log(0.1 + 1.0047655)] = 1.0049816$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f\{x_1, y_1^{(3)}\}]$$

$$= 1 + \frac{(0.1)}{2}\,[0 + \log(0.1 + 1.0049816)] = 1.0049914$$

Since $y_1^{(3)}$ and $y_1^{(4)}$ are same up to four decimal places

$\therefore \quad y_1 = y(0.1) = 1.0049$

Again, $x_1 = 0.1,\ y_1 = 1.0049,\ h = 0.1$

Using modified Euler's method, we get

$$y_2 = y_1 + h\, f\left\{x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f(x_1, y_1)\right\}$$

$$= 1.0049 + (0.1)\log\left\{x_1 + \frac{h}{2} + y_1 + \frac{h}{2} f(x_1, y_1)\right\}$$

$$= 1.0049 + (0.1)\log[0.1 + 0.05 + 1.0049 + (0.05)\log(0.1 + 1.0049)]$$

$$= 1.0197323$$

$\therefore \quad y(0.2) = 1.0197323$

Example 6. *Apply modified Euler's method to solve*

$$\frac{dy}{dx} = e^x + xy,\ y(0) = 0 \ \text{ to compute } y(0.1) \text{ and } y(0.2).$$

Sol. Here, $f(x, y) = e^x + xy$

We have, $x_0 = 0, y_0 = 0, h = 0.1$

Using modified Euler's method, we obtain

$$y_1 = y_0 + h\, f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$$

$$= 0 + (0.1)\ [e^{0.05} + (0.05)\ (0.05)\ (e^0 + 0)] = 0.105377$$

$\therefore \quad y(0.1) = 0.105377$

Again, $$y_2 = y_1 + h\, f\left\{x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right\}$$

$$= 0.105377 + (0.1)\ [e^{0.15} + (0.15)\ \{(0.105377) + (0.05)\ (e^{0.1} + 0.0105377)\}]$$

$$= 0.105377 + (0.1)\ [e^{0.15} + (0.15)\ (0.16116243)] = 0.22397786$$

$\therefore \quad y(0.2) = 0.22397786.$

ASSIGNMENT

1. Using Euler's method, compute $y(0.04)$ for the differential equation

   $\frac{dy}{dx} = -y; \quad y(0) = 1. \qquad$ Take $h = 0.01.$ (*U.P.T.U. 2007*)

2. Apply Euler's method to solve $\frac{dy}{dx} = x + y$; $y(0) = 0$ choosing $h = 0.2$ and compute $y(0.4)$, $y(0.6)$, $y(0.8)$ and $y(1.0)$.

3. If $\frac{dy}{dx} = 1 + y^2$, $y(0) = 1$, find $y(0.4)$ by using Euler's method. Take $h = 0.2$.

   [*G.B.T.U. (C.O.) 2011*]

4. Given that $\frac{dy}{dx} = \log_{10}(x + y)$; $y(0) = 1$

   Find $y(0.2)$ and $y(0.5)$ using improved Euler's method.

5. Using improved Euler's method, obtain a solution of the equation $\frac{dy}{dx} = x + |\sqrt{y}| = f(x, y)$ with initial condition $y = 1$ at $x = 0$ for the range $0 \le x \le 0.6$ in steps of 0.2.

6. Use Euler's modified method to obtain $y(0.25)$ given that $y' = 2xy$, $y(0) = 1$

7. Solve $y' = x^2 + y$, $y(0) = 1$ to obtain $y(0.02)$ and $y(0.04)$ using Euler's modified method.

8. Solve $\frac{dy}{dx} = y - \frac{2x}{y}$; $y(0) = 1$ in the range $0 \le x \le 0.2$ using

   (*i*) Improved Euler's method (*ii*) Modified Euler's method.

Answers

1. $y(0.04) = 0.960596$

2. $y(0.4) = 0.04$, $y(0.6) = 0.128$, $y(0.8) = 0.2736$, $y(1.0) = 0.48832$

3. $y(0.4) = 1.992$ **4.** $y(0.2) = 1.0082$, $y(0.5) = 1.0490$

5. $y(0.2) = 1.2309$, $y(0.4) = 1.5253$, $y(0.6) = 1.8861$

6. $y(0.25) = 1.0625$ **7.** $y(0.02) = 1.0202$, $y(0.04) = 1.0408$

8.

x	*Improved Euler's method*	*Modified Euler's method*
0	1	1
0.1	1.095	1.0954762
0.2	1.1828	1.1832984

5.30 RUNGE-KUTTA METHODS

More efficient methods in terms of accuracy were developed by two German Mathematicians **Carl Runge** (1856–1927) and **Wilhelm Kutta** (1867–1944). These methods are well-known as Runge-Kutta methods. They are distinguished by their orders in the sense that they agree with Taylor's series solution upto terms of h^r, where r is the order of the method.

These methods donot demand prior computation of higher derivatives of $y(x)$ as in Taylor's method. In place of these derivatives, extra values of the given function $f(x, y)$ are used.

Fourth order Runge-Kutta method is widely used for finding the numerical solutions of linear or non-linear ordinary differential equations.

Runge-Kutta methods are referred to a single step methods. The major disadvantage of Runge-Kutta methods is that they use many more evaluations of the derivative $f(x, y)$ to obtain the same accuracy compared with multi-step methods. A class of methods known as Runge-Kutta methods combines the advantage of high order accuracy with the property of being one step.

5.30.1. First Order Runge-Kutta Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y); \; y(x_0) = y_0 \qquad ...(1)$$

Euler's method gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \qquad ...(2)$$

Expanding by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + ... \qquad ...(3)$$

Comparing (2) and (3), it follows that Euler's method agrees with Taylor's series solution up to the term in h. Hence *Euler's method is the first order Runge-Kutta method*.

5.30.2. Second Order Runge-Kutta Method

Consider the differential equation

$$y' = f(x, y) \text{ with the initial condition } y(x_0) = y_0$$

Let h be the interval between equidistant values of x then in second order Runge-Kutta method, the first increment in y is computed from the formulae

$$\left.\begin{aligned} k_1 &= hf(x_0, y_0) \\ k_2 &= hf(x_0 + h, y_0 + k_1) \\ \Delta y &= \tfrac{1}{2}(k_1 + k_2) \end{aligned}\right\}$$

taken in the given order.

Then,
$$x_1 = x_0 + h$$
$$y_1 = y_0 + \Delta y = y_0 + \tfrac{1}{2}(k_1 + k_2)$$

In a similar manner, the increment in y for the second interval is computed by means of the formulae,

$$k_1 = hf(x_1, y_1)$$
$$k_2 = hf(x_1 + h, y_1 + k_1)$$
$$\Delta y = \tfrac{1}{2}(k_1 + k_2)$$

and similarly for the next intervals.

The inherent error in the second order Runge-Kutta method is of order h^3.

5.30.3. Third Order Runge-Kutta Method

This method gives the approximate solution of the initial value problem

$$\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0 \text{ as}$$

$$y_1 = y_0 + \Delta y \qquad \text{...(1)}$$

where
$$\Delta y = \frac{h}{6}(k_1 + 4k_2 + k_3)$$

Here,
$$k_1 = f(x_0, y_0)$$
$$k_2 = f\left\{x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right\}$$
$$k_3 = f(x_0 + h, y_0 + k');\ k' = hf(x_0 + h, y_0 + k_1)$$

Formula (1) can be generalised for successive approximations. Expression in (1) agrees with Taylor's series expansion for y_1 up to and including terms in h^3. This method is also known as **Runge's method**.

5.31 FOURTH ORDER RUNGE-KUTTA METHOD (*U.P.T.U. 2015*)

It is one of the most widely used methods and is particularly suitable in cases when the computation of higher derivatives is complicated.

Consider the differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$. Let h be the interval between equidistant values of x then the first increment in y is computed from the formulae

$$\left.\begin{aligned} k_1 &= hf\,(x_0, y_0)\\ k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)\\ k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)\\ k_4 &= hf\,(x_0 + h, y_0 + k_3) \end{aligned}\right\} \quad ....(1)$$

$$\Delta y = \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)$$

taken in the given order.

Then, $\qquad x_1 = x_0 + h \quad$ and $\quad y_1 = y_0 + \Delta y$

In a similar manner, the increment in y for the second interval is computed by means of the formulae

$$\begin{aligned} k_1 &= hf\,(x_1, y_1)\\ k_2 &= hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)\\ k_3 &= hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)\\ k_4 &= hf\,(x_1 + h, y_1 + k_3)\\ \Delta y &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

and similarly for the next intervals.

This method is also termed as *Runge-Kutta's* method simply.

It is to be noted that the calculations for the first increment are exactly the same as for any other increment. The change in the formula for the different intervals is only in the values of x and y to be substituted. Hence to obtain Δy for the n^{th} interval, we substitute $x_{n-1}, y_{n-1,}$ in the expressions for k_1, k_2, etc.

The inherent error in the fourth order Runge-Kutta method is of order h^5.

5.32 RUNGE-KUTTA METHOD FOR SIMULTANEOUS FIRST ORDER EQUATIONS

Consider the simultaneous equations

$$\frac{dy}{dx} = f_1(x, y, z) \quad ...(1)$$

$$\frac{dz}{dx} = f_2\,(x, y, z) \quad ...(2)$$

with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. Now, starting from (x_0, y_0, z_0), the increments k and l in y and z are given by the following formulae:

$$k_1 = hf_1(x_0, y_0, z_0); \qquad l_1 = hf_2(x_0, y_0, z_0)$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right); \qquad l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right); \qquad l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3); \qquad l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); \qquad l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Hence, $\quad y_1 = y_0 + k, \qquad z_1 = z_0 + l$

To compute y_2, z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

5.33 RUNGE-KUTTA METHOD FOR SECOND ORDER DIFFERENTIAL EQUATIONS

Consider a second order differential equation

$$\frac{d^2y}{dx^2} = \phi\left(x, y, \frac{dy}{dx}\right) \qquad ...(1)$$

with initial conditions $\quad y(x_0) = y_0, y'(x_0) = y_0'$ $\qquad$ ...(2)

Let $\quad \frac{dy}{dx} = z$ so that $\frac{d^2y}{dx^2} = \frac{dz}{dx}$.

Substituting in equation (1), we get

$$\frac{dz}{dx} = \phi\,(x, y, z)$$

with initial conditions $\quad y(x_0) = y_0, z(x_0) = z_0$

Hence, the problem is reduced to solving the simultaneous equations

$$\frac{dy}{dx} = z = f_1\,(x, y, z) \quad \text{and} \quad \frac{dz}{dx} = f_2\,(x, y, z)$$

subject to $y(x_0) = y_0, z(x_0) = z_0$.

Above simultaneous equations can be solved as explained in art. 5.32.

EXAMPLES

Example 1. *Solve the equation* $\frac{dy}{dx} = x + y$ *with initial condition* $y(0) = 1$ *by Runge-Kutta rule, from* $x = 0$ *to* $x = 0.4$ *with* $h = 0.1$. (M.T.U. 2014)

Sol. Here $\quad f(x, y) = x + y, h = 0.1, x_0 = 0, y_0 = 1$

We have,

$$k_1 = hf\,(x_0, y_0) = 0.1\,(0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, \quad y_0 + \frac{k_1}{2}\right) = 0.1\,(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, \quad y_0 + \frac{k_2}{2}\right) = 0.1105$$

$$k_4 = hf\,(x_0 + h, \quad y_0 + k_3) = 0.12105$$

$\therefore \qquad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.11034$

Thus, $\quad x_1 = x_0 + h = 0.1 \quad$ and $\quad y_1 = y_0 + \Delta y = 1.11034$

Now for the second interval, we have

$$k_1 = hf(x_1, y_1) = 0.1\,(0.1 + 1.11034) = 0.121034$$

$$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right) = 0.13208$$

$$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right) = 0.13263$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.14429$$

$\therefore \qquad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.132460$

Hence $\quad x_2 = 0.2 \quad$ and $\quad y_2 = y_1 + \Delta y = 1.11034 + 0.13246 = 1.24280$

Similarly, for finding y_3, we have

$$k_1 = hf(x_2, y_2) = 0.14428,\ k_2 = 0.15649,\ k_3 = 0.15710,\ k_4 = 0.16999$$

$\therefore \qquad y_3 = 1.3997$

and $\quad$ for $y_4 = y(0.4)$, we calculate

$$k_1 = 0.16997,\ k_2 = 0.18347,\ k_3 = 0.18414,\ k_4 = 0.19838$$

$\therefore \qquad y_4 = 1.5836.$

Example 2. *Given* $\frac{dy}{dx} = y - x$, $y(0) = 2$. *Find* $y(0.1)$ *and* $y(0.2)$ *correct to four decimal places. (Use both second and fourth order methods)* (U.P.T.U. 2014, 2015)

Sol. By second order Method

To find y(0.1)

Here $\qquad y' = f(x, y) = y - x,\ x_0 = 0,\ y_0 = 2$ and $h = 0.1$

Now, $\qquad k_1 = hf(x_0, y_0) = 0.1(2 - 0) = 0.2$

$$k_2 = hf(x_0 + h, y_0 + k_1) = 0.21$$

$\therefore \qquad \Delta y = \frac{1}{2}(k_1 + k_2) = 0.205$

Thus, $\quad x_1 = x_0 + h = 0.1 \quad$ and $\quad y_1 = y_0 + \Delta y = 2.205$

To find y(0.2) we note that,

$$x_1 = 0.1,\ y_1 = 2.205,\ h = 0.1$$

For second interval, we have,

$$k_1 = hf(x_1, y_1) = 0.2105$$

$$k_2 = hf(x_1 + h, y_1 + k_1) = 0.22155$$

$\therefore \qquad \Delta y = \frac{1}{2}(k_1 + k_2) = 0.216025$

Thus, $\quad x_2 = x_1 + h = 0.2 \quad$ and $\quad y_2 = y_1 + \Delta y = 2.4210$

Hence $\qquad y(0.1) = 2.205,\ y(0.2) = 2.421.$

By fourth order method

$$y' = f(x, y) = y - x, x_0 = 0, \ y_0 = 2 \ \text{ and } \ h = 0.1$$

$$k_1 = hf(x_0, y_0) = 0.1\ (2 - 0) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.205$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.20525$$

and $$k_4 = hf\,(x_0 + h, y_0 + k_3) = 0.210525$$

$\therefore$ $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2052$$

Thus, $$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y_0 + \Delta y = 2 + 0.2052 = 2.2052$$

Now to determine $y_2 = y(0.2)$, we note that

$$x_1 = x_0 + h = 0.1, y_1 = 2.2052, h = 0.1$$

For II interval, $$k_1 = hf\,(x_1, y_1) = 0.21052$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.21605$$

$$k_3 = hf\left(x_1 + h/2, y_1 + \frac{k_2}{2}\right) = 0.216323$$

and $$k_4 = hf\,(x_1 + h, y_1 + k_3) = 0.221523$$

$\therefore$ $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.21613$$

Thus, $$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

and $$y_2 = y_1 + \Delta y = 2.2052 + 0.21613 = 2.4213$$

Hence, $y(0.1) = 2.2052, \quad y(0.2) = 2.4213.$

Example 3. *Given the initial value problem: $y' = 1 + y^2$, $y(0) = 0$*

Find $y(0.6)$ by Runge-Kutta fourth order method taking $h = 0.2$. (U.P.T.U. 2008, 2006)

Sol. Here, $f(x, y) = 1 + y^2, h = 0.2, x_0 = 0, y_0 = 0$

We have, $$k_1 = h\,f\,(x_0, y_0) = 0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.202$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2020402$$

$$k_4 = h\,f\,(x_0 + h, y_0 + k_3) = 0.2081640$$

$\therefore$ $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2027074$$

Thus, $y_1 = y(0.2) = y_0 + \Delta y = 0.2027074,\ x_1 = x_0 + h = 0.2$

Again, $f(x, y) = 1 + y^2,\ h = 0.2,\ x_1 = 0.2,\ y_1 = 0.2027074$

We have, $k_1 = h\, f(x_1, y_1) = 0.208218$

$$k_2 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.218827$$

$$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.219484$$

$$k_4 = h\, f(x_1 + h, y_1 + k_3) = 0.235649$$

$$\therefore \quad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.22008$$

Thus, $y_2 = y(0.4) = y_1 + \Delta y = 0.4227874,\ x_2 = x_1 + h = 0.4$

Again, $f(x, y) = 1 + y^2,\ h = 0.2,\ x_2 = 0.4,\ y_2 = 0.4227874$

We have, $k_1 = h\, f(x_2, y_2) = 0.235749$

$$k_2 = h\, f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.258463$$

$$k_3 = h\, f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.260945$$

$$k_4 = h\, f(x_2 + h, y_2 + k_3) = 0.293498$$

$$\therefore \quad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.261344$$

Thus, $y_3 = y(0.6) = y_2 + \Delta y = 0.6841314.$

Example 4. *Use the Runge-Kutta fourth order method to find the value of y when x = 1 given that y = 1 when x = 0 (taking n = 2) and* $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$. [U.P.T.U. (MCA) 2008]

Sol. Here, $f(x, y) = \dfrac{y - x}{y + x},\ x_0 = 0,\ y_0 = 1,\ h = 0.5$

We have, $k_1 = h\, f(x_0, y_0) = 0.5$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.333$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.3235$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = 0.2258$$

$$\therefore \quad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.3398$$

Thus, $y_1 = y(0.5) = y_0 + \Delta y = 1.3398,\ x_1 = x_0 + h = 0.5$

Again, $k_1 = h\, f(x_1, y_1) = 0.22823$

$$k_2 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.15969$$

$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.15432$$

$$k_4 = h\,f\,(x_1 + h, y_1 + k_3) = 0.09906$$

$\therefore$
$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.159218$$

Thus, $y_2 = y(1.0) = y_1 + \Delta y = 1.499018.$

Example 5. *Use Runge-Kutta method of fourth order to approximate y when x = 0.1 given that y = 1 at x = 0 and* $\frac{dy}{dx} = 3x + y^2$. [U.P.T.U. MCA (C.O.) 2008]

Sol. Here, $f(x, y) = 3x + y^2,\ h = 0.1,\ x_0 = 0,\ y_0 = 1$

We have,
$$k_1 = h\,f\,(x_0, y_0) = 0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.12525$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.127917$$

$$k_4 = h\,f\,(x_0 + h, y_0 + k_3) = 0.15722$$

$\therefore$
$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.127259$$

Thus, $y(0.1) = y_0 + \Delta y = 1.127259.$

Example 6. *Find the value of y(1.1) using Runge-Kutta method of fourth order, given that* $\frac{dy}{dx} = y^2 + xy$, *y(1) = 1.0, take h = 0.05.* (G.B.T.U. 2011, 2012; M.T.U. 2013)

Sol. Here, $f(x, y) = y^2 + xy,\ x_0 = 1,\ y_0 = 1,\ h = 0.05$

We have,
$$k_1 = h\,f\,(x_0, y_0) = 0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.10894$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.109637$$

$$k_4 = h\,f\,(x_0 + h, y_0 + k_3) = 0.119821$$

$\therefore$
$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.109496$$

Thus, $y_1 = y(1.05) = y_0 + \Delta y = 1.109496,\ x_1 = x_0 + h = 1.05$

Again,
$$k_1 = h\,f\,(x_1, y_1) = 0.119798$$

$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.13123$$

$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.13221$$

$$k_4 = h\,f\,(x_1 + h, y_1 + k_3) = 0.145385$$

$\therefore \qquad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.13201$

Thus, $\qquad y_2 = y(1.1) = y_1 + \Delta y = 1.241506$

Example 7. *Use Runge-Kutta IV order method with $h = 0.1$ to find $x(0.1)$ and $x(0.2)$ where $\frac{dx}{dt} = t - x$ and $x(0) = 0$.*

Sol. Here, $\qquad f(t, x) = t - x,\ t_0 = 0,\ x_0 = 0$

$$k_1 = hf(t_0, x_0) = 0$$

$$k_2 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right) = 0.005$$

$$k_3 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right) = 0.00475$$

$$k_4 = hf(t_0 + h, x_0 + k_3) = 0.009525$$

$\therefore \qquad \Delta x = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.0048375$

Thus, $\qquad x(0.1) = x_0 + \Delta x = 0.0048375,\ t_1 = t_0 + h = 0.1$

Again, $\qquad k_1 = hf(t_1, x_1) = 0.00951625$

$$k_2 = hf\left(t_1 + \frac{h}{2}, x_1 + \frac{k_1}{2}\right) = 0.014040$$

$$k_3 = hf\left(t_1 + \frac{h}{2}, x_1 + \frac{k_2}{2}\right) = 0.013814$$

$$k_4 = hf(t_1 + h, x_1 + k_3) = 0.018135$$

$\therefore \qquad \Delta x = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.013893$

Thus, $\qquad x(0.2) = x_1 + \Delta x = 0.0187305.$

Example 8. *Solve the initial value problem $u' = -2tu^2$, $u(0) = 1$ with $h = 0.2$ on the interval [0, 0.4]. Use Runge-Kutta fourth order method and compare your result with exact solution.* [U.P.T.U. 2009, U.P.T.U. (MCA) 2006]

Sol. Here, $\qquad f(t, u) = -2t\,u^2,\ t_0 = 0,\ u_0 = 1,\ h = 0.2$

We have, $\qquad k_1 = h\,f(t_0, u_0) = 0$

$$k_2 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{k_1}{2}\right) = -0.04$$

$$k_3 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{k_2}{2}\right) = -0.038416$$

$$k_4 = hf(t_0 + h, u_0 + k_3) = -0.0739715$$

$\therefore \qquad \Delta u = -0.03846725$

Thus, $\qquad u_1 = u(0.2) = u_0 + \Delta u = 0.9615328$

Again, $k_1 = hf(t_1, u_1) = -0.0739636$

$$k_2 = hf\left(t_1 + \frac{h}{2}, u_1 + \frac{k_1}{2}\right) = -0.1025753$$

$$k_3 = hf\left(t_1 + \frac{h}{2}, u_1 + \frac{k_2}{2}\right) = -0.0994255$$

$$k_4 = hf(t_1 + h, u_1 + k_3) = -0.1189166$$

$\therefore$ $\Delta u = -0.0994803$

Thus, $u_2 = u(0.4) = u_1 + \Delta u = 0.8620525$

Absolute errors in numerical solutions are

$$\in(0.2) = |\ 0.961539 - 0.961533\ | = 0.000006$$

$$\in(0.4) = |\ 0.862069 - 0.862053\ | = 0.000016$$

Example 9. *Using Runge-Kutta method of fourth order, find y(0.8) correct to 4 decimal places if* $y' = y - x^2$, $y(0.6) = 1.7379$, *taking* $h = 0.1$. (U.P.T.U. 2007)

Sol. Here, $f(x, y) = y - x^2$, $x_0 = 0.6$, $h = 0.1$, $y_0 = 1.7379$

We have, $k_1 = hf(x_0, y_0) = h(y_0 - x_0^2) = 0.13779$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left[\left(y_0 + \frac{k_1}{2}\right) - \left(x_0 + \frac{h}{2}\right)^2\right] = 0.1384295$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\left[\left(y_0 + \frac{k_2}{2}\right) - \left(x_0 + \frac{h}{2}\right)^2\right]$$

$$= 0.138461475$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = h[(y_0 + k_3) - (x_0 + h)^2] = 0.1386361475$$

$\therefore$ $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.138368$$

Thus, $y_1 = y(0.7) = y_0 + \Delta y = 1.876268$, $x_1 = x_0 + h = 0.7$

Now, $x_1 = 0.7$, $y_1 = y(0.7) = 1.876268$, $h = 0.1$

Again, $k_1 = hf(x_1, y_1) = h(y_1 - x_1^2) = 0.1386268$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = h\left[\left(y_1 + \frac{k_1}{2}\right) - \left(x_1 + \frac{h}{2}\right)^2\right] = 0.13830814$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = h\left[\left(y_1 + \frac{k_2}{2}\right) - \left(x_1 + \frac{h}{2}\right)^2\right]$$

$$= 0.138292207$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = h[(y_1 + k_3) - (x_1 + h)^2] = 0.1374560207$$

$\therefore$ $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.138213919$$

Thus, $y_2 = y(0.8) = y_1 + \Delta y = 2.0145$, $x_2 = x_1 + h = 0.8$.

Example 10. *Solve:* $y' = \dfrac{1}{x+y}$, *y(0) = 1 for x = 0.5 to x = 1 by Runge-Kutta method. Take h = 0.5.*

Sol. Here, $f(x, y) = \dfrac{1}{x+y}$, $x_0 = 0$, $h = 0.5$, $y_0 = 1$

We have, $k_1 = hf(x_0, y_0) = h\left(\dfrac{1}{x_0 + y_0}\right) = 0.5$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h \frac{1}{\left(x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}\right)} = 0.3333$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h \,.\, \frac{1}{\left(x_0 + \frac{h}{2} + y_0 + \frac{k_2}{2}\right)} = 0.3529$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = h \frac{1}{(x_0 + h + y_0 + k_3)} = 0.2698$$

$\therefore$ $\Delta y = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.3570$

Thus, $y_1 = y(0.5) = y_0 + \Delta y = 1.3570$, $x_1 = x_0 + h = 0.5$

Now, $x_1 = 0.5$, $y_1 = 1.3570$, $h = 0.5$

Again, $k_1 = hf(x_1, y_1) = \dfrac{h}{x_1 + y_1} = 0.2692$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = \frac{h}{x_1 + \frac{h}{2} + y_1 + \frac{k_1}{2}} = 0.2230$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = \frac{h}{x_1 + \frac{h}{2} + y_1 + \frac{k_2}{2}} = 0.2254$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = \frac{h}{x_1 + h + y_1 + k_3} = 0.1936$$

$\therefore$ $\Delta y = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2266$

Thus, $y_2 = y(1) = y_1 + \Delta y = 1.5836$, $x_2 = x_1 + h = 1.$

Example 11. *If* $\dfrac{dy}{dx} = x + y^2$, *use Runge-Kutta method of fouth order to find an approximate value of y for x = 0.2 given that y = 1 when x = 0. (Take h = 0.1)* (U.P.T.U. 2009)

Sol. Here, $f(x, y) = x + y^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

We have, $k_1 = hf(x_0, y_0) = h(x_0 + y_0^2) = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left[x_0 + \frac{h}{2} + \left(y_0 + \frac{k_1}{2}\right)^2\right] = 0.11525$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\left[x_0 + \frac{h}{2} + \left(y_0 + \frac{k_2}{2}\right)^2\right] = 0.116857$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = h[x_0 + h + (y_0 + k_3)^2] = 0.134737$$

$\therefore$ $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.116492$$

Thus, $y_1 = y(0.1) = y_0 + \Delta y = 1.116492$, $x_1 = x_0 + h = 0.1$

Now, $x_1 = 0.1$, $y_1 = 1.116492$, $h = 0.1$

Again, $k_1 = hf(x_1, y_1) = h(x_1 + y_1^2) = 0.134655$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = h\left[x_1 + \frac{h}{2} + \left(y_1 + \frac{k_1}{2}\right)^2\right] = 0.155143$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = h\left[x_1 + \frac{h}{2} + \left(y_1 + \frac{k_2}{2}\right)^2\right] = 0.157579$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = h[x_1 + h + (y_1 + k_3)^2] = 0.1823257$$

$\therefore$ $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.15707$$

Thus, $y_2 = y(0.2) = y_1 + \Delta y = 1.273562$, $x_2 = x_1 + h = 0.2.$

Example 12. *Solve:* $\dfrac{dy}{dx} = yz + x$, $\dfrac{dz}{dx} = xz + y$ *; given that* $y(0) = 1$, $z(0) = -1$ *for* $y(0.1)$, $z(0.1)$.

Sol. Here, $f_1(x, y, z) = yz + x$, $f_2(x, y, z) = xz + y$

$$h = 0.1, x_0 = 0, y_0 = 1, z_0 = -1$$

$$k_1 = hf_1(x_0, y_0, z_0) = h(y_0 z_0 + x_0) = -0.1$$

$$l_1 = hf_2(x_0, y_0, z_0) = h(x_0 z_0 + y_0) = 0.1$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = -0.08525$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = 0.09025$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = -0.0864173$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = 0.090963125$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3) = -0.073048$$

$$l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.0822679$$

$\therefore$ $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0860637$$

and $$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 0.0907823$$

Thus, $y_1 = y(0.1) = y_0 + k = 1 - 0.0860637 = 0.9139363$

$$z_1 = z(0.1) = z_0 + l = -1 + 0.0907823 = -0.9092176.$$

ASSIGNMENT

1. Use Runge-Kutta method to approximate y when $x = 0.1$ given that $x = 0$ when $y = 1$ and $\frac{dy}{dx} = x + y$.

2. Use Runge-Kutta Fourth order formula to find $y(1.4)$ if $y(1) = 2$ and $\frac{dy}{dx} = xy$. Take $h = 0.2$.

3. Use Runge-Kutta method to find y when $x = 1.2$ in steps of 0.1 given that

$$\frac{dy}{dx} = x^2 + y^2 \quad \text{and} \quad y(1) = 1.5$$

4. (*i*) Write the main steps to be followed in using the Runge-Kutta method of fourth order to solve an ordinary diff. equation of first order. Hence solve $\frac{dy}{dx} = x^3 + y^3$, $y(0) = 1$ and step length $h = 0.1$ up to three iterations.

   (*ii*) State Runge-Kutta method of fourth order. Using this method, find the values of $y(0.2)$, $y(0.4)$ and $y(0.6)$ for the following initial value problem $\frac{dy}{dx} = x^3 - y^3$ with condition that $y(0) = 1$.

   (*U.P.T.U. 2015*)

5. Use classical Runge-Kutta method of fourth order to find the numerical solution at $x = 1.4$ for $\frac{dy}{dx} = y^2 + x^2$, $y(1) = 0$. Assume step size $h = 0.2$.

6. Given that $y' = x^2 - y$, $y(0) = 1$, find $y(0.1)$, $y(0.2)$ using Runge-Kutta method of fourth order.

7. (*i*) Estimate $y(1)$ if $2yy' = x^2$ and $y(0) = 2$ using Runge-Kutta method of fourth order by taking $h = 0.5$. Also compare the result with exact value. (*G.B.T.U. 2011*)

   (*ii*) Estimate $y(0.8)$ using Runge-Kutta method of fourth order (perform two iterations) for the differential equation $2y\frac{dy}{dx} = x^2$, $y(0) = 2$. Also compare the result with exact value.

   (*G.B.T.U. 2013*)

8. Using Runge-Kutta method, find $y(0.2)$ given that $\frac{dy}{dx} = 3x + \frac{1}{2}y$, $y(0) = 1$ taking $h = 0.1$.

9. Given that $\frac{dy}{dx} = 1 + xy$; $y(0) = 2$. Using Runge-Kutta fourth order method, find $y(0.1)$, $y(0.2)$.

   [*G.B.T.U. MCA (SUM) 2010*]

10. Use Runge-Kutta method of fourth order to solve the following differential equation in the interval [0, 0.4]:

$$\frac{dy}{dx} = \frac{y + x}{y - x}, \quad y(0) = 1. \quad \text{Take } h = 0.2.$$ [*G.B.T.U. (C.O.) 2011*]

11. Using Runge-Kutta method of fourth order, solve the following differential equation:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2} \quad \text{with} \quad y(0) = 1 \quad \text{at } x = 0.2, 0.4.$$ (*G.B.T.U. 2010*)

12. Use Runge-Kutta formula of fourth order to find the numerical solution at $x = 0.6$ and 0.8 for the differential equation $y' = \sqrt{x + y}$, $y(0.4) = 0.41$. Assume the step length $h = 0.2$.
[*M.T.U. 2012, G.B.T.U. (MCA) 2011*]

13. (*i*) Apply Runge-Kutta fourth order method to solve $10 \dfrac{dy}{dx} = x^2 + y^2$; $y(0) = 1$ for $0 < x \le 0.4$ and $h = 0.1$. (*G.B.T.U. 2013, 2011*)

(*ii*) Apply Runge-Kutta method of fourth order to solve the initial value problem:

$$5 \frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1$$

and find y in the interval $0 \le x \le 0.2$, taking $h = 0.1$. (*G.B.T.U. 2012*)

14. Using Runge-Kutta method of fourth order, solve for $y(0.1)$, $y(0.2)$ and $y(0.3)$ given that $y' = xy + y^2$, $y(0) = 1$. [*M.T.U. (MCA) 2012, G.B.T.U. (C.O.) 2010*]

15. Using fourth order Runge-Kutta method, solve the initial value problem $\dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + y^2 = 0$ with initial conditions $y = 1$ and $\dfrac{dy}{dx} = 0$ when $x = 0$ in the interval [0, 0.2] and step size $h = 0.1$.
(*M.T.U. 2013*)

16. Solve $\dfrac{dy}{dx} = x + z$, $\dfrac{dz}{dx} = x - y^2$ for $y(0.1)$, $z(0.1)$ given that $y(0) = 2$, $z(0) = 1$ by Runge-Kutta IV order method.

17. Given the initial value problem $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - y^2}$, $y(1) = 3$,

Find the numerical solution at $x = 1.2$ and $x = 1.5$ by using Runge-Kutta method of fourth order.
(*U.P.T.U. 2014*)

Answers

1. 1.11034 **2.** $y(1.2) = 2.4921$, $y(1.4) = 3.2320$

3. $y(1.1) = 1.8955$, $y(1.2) = 2.5041$

4. (*i*) 1.118057, 1.291457, 1.584057 (*ii*) 0.8456, 0.7510, 0.7015

5. $y(1.2) = 0.246326$, $y(1.4) = 0.622751489$

6. $y(0.1) = 0.9051627$, $y(0.2) = 0.8212695$

7. (*i*) 2.0816705

(*ii*) $y(0.4) = 2.005326$, $y(0.8) = 2.042221$; $y(0.8)$ [Exact value] = 2.042221829

8. 1.1749 **9.** 2.110359, 2.24309

10. $y(0.2) = 1.23923$, $y(0.4) = 1.54892$ **11.** 1.195999, 1.375269

12. $y(0.6) = 0.61035$, $y(0.8) = 0.84899$

13. (*i*) $y(0.1) = 1.0101$, $y(0.2) = 1.0206$, $y(0.3) = 1.0317$, $y(0.4) = 1.0437$

(*ii*)

x:	0	0.1	0.2
y:	1	1.020475	1.04221

14. $y(0.1) = 1.1168873$, $y(0.2) = 1.2773914$, $y(0.3) = 1.50412$

15.

x:	0	0.1	0.2
y:	1	0.995	0.98

16. 2.0845, 0.586.

17. $y(1.2) = 2.8233$

$y(1.5) = 2.39265$

TEST YOUR KNOWLEDGE

1. What do you mean by initial value problem? (*U.P.T.U. 2014*)
2. What do you mean by numerical differentiation? Explain in brief. (*M.T.U. 2012*)
3. Let I = $\int_{x_0}^{x_3} f(x)\,dx$ where $f(x)$ is a third degree polynomial. Write the formula you will like to use to find the approximate value of I. It is given that the data are equispaced. (*M.T.U. 2012*)
4. Derive Newton-Cote's quadrature formula for numerical integration. (*M.T.U. 2013*)
5. Show that $y' = \frac{1}{h}\left[\Delta y - \frac{1}{2}\Delta^2 y + \frac{1}{3}\Delta^3 y - \frac{1}{4}\Delta^2 y + \ldots\right]$. (*M.T.U. 2013*)
6. Calculate the value of $\int_4^{5.2} \log_e x\,dx$ by trapezoidal rule. (*M.T.U. 2013*)
7. Explain two types of errors in numerical differentiation. [*M.T.U. (MCA) 2012*]
8. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Simpson's one-third rule. [*M.T.U. (MCA) 2012*]
9. Discuss and explain the working of Runge-Kutta's II and IV order methods.
10. What is the disadvantage of Picard's method?
11. Given the table:

x:	0	0.5	1
y:	1	0.8	0.5

then find $\int_0^1 y\,dx$ by trapezoidal rule.

12. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ (taking h = 1/4) by Simpson's 1/3 rule.
13. Derive Picard's method of successive approximations.
14. Derive Simpson's 1/3rd rule from Newton-Cote's quadrature formula. Explain its limitations and usefulness.
15. Derive Simpson's 3/8th rule from Newton-Cote's quadrature formula. (*M.T.U. 2014*)
16. Derive Trapezoidal rule from Newton-Cote's quadrature formula.
17. What are the single-step and multi-step methods? Differentiate.
18. What do you mean by a boundary value problem?
19. Explain Milne's Simpson predictor-corrector method for solution of an ordinary differential equations numerically.
20. Write a short note on numerical differentiation.

Answers

3. Simpson's 3/8th rule **6.** 1.82765 **8.** 1.366173413 **11.** 0.775
12. 0.7854.

EXAMINATION PAPERS

U.P.T.U., LUCKNOW

NAS–301

B. Tech. [SEMESTER-III]

(ODD SEM.) THEORY EXAMINATION, 2014–2015
MATHEMATICS–III

(*PAPER ID: 199320*)

Time: 3 Hours *Total Marks: 100*

Note: Attempt All Questions. All Questions carry equal marks.

1. Attempt **any four** parts of the following: **(5 × 4 = 20)**

(*a*) State Cauchy-Riemann theorem for an analytic function. Test the analyticity of the following function :

$$f(z) = \begin{cases} \dfrac{(x^3 - y^3) + i\,(x^3 + y^3)}{x^2 + y^2}, & \text{if } x \neq 0 \text{ and} \\ 0, & \text{if } z = 0 \end{cases}$$

(*b*) State Cauchy-integral theorem for an analytic function. Verify this theorem by integrating the function $z^3 + iz$ along the boundary of the rectangle with vertices $+1, -1, i, -i$.

(*c*) Show that the function $u = \dfrac{1}{2}\log(x^2 + y^2)$ is harmonic. Find the harmonic conjugate of u.

(*d*) Evaluate the integral $\displaystyle\int \frac{e^{2z}}{(z+1)^5}\,dz$, around the boundary of the circle $|z| = 2$.

(*e*) Find the Taylor series expansion of the function $\tan^{-1} z$ about the point $z = \pi/4$.

(*f*) Evaluate the integral $\displaystyle\int_0^{\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\,d\theta$.

2. Attempt **any two** parts of the following: **(10 × 2 = 20)**

(*a*) Find the Fourier transform of the following function $f(x) = 1 - x^2$, if $|x| \le 1$ and $f(x) = 0$, if $|x| > 1$.

(*b*) Using z-transform, solve the following difference equation

$Y_{n+2} - (2\cos\alpha)Y_{n+1} + Y_n = 7^n$ with the conditions that $Y_0 = 5$, $Y_1 = 1$.

(*c*) State the convolution theorem for Fourier transform. Prove that the Fourier transform of the convolution of the two functions is equal to the product of their Fourier transforms.

3. Attempt **any two** parts of the following: **(10 × 2 = 20)**

(*a*) Define skewness and kurtosis of a distribution. The first four moments of a distribution are 0, 2.5, 0.7, and 18.71. Find the coefficient of skewness and kurtosis.

(*b*) Fit a second degree parabola to the following data:

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	8	13	5

(*c*) Define coefficient of correlation and regression. If θ is the acute angle between the two lines of regression then prove that $\tan\theta = \dfrac{1-r^2}{r}\dfrac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$

where r, σ_x, σ_y have their usual meanings. Give the significance of the formula when $r = 0$ and $r = \pm 1$.

4. Attempt any two parts of the following: **(10 × 2 = 20)**

(*a*) Derive Newton-Raphson's method to find a root of the equation $f(x) = 0$. Prove that this method has quadratic convergence.

(*b*) Apply Newton's divided difference method to obtain an interpolatory polynomial for the following data:

x	3	5	7	9	11	13
$f(x)$	31	51	17	19	90	110

(*c*) Obtain Lagrange's Interpolatory for the following data:

x	1	3	5	7	10
$f(x)$	13	31	25	37	101

Find the values of $f(4)$ and $f(8.5)$.

5. Attempt any two parts of the following: **(10 × 2 = 20)**

(*a*) Solve the following system of linear equations using Gauss-Seidel method

$$10x + 3y + 7z = 41$$
$$3x + 20y + 17z = 101$$
$$x + 19y + 23z = 201$$

Perform three iterations.

(*b*) State Simpson's three-eighth rule. Using this rule, evaluate the following integral

$$\int_0^6 \frac{x}{1+x^5}\,dx.$$

(*c*) State Runge-Kutta method of fourth order. Using this method, find the values of $y(0.2)$, $y(0.4)$ and $y(0.6)$ for the following initial value problem $\dfrac{dy}{dx} = x^3 - y^3$ with condition that $y(0) = 1$.

U.P.T.U., LUCKNOW

AS–303

B. Tech. [SEMESTER-III]

(ODD SEM.) THEORY EXAMINATION, 2014–2015
ENGINEERING MATHEMATICS–III

(*PAPER ID: 990303*)

Time: 3 Hours *Total Marks: 100*

UNIT–1

1. Answer **any four** from the following: **(4 × 5 = 20)**

(*a*) If $f(z)$ is a regular function of z, then prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$.

(*b*) Find the analytic function $f(z) = u + iv$, given that $v = e^x(x\sin y + y\cos y)$.

(*c*) Evaluate the following integral using Cauchy's integral formula $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$

where C is the circle $|z| = \frac{3}{2}$.

(*d*) Expand $f(z) = \frac{1}{(z-1)(z-2)}$ for $1 < |z| < 2$.

(*e*) Determine the poles of the following function and residue at each pole:

$f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and hence evaluate $\int_C \frac{z^2 dz}{(z-1)^2(z+2)}$ where C $|z| = 3$.

(*f*) Evaluate $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$ by contour integration in the complex plane.

UNIT–2

2. Answer **any four** from the following: **(4 × 5 = 20)**

(*a*) Find Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$.

(*b*) Using Parseval's identity, show that $\int_0^{\infty}\frac{x^2 dx}{(x^2+1)^2} = \frac{\pi}{4}$.

(*c*) Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$ subject to the condition:

(*i*) $u = 0$ when $x = 0, t > 0$ (*ii*) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$ when $t = 0$

(*iii*) $u(x, t)$ is bounded.

(*d*) Solve the difference equation $y_{k+1} - 2y_{k-1} = 0$, $k \geq 1$, $y_{(0)} = 1$.

(*e*) Find the Z-transform of $\sin \alpha k$, $k \geq 0$.

(*f*) Find $Z^{-1} \dfrac{9z^3}{(3z-1)^2 (z-2)}$.

3. Answer **any four** from the following: **(4 × 5 = 20)**

(*a*) Three urns contains 6 red, 4 black; 4 red, 6 black, 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the first urn.

(*b*) Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well-shuffled cards at least once in 104 consecutive trials.

(*c*) Find the mean and standard deviation of Normal distribution.

(*d*) A manufacturer of envelopes knows that the weight of the envelopes is normally distributed with mean 1.9 gm and variance 0.01 square gm. Find how many envelopes weighing (*i*) 2 gm or more, (*ii*) 2.1 gm or more, can be expected in a given packet of 1000 envelopes.

[Given: If t is the normal variable, then $\phi(0 \leq t \leq 1) = 0.3413$ and $\phi(0 \leq t \leq 2) = 0.4772$]

(*e*) Find the moment generating function of Binomial distribution about its mean.

(*f*) If the probability density function of a random variable x is

$$f(x) = \begin{cases} kx^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1, \alpha > 0, \beta > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find k and mean of x.

4. Answer **any two** from the following: **(2 × 10 = 20)**

(*a*) If an approximate root of the equation $x(1 - \log_e x) = 0.5$ lies between 0.1 and 0.2, find the value of the root correct to three decimal places by Newton-Raphson method.

(*b*) Solve the system of equations

$$x + y + 54z = 110$$
$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$

Using Gauss-Seidel iteration method.

(*c*) Find the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0 = y_3 = 0$

x	– 1	0	1	2
y	– 1	1	3	35

5. Answer **any two** from the following: **(2 × 10 = 20)**

(*a*) The velocity V of a particle at distances from a point on its path is given by the table:

S	0	10	20	30	40	50	60	feet
V	47	58	64	65	61	52	38	feet/sec

Estimate the time taken to travel 60 feet by using Simpson's one-third rule. Compare the result with Simpson's $\frac{3}{8}$ rule.

(*b*) By applying the fourth order Runge-Kutta Method find $y(0.2)$ from $y' = y - x$, $y(0) = 2$ taking $h = 0.1$

(*c*) The differential equation $\frac{dy}{dx} = y - x^2$ is satisfied by $y(0) = 1$, $y(0.2) = 1.12186$, $y(0.4) = 1.46820$, $y(0.6) = 1.7379$. Compute the value of $y(0.8)$ by Milne's predictor-corrector formula.

U.P.T.U., LUCKNOW

B. Tech. (SEM. III) ODD SEMESTER THEORY EXAMINATION 2013–2014 AS–303

MATHEMATICS –III

(*PAPER ID: 1226*)

Time: 3 Hours *Total Marks: 100*

Note: Attempt **all** questions from each Section as indicated. The symbols have their usual meaning.

SECTION–A

1. Attempt **all** parts of this Section. Each part carries 2 marks: **(2 × 10 = 20)**

(*a*) Define Conformal Transformation.

(*b*) Find residue of $f(z) = \dfrac{z^2}{z^2 + 3z + 2}$ at the pole – 1.

(*c*) Define Fourier Transform of a function $f(x)$.

(*d*) Find the Z-Transform of $\{a^k\}$, $k \geq 0$.

(*e*) Define coefficients of Skewniess.

(*f*) What is Total Probability Theorem?

(*g*) Define Spline Function.

(*h*) Show that $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$.

(*i*) Define rate of convergence.

(*j*) What do you mean by initial value problem?

SECTION–B

2. Attempt **any three** parts of this Section. **(10 × 3 = 30)**

(*a*) State and prove Cauchy integral formula. Also evaluate $\oint_C \dfrac{z^2+1}{z^2-1}\,dz$, where C is the circle:

(*i*) $|z - 1| = 1$ (*ii*) $|z| = \dfrac{1}{2}$.

(*b*) Using Fourier Transform, solve $\dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$; $y(x, 0) = f(x)$.

(*c*) The first four moments of a distribution about the value '4' of the variables are – 1.5, 17, – 30 and 108. Find the moments about mean and about origin. Also find Skewness and Kurtosis.

(*d*) Use Gauss-Seidel method to solve the following system of simultaneous equations:

$$9x + 4y + z = -17$$
$$x - 2y - 6z = 14$$
$$x + 6y = 4$$

Perform four iterations.

(*e*) Given $\frac{dy}{dx} = y - x$, $y(0) = 2$. Find $y(0.1)$ and $y(0.2)$ correct to four decimal places by Runge-Kutta fourth order method.

SECTION–C

Note: All questions of this Section are compulsory. Attempt **any two** part from each question: **(5 × 2 × 5 = 50)**

3. (*a*) Verify Cauchy's theorem by integrating z^3 along the boundary of a square with vertices at $1 + i$, $1 - i$, $-1 + i$ and $-1 - i$.

(*b*) Evaluate the following integral by using complex integration $\int_0^{\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta$.

(*c*) Determine the analytic function $f(z) = u + iv$, in terms of z, whose real part is $e^{-x}(x \sin y - y \cos y)$.

4. (*a*) Find the Fourier transform of: $F(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$. Hence evaluate

(*i*) $\int_{-\infty}^{\infty} \frac{\sin ap \cos px}{p}\, dp$ (*ii*) $\int_0^{\infty} \frac{\sin p}{p}\, dp$.

(*b*) Find the inverse Z-transform of: $F(z) = \frac{1}{(z-3)(z-2)}$ for

(*i*) $|z| < 2$ (*ii*) $2 < |z| < 3$ (*iii*) $|z| > 3$.

(*c*) Solve by Z-transform the difference equation:
$y_{k+2} + 6y_{k+1} + 9y_k = 2^k$; $(y_0 = y_1 = 0)$.

5. (*a*) State and prove Baye's theorem.

(*b*) A continuous random variable X has a p.d.f. $f(x) = 3x^2$, $0 \le x \le 1$. Find a and b such that:

(*i*) $P(x \le a) = P(x > a)$, and (*ii*) $P(x > b) = 0.05$.

(*c*) Fit a Poisson distribution to the following data and calculate theoretical frequencies:

Death	0	1	2	3
Frequencies	122	260	15	2

6. (*a*) Find a positive value of $(17)^{1/3}$ correct to four decimal places by Newton-Raphson method.

(*b*) Obtain cubic spline for the following data:

x	0	1	2	3
$f(x)$	1	2	32	244

With the end conditions $M_0 = M_3 = 0$ for (0, 1). Hence compute $f(0.5)$.

(*c*) From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46.

Age	45	50	55	60	65
Premium (in rupees)	114.84	96.16	83.32	74.48	68.48

7. (*a*) The table given below reveals the velocity 'v' of a body during the time 't' specified. Find its acceleration at $t = 1.1$.

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

(*b*) Evaluate $\int_0^6 \frac{e^x}{x+1}\, dx$ by Simpson's 3/8$^{\text{th}}$ rule.

(*c*) Using Milne's method, solve $\frac{dy}{dx} = 1 + y^2$ with initial condition $y(0) = 0$, $y(0.2) = 0.2027$, $y(0.4) = 0.4228$, $y(0.6) = 0.6841$, obtain $y(0.8)$.

B. Tech. [SEMESTER III/IV]

THEORY EXAMINATION, 2013–2014
MATHEMATICS –III

Time: 3 Hours | *Total Marks: 100*

Note: Attempt all questions.

1. Attempt **any four** parts of the question: **(5 × 4 = 20)**

(*a*) Define analytic function. Discuss the analyticity of $f(z) = \text{Re}(z^3)$ in the complex plane.

(*b*) Show that $v(x, y) = e^{-x}(x \cos y + y \sin y)$ is harmonic. Find its harmonic conjugate.

(*c*) Integrate $f(z) = \text{Re}(z)$ from $z = 0$ to $z = 1 + 2i$.

(*i*) along straight line joining $z = 0$ to $z = 1 + 2i$.

(*ii*) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + 2i$.

(*d*) Evaluate $\int_C \frac{(1+z)\sin z}{(2z-3)^2}\,dz$, where C is the circle $|z - i| = 2$ counter-clockwise.

(*e*) Find all Taylor and Laurent's series expansion of the following function about $z = 0$:

$$f(z) = \frac{-2z+3}{z^2 - 3z + 2}$$

(*f*) Use contour integration to evaluate $\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

2. Attempt **any two** parts of the following: **(2 × 10 = 20)**

(*a*) Define moment generating function. Why is it called moment generating function?

If $P(X = x) = \frac{1}{2^x}$, $x = 1, 2, 3, \ldots$ find the moment generating function of x. Hence obtain the variance.

(*b*) Determine the normal equation if the curve $y = ax + bx^2$ is fitted to the data (x_i, y_i), $i = 1, 2, \ldots, m$. Hence fit this curve to the data:

x :	1	2	3	4	5
y :	1.8	5.1	8.9	14.1	19.8

(*c*) Calculate the coefficient of correlation between the following ages of husband (x) and wife (y) by taking 30 and 28 as assumed mean in case of x and y respectively:

x :	24	27	28	28	29	30	32	33	35	35	40
y :	18	20	22	25	22	28	28	30	27	30	32

3. Attempt **any two** parts of the following: **(2 × 10 = 20)**

(*a*) Out of 800 families with four children each, how many families would be expected to have

(*i*) 2 boys and 2 girls (*ii*) at least one boy

(*iii*) no girl (*iv*) atmost two girls.

Assume equal probabilities for boys and girls.

(*b*) The groups of 100 people each were taken for testing the use of a vaccine. 15 persons contracted the disease out of the inoculated persons, while 25 contracted the disease in the other group. Test the efficiency of the vaccine using chi-square test. (the value of χ^2 for one degree of freedom at 5% level of significance is 3.84)

(*c*) Calculate the trend values by the method of least square fit to a straight line and hence estimate profit for 1981:

Year :	1971	1972	1973	1974	1975	1976	1977
Profit (in thousands) :	60	72	75	65	80	85	95

4. Attempt **any four** parts of the following:

(*a*) Explain Newton-Raphson method and use it to find the positive root of $x^4 = x + 10$ correct to three decimal places.

(*b*) Find the root of the equation $2x\,(1 - x^2 + x) \ln x = x^2 - 1$ lying in the interval [0, 1] using Regula-Falsi method.

(*c*) Prove that $1 + \delta^2\mu^2 = \left(1 + \frac{1}{2}\delta^2\right)^2$, where symbols have their usual meanings for finite differences.

(*d*) Use Newton-Gregory formula to interpolate the value of y at $x = 36$ from the following data:

x :	21	25	29	33	37
y :	18.4	17.8	17.1	16.3	15.5

(*e*) Find $f(x)$ as a polynomial in x for the following data using Newton's divided difference formula:

x :	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

(*f*) Using Lagrange's interpolation formula, find polynomial which takes the values 3, 12, 15, – 21 when x has the values 3, 2, 1, – 1.

5. Attempt **any two** parts of the following:

(*a*) Decompose $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & -2 \\ 3 & 2 & -4 \end{bmatrix}$ in the form LU, where L is lower triangular matrix and U is the upper triangular matrix and hence solve the system of equations:

$$4x_1 + x_2 + x_3 = 6$$
$$x_1 + 4x_2 - 2x_3 = 4$$
$$3x_1 + 2x_2 - 4x_3 = 6$$

(*b*) (*i*) A slider in a machine moves along fixed straight rod. Its distances x(m) along the rod are given at various times (sec.)

t :	1	1.1	1.2	1.3	1.4	1.5
x :	16.40	19.01	21.96	25.29	29.03	33.21

Find the velocity of the slider at $t = 1.1$ sec.

(*ii*) Evaluate $\int_0^{2\pi} (e^{-t} \sin 10t)\, dt$ using Simpson's rule with eight intervals.

(*c*) Given the initial value problem $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$, $y(1) = 3$

Find the numerical solution of $x = 1.2$ and $x = 1.5$ by using Runge-Kutta method of fourth order.

B. Tech. (SEM. III) ODD SEMESTER THEORY EXAMINATION, 2013–2014 AS301

MATHEMATICS–III

Time: 3 Hours *Total Marks: 100*

Note: Attempt **all** questions from each Section as indicated. The symbols have their usual meaning.

SECTION–A

1. Attempt **all** parts of this Section. Each part carries 2 marks: **(2 × 10 = 20)**

(*a*) Find residue of $f(z) = \dfrac{2z+1}{z^2 - z - 2}$ at the pole $z = -1$.

(*b*) Define harmonic function.

(*c*) State Convolution theorem for Fourier Transform.

(*d*) Find the Z-Transform of $\{^nC_k\}$, $0 \le k \le n$.

(*e*) Define coefficients of kurtosis.

(*f*) Define marginal and conditional distribution.

(*g*) Prove that: $|(X, Y)| < \|X\| \, \|Y\|$.

(*h*) Define Abelian group.

(*i*) Define rate of convergence.

(*j*) Write the formula for Simpson's 3/8 rule.

SECTION–B

Note: Attempt any **three** parts of this Section. **(10 × 3 = 30)**

2. (*a*) Apply calculus residues to prove that: $\displaystyle\int_0^\infty \frac{\cosh ax}{\cosh \pi x}\, dx = \frac{1}{2} \sec \frac{a}{2}$.

(*b*) If $F_c(p) = \dfrac{1}{2} \tan^{-1} \dfrac{2}{p^2}$, then find $f(x)$.

(*c*) Show that Poisson distribution is a limiting form of binomial distribution when p is a very small and n is very large. Also find mean and variance of Poisson distribution.

(*d*) If $p = p(x) = p_0 + p_1x + p_2x^2$ and $q = q(x) = q_0 + q_1x + q_2x^2$, then the inner product is defined by:

$(p, q) = p_0q_0 + p_1q_1 + p_2q_2$ for the vectors $X_1 = 1 + 2x + 3x^2$, $X_2 = 3 + 5x + 5x^2$, $X_3 = 2 + x + 8x^2$. Find the orthogonal vectors.

(*e*) Use Gauss-Seidel method to solve the following system of simultaneous equations:

$$83x + 11y - 4z = 95$$
$$7x + 52y + 13z = 104$$
$$3x + 8y + 29z = 71$$

Perform four iterations.

SECTION–C

Note: All questions of this Section are compulsory. Attempt any **two** parts from each question:

(10 × 5 = 50)

3. (*a*) In a two-dimensional fluid flow, the stream function is

$\psi = -\dfrac{y}{x^2 + y^2}$, find the velocity potential Φ.

(*b*) Expand $f(z) = \dfrac{z}{(z-1)(z-2)}$ in Laurent series vaild for region:

(*i*) $|z-1| > 1$ (*ii*) $0 < |z-2| < 1$.

(*c*) State and prove Cauchy's Theorem.

4. (*a*) Find Fourier cosine transform of $\dfrac{1}{1+x^2}$ and hence find Fourier sine Transform of $\dfrac{x}{1+x^2}$.

(*b*) Find the inverse Z-transform of $F(z) = \dfrac{9z^3}{(z-2)(3z-1)^3}$.

(*c*) Solve by Z-transform the difference equation:

$y_{k+2} - 2y_{k+1} + y_k = 3k + 5, y(0) = 0, y(1) = 1.$

5. (*a*) In a certain factory manufacturing razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Use suitable distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 20,000 packets.

(*b*) Find the moment generating function of the exponential distribution:

$f(x) = \dfrac{1}{c} e^{-x/c}, 0 \le x \le \infty, c > 0$. Hence find its mean and S.D.

(*c*) Calculate the first four moments about the mean for the following data:

Class-interval:	0–10	10–20	20–30	30–40	40–50
Frequency:	10	20	40	20	10

6. (*a*) Examine the following vectors for linear dependence and find the relation, if it exists:

$X_1 = (1, 2, 1), X_2 = (3, 1, 5), X_3 = (3, -4, 5)$.

(*b*) Let V be the vector space of all real valued continuous functions over R. Then show that the solutions set W of the differential equation:

$3\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 4y = 0$, is a subspace of V.

(*c*) Show that the intersection of any two subspaces of a vector space is also a space of the same.

7. (*a*) Use Newton's Divided difference formula to find $f(x)$ from the following data:

x :	0	1	2	4	5	6
$f(x)$:	1	14	15	5	6	19

(*b*) Compute the rate of convergence of Newton-Raphson method.

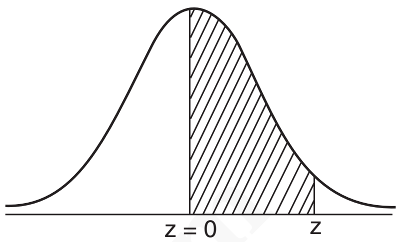
(*c*) Apply Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$, given that $\dfrac{dy}{dx} = x + y$ with initial condition $y = 1$ at $x = 0$.

APPENDIX

Table 1: NORMAL TABLE

AREAS UNDER THE STANDARD NORMAL

$$\textbf{CURVE} = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{z^2}{2}}\, dz$$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2485	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.8729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4255	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4930	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4999	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

www.cgaspirants.com

Table 2 : SIGNIFICANT VALUES t_v (α) OF t-DISTRIBUTION (TWO TAIL AREAS) [$|t| > t_v(\alpha)$] = α

d.f.	*Probability (Level of Significance)*					
(ν)	*0.50*	*0.10*	*0.05*	*0.02*	*0.01*	*0.001*
1	1.00	6.31	12.71	31.82	63.66	636.62
2	0.82	0.92	4.30	6.97	6.93	31.60
3	0.77	2.32	3.18	4.54	5.84	12.94
4	0.74	2.13	2.78	3.75	4.60	8.61
5	0.73	2.02	2.57	3.37	4.03	6.86
6	0.72	1.94	2.45	3.14	3.71	5.96
7	0.71	1.90	2.37	3.00	3.50	5.41
8	0.71	1.80	2.31	2.90	3.36	5.04
9	0.70	1.83	2.26	2.82	3.25	4.78
10	0.70	1.81	2.23	2.76	3.17	4.59
11	0.70	1.80	2.20	2.72	3.11	4.44
12	0.70	1.78	2.18	2.68	3.06	4.32
13	0.69	1.77	2.16	2.05	3.01	4.22
14	0.69	1.76	2.15	2.62	2.98	4.14
15	0.69	1.75	2.13	2.60	2.95	4.07
16	0.69	1.75	2.12	2.58	2.92	4.02
17	0.69	1.74	2.11	2.57	2.90	3.97
18	0.69	1.73	2.10	2.55	2.88	3.92
19	0.69	1.73	2.09	2.54	2.86	3.88
20	0.69	1.73	2.09	2.53	2.85	3.85
21	0.69	1.72	2.08	2.52	2.83	3.83
22	0.69	1.72	2.07	2.51	2.42	3.79
23	0.69	1.71	2.07	2.50	2.81	3.77
24	0.69	1.71	2.06	2.49	2.80	3.75
25	0.68	1.71	2.06	2.49	2.79	3.73
26	0.68	1.71	2.06	2.48	2.78	3.71
27	0.68	1.70	2.05	2.47	2.77	3.69
28	0.68	1.70	2.05	2.47	2.76	3.67
29	0.68	1.70	2.05	2.46	2.76	3.66
30	0.68	1.70	2.04	2.46	2.75	3.65
∞	0.67	1.65	1.96	2.33	2.58	3.29

Table 3 : F-Distribution

Values of F for F-Distributions with 0.05 of the Area in the Right Tail

	Degrees of freedom for numerator																		
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*12*	*15*	*20*	*24*	*30*	*40*	*60*	*120*	∞
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	9.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	3.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	3.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.17	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.98	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.94	2.40	2.34	2.28	2.24	2.16	2.29	2.01	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.64	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Table 4 : CHI-SQUARE

Significant Values χ^2 (α) of Chi-Square Distribution Right Tail Areas for Given Probability α,

$P = P_r(\chi^2 > \chi^2(\alpha)) = \alpha$

And v is Degrees of Freedom (d.f.)

Degree of freedom (ν)	*Probability (Level of Significance)*						
	0.99	*0.95*	*0.50*	*0.10*	*0.05*	*0.02*	*0.01*
1	.000157	.00393	.455	2.706	3.841	5.214	6.635
2	.0201	.103	1.386	4.605	5.991	7.824	9.210
3	.115	.352	2.366	6.251	7.815	9.837	11.341
4	.297	.711	3.357	7.779	9.488	11.668	13.277
5	.554	1.145	4.351	9.236	11.070	13.388	15.086
6	.872	2.635	5.348	10.645	12.592	15.033	16.812
7	1.239	2.167	6.346	12.017	14.067	16.622	18.475
8	1.646	2.733	7.344	13.362	15.507	18.168	20.090
9	2.088	3.325	8.343	14.684	16.919	19.679	21.669
10	2.558	3.940	9.340	15.987	18.307	21.161	23.209
11	3.053	4.575	10.341	17.275	19.675	22.618	24.725
12	3.571	5.226	11.340	18.549	21.026	24.054	26.217
13	4.107	5.892	12.340	19.812	22.362	25.472	27.688
14	4.660	6.571	13.339	21.064	23.685	26.873	29.141
15	4.229	7.261	14.339	22.307	24.996	28.259	30.578
16	5.812	7.962	15.338	23.542	26.296	29.633	32.000
17	6.408	8.672	15.338	24.769	27.587	30.995	33.409
18	7.015	9.390	17.338	25.989	28.869	32.346	34.805
19	7.633	10.117	18.338	27.204	30.144	33.687	36.191
20	8.260	10.851	19.337	28.412	31.410	35.020	37.566
21	8.897	11.591	20.337	29.615	32.671	36.343	38.932
22	9.542	12.338	21.337	30.813	33.924	37.659	40.289
23	10.196	13.091	22.337	32.007	35.172	38.968	41.638
24	10.856	13.848	23.337	32.196	36.415	40.270	42.980
25	11.524	14.611	24.337	34.382	37.65	41.566	44.314
26	12.198	15.379	25.336	35.363	38.885	41.856	45.642
27	12.879	16.151	26.336	36.741	40.113	41.140	46.963
28	13.565	16.928	27.336	37.916	41.337	45.419	48.278
29	14.256	17.708	28.336	39.087	42.557	46.693	49.588
30	14.933	18.493	29.336	40.256	43.773	47.962	50.892

Note. For degrees of freedom (ν) greater than 30, the quantity $\sqrt{2\chi^2} - \sqrt{2\nu - 1}$ may be used as a normal variate with unit variance.

ABOUT THE BOOK

The book has been renovated in the light of the latest modified syllabus. Thus the book will work as the latest ready reckoner for the readers. All the questions of the latest university papers have been mostly marked and rarely added as per the requirement. The book has been thoroughly revised and stirred. A large number of new questions have been added as solved examples. All the answers have been checked and verified. The subject matter has been made more lucid and easier to understand.

ABOUT THE AUTHORS

N.P. Bali is a prolific author of over 100 books for degree and engineering students. He has been writing books for more than forty years.

His books on the following topics are well known for their easy comprehension and lucid presentation: Algebra, Trigonometry, Differential Calculus, Integral Calculus, Real Analysis, Co-ordinate Geometry, Statics, Dynamics, etc.

Dr. Manish Goyal has been associated with the G.L.A. University, Mathura for the last one and a half decade as a competent, dynamic, successful and responsible faculty member in Mathematics. He is working as an **Associate Professor** in the Institute of Applied Sciences and Humanities there.

He is equipped with an extraordinary calibre and appreciable academic potency. He has always been synonymous with the word merit and excellence. He is awarded with the degree of Ph.D. by Dr. B.R. Ambedkar University, (Formerly Agra University), Agra in 2009. He has also qualified CSIR-NET Examination in Mathematical Sciences with an all India rank (A.I.R.)—09 in June 2011.

He has authored **71 (SEVENTY ONE)** books during this span. Out of these, he has written about **FOUR DOZEN** books on Engineering Mathematics for various Technical Universities, the prominent among them are U.P.T.U., Lucknow; R.G.P.V., Bhopal; Anna University, Chennai; P.T.U. Jalandhar; Uttarakhand Technical University; M.D.U., Rohtak, K.U., Kurukshetra, etc. His book titled **'Computer Based Numerical and Statistical Techniques'** has widely been acclaimed.

He is an ardent researcher. He has, to his credit, **EIGHT** research papers published in a leading journal of Mathematics. He has attended **EIGHT** short term courses at IITK, IITR, NITW, SVNIT and MANIT, etc., sponsored by AICTE and TEQIP during the last five years. He has also participated in various workshops. He is a reviewer of **TWO** international journals.

ISBN 978-93-80386-53-9
9789380386539- 0475

UNIVERSITY SCIENCE PRESS
(An Imprint of Laxmi Publications Pvt. Ltd.)
(An ISO 9001:2015 Company)

UEM-9443-475-ENGG MATH III (UP)-GOY

www.cgaspirants.com